Math 300 Assignment 1

Due Tuesday, September 19

1) Which of the following sentences are statements? Determine, if possible, whether the statement is conditional or not and the truth value of each statement, but DO NOT attempt any proofs.

a) There exist nonzero integers a, b, and c with $a^3 + b^3 = c^3$.

b) If p is a prime number, then $p + 2$ is a prime number.

c) For infinitely many prime numbers p, $p + 2$ is also a prime number.

d) Let M be a manifold, and let $\phi : M \to S^3$ be a homeomorphism, where S^3 is the three sphere.

e) If L is a linear operator between finite-dimensional vector spaces, then L admits a nontrivial invariant subspace.

2) (#6, Section 1.1) The following is the statement of a theorem which can be proved using calculus or precalculus mathematics. For this theorem, a, b, and c are real numbers.

Theorem 0.1. *If f is a quadratic function of the form $f(x) = ax^2 + bx + c$ and $a < 0$, then the function f has a maximum value when $x = -\dfrac{b}{2a}$.*

Using **only** this theorem, what can be concluded about the functions given by the following formulas?

a) $g(x) = -8x^2 + 5x - 2$ b) $h(x) = -\frac{1}{3}x^2 + 3x$

c) $k(x) = 8x^2 - 5x - 7$ d) $F(x) = -x^4 + x^3 + 9$

3) (#4, Section 5.1) One of the properties of real numbers is the so-called **Law of Trichotomy**, which states that if $a, b \in \mathbb{R}$, then exactly one of the following is true:

$$a < b; \quad a = b; \quad b < a$$

Is the following proposition concerning sets true or false? Either provide a proof that it is true or a counterexample showing it is false. If A and B are subsets of some universal set, then exactly one of the following is true:

$$A \subset B; \quad A = B; \quad B \subset A$$

4) Prove the quadratic formula: if a, b, and c are real numbers, then the roots of the quadratic polynomial $ax^2 + bx + c$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

5) Let $S = \{1, \{2, 3\}, 4\}$. Indicate whether each statement is true or false.

(a) $\text{card}(S) = 4$

(b) $\{1\} \in S$

(c) $\{2, 3\} \in S$

(d) $\{1, 4\} \subseteq S$

(e) $2 \in S$.

(f) $S = \{1, 4, \{2, 3\}\}$

(g) $\emptyset \subseteq S$

6) Let $U = \{1, 2, 3, ..., 14, 15\}$ and $E^c = U \setminus E$. Let $A = \{1, 5, 9, 13\}$ and $B = \{3, 9, 13\}$. Determine the following:

(a) $A \cup B$

(b) $A \cap B$

(c) $A \setminus B$

(d) $B \setminus A$

(e) A^c