Math 300 Assignment 6

Due Thursday, November 30

1) (#6, Section 6.2) Let $D = \mathbb{N} - \{1, 2\}$ and define $d : D \to \mathbb{N} \cup \{0\}$ by $d(n) =$ the number of diagonals of a convex polygon with n sides. Use mathematical induction to prove that for all $n \in D$,

$$d(n) = \frac{n(n-3)}{2}.$$

Hint: To get an idea of how to handle the inductive step, use a pentagon. First, form all the diagonals that can be made from four of the vertices. Then consider how to make new diagonals when the fifth vertex is used. This may generate an idea of how to proceed from a polygon with k sides to a polygon with $k + 1$ sides.

2) (#2a) and #3, Section 4.3) Assume that $f_1, f_2, \ldots, f_n \ldots$ are the Fibonacci numbers.

a) Prove that for each $n \in \mathbb{N}$, $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$.

b) Use the result in part a) to prove that

$$\frac{f_1^2 + f_2^2 + \cdots + f_n^2 + f_{n+1}^2}{f_1^2 + f_2^2 + \cdots + f_n^2} = 1 + \frac{f_{n+1}}{f_n}$$

3) Let $M_2(\mathbb{R})$ denote all 2×2 matrices with entries from $\mathbb{R}$, and let U be the subset of invertible matrices (note that every invertible matrix is bijective). Prove that the relation $\sim$ given by

$$S \sim T \text{ if } \det(ST^{-1}) > 0$$

for $S, T \in U$ is an equivalence relation.

4) (#3, Section 5.3) Let A, B, and C be subsets of some universal set U. As part of Theorem 5.20, we proved one of De Morgan's Laws. Prove the other one. That is, prove that

$$(A \cap B)^c = A^c \cup B^c.$$

5) Let $f(1) = 1$, $f(n+1) = 1 + \frac{1}{f(n)}$ for all $n \in \mathbb{N}$, $n \geq 1$. Prove that $f(n) < \frac{1+\sqrt{5}}{2}$ for all ODD $n \geq 1$.

6) Prove or disprove: Suppose $f : A \to B$ and $g : B \to C$ are functions. Then $g \circ f$ is bijective if and only if f is injective and g is surjective.

7) Let $f : [0,1] \times [0,1] \to [0,1]$ be defined by

$$f(.a_1a_2a_3\ldots, .b_1b_2b_3\ldots) = .a_1b_1a_2b_2a_3b_3\ldots$$

where $.a_1a_2a_3\ldots$ and $.b_1b_2b_3\ldots$ are the usual (base-10) decimal expansions of elements in $[0,1]$. Assuming f is well-defined, prove that f is a bijection.