Math 300 Final

Thursday, December 14th

The even-numbered problems are definitions meant to aid you in the subsequent odd-numbered problem. Use them wisely.

1) a) Negate the statement "$P \Rightarrow (Q \vee \neg R)$".

b) Show that the statement "$P \Rightarrow (Q \vee \neg R)$" is logically equivalent to the statement "$\neg P \vee Q \vee \neg R$".

2) a) Define what it means for $m \in \mathbb{N}$ to divide $n \in \mathbb{N}$.

b) Define the set $\mathbb{Q}$ of rational numbers.

3) a) Prove that, for all $n \in \mathbb{N}$, 3 divides $n^3 - n$.

b) Prove that $\sqrt{45}$ is irrational.

4) Define an equivalence relation "$\sim$" on a set S (alternatively, you may define an equivalence relation as a subset of $S \times S$).

5) Define a relation " $\sim$ " on $\mathbb{R}$ by

$$x \sim y \text{ if } x - y = n\pi$$

for some $n \in \mathbb{Z}$. Prove that "$\sim$" is an equivalence relation.

6) Let S and T be sets. Let $\phi : S \to T$ be a function.

a) What does it mean for ϕ to be injective?

b) What does it mean for ϕ to be surjective?

c) What does it mean for ϕ to be bijective?

7) Suppose $\psi : A \to B$ is a bijection and $\gamma : S \to T$ is a bijection. Prove that the map $\phi : A \times S \to B \times T$ given by

$$\phi(a, s) = (\psi(a), \gamma(s))$$

is a bijection.

8) a) Define what it means for a set S to be countably infinite.

b) Give an explicit example of a countably infinite set that is not $\mathbb{N}$.

c) In $\mathbb{Z}_4$, define $[1]_4$.

9) Choose one of the following two problems. If you attempt both, I will grade the one you do WORSE on.

a) Let $\mathcal{S}$ be the set whose elements are the FINITE subsets of $\mathbb{N}$. Prove that $\mathcal{S}$ is countably infinite. You may use the fact that a countable union of countable sets is countable.

-OR-

b) Show if $\phi : \mathbb{Z}_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ is a bijection, then ϕ can never satisfy

$$\phi([x]_4) + \phi([y]_4) = \phi([x+y]_4).$$

EXTRA CREDIT: Prove that the first principle of mathematical induction is equivalent to the well-ordering principle, where the former is the statement

- If $S \subseteq \mathbb{N}$ is such that $1 \in S$ and if $n \in S$, then $n+1 \in S$, we must have $S = \mathbb{N}$

  and the latter is the statement

- Every nonempty subset of $\mathbb{N}$ contains a least element.