Math 300 Assignment 8

Due Thursday, November 28

1) (#6, Section 6.6) Prove part (1) of Theorem 6.35.

Let $f : S \to T$ be a function and let C and D be subsets of T. Then $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

2) (#3, Section 9.2) Prove part (2) of Theorem 9.10.

Let A and B be sets. If A is infinite and $A \subseteq B$, then B is infinite.

3) (#16, Section 8.3) The Twin Prime Conjecture states that there are infinitely many twin primes, but it is not known if this conjecture is true or false. The answers to the following questions, however, can be determined.

(a) How many pairs of primes p and q exist whereq $q - p = 3$? That is, how many pairs of primes are there that differ by 3? Prove that your answer is correct. (One such pair is 2 and 5.)

(b) How many triplets of primes of the form p, $p + 2$, and $p + 4$ are there? That is, how many triplets of primes exist where each prime is 2 more than the preceding prime? Prove that your answer is correct. Notice that one such triplet is 3, 5, and 7. **Hint**: Try setting up cases using congruence modulo 3.

4) Let $n \in \mathbb{N}$ and let $A \subset \mathbb{N}$ be a set of $2n$ consecutive integers. Let $B \subset A$ with $\text{card}(B) = n + 1$ Must there always exist $s, t \in B$ with $\gcd(s, t) = 1$?

5)

(a) Show that, for any $n \in \mathbb{N}$, $\times_{i=1}^{n} \mathbb{N}$ is countable.

(b) Explain how part (a) proves the following statement: a

finite directproduct of countable sets is countable.

(c) (Extra Credit) Prove that the countable union of countable sets is countable.