Math 300 Assignment 5

Due Tuesday, November 14

1) (#3(c), Section 4.1) Use mathematical induction to prove that for each natural number n,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

2) For $f, g : \mathbb{R} \to \mathbb{R}$, define $f \sim g$ if $f(1) = g(1)$.

a) Let $f \in [x^2 + 1]$. What is the value of $f(1)$?

b) Prove that "$\sim$" is an equivalence relation.

3) Prove or disprove: Suppose $f : A \to B$ and $g : B \to C$ are functions. Then $g \circ f$ is bijective if and only if f is injective and g is surjective.

4) (#3, Section 6.3) For each of the following functions, determine if the function is an injection and determine if the function is a surjection. Justify all assertions. Note $\mathbb{R}^* = \{x \in \mathbb{R} \mid x \geq 0\}$

(a) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 3x + 1$ for all $x \in \mathbb{Z}$.

(b) $F : \mathbb{Q} \to \mathbb{Q}$ defined by $F(x) = 3x + 1$ for all $x \in \mathbb{Q}$.

(c) $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^3$ for all x in $\mathbb{R}$.

(d) $G : \mathbb{Q} \to \mathbb{Q}$ defined by $g(x) = x^3$ for all $x \in \mathbb{Q}$.

(e) $k : \mathbb{R} \to \mathbb{R}$ defined by $k(x) = e^{-x^2}$ for all $x \in \mathbb{R}$.

(f) $K : \mathbb{R}^* \to \mathbb{R}$ defined by $K(x) = e^{-x^2}$ for all $x \in \mathbb{R}^*$.

(g) $K_1 : \mathbb{R}^* \to T$ defined by $K_1(x) = e^{-x^2}$ for all $x \in \mathbb{R}^*$, where $T = \{y \in \mathbb{R} \mid 0 < y \leq 1\}$.

(h) $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = \dfrac{2x}{x^2 + 4}$ for all $x \in \mathbb{R}$.

(i) $H : \mathbb{R}^* \to \{x \in \mathbb{R} \mid 0 \le y \le 1/2\}$ defined by $H(x) = \dfrac{2x}{x^2+4}$ for all $x \in \mathbb{R}^*$.

5) a) Prove that if X is a set with n elements, $\mathcal{P}(X)$ has 2^n elements, so that

$$\text{card}(\mathcal{P}(X)) > \text{card}(X) \tag{1}$$

b) (Extra Credit) Prove equation (1) without the assumption that card(X) is finite. I will accept no written arguments; you must present your solution on the board in my office.

6) For each $n \in \mathbb{Z}$, define the set $A_n = \mathbb{R} - [n, n+1]$. Use a contradiction argument to prove that

$$\bigcap_{n \in \mathbb{Z}} A_n = \emptyset.$$