Announcements

1) Exam 1 Thursday next week

Proof By Cases

(Section 3.4)

Dividing the proof of a result into smaller pieces that are easier to establish.

Caution: make sure you have all the pieces!

Example 1: (The triangle inequality)

$\forall\ x, y \in \mathbb{R},$

$$|x+y| \leq |x| + |y| .$$

proof: By cases!

1) $x \geq 0,\ y \geq 0.$

In this case,

$$|x| = x$$

$$|y| = y$$

$$|x+y| = x+y$$

and we have equality.

2) $x \geq 0,\ y < 0$

$$|x| = x$$

$$|y| = -y$$

$$\begin{aligned} |x+y| &\leq \max\{|x|, |y|\} \\ &\leq |x| + |y| \\ &= x + (-y) \end{aligned}$$

3) $y \geq 0,\ x < 0$

Same as case 2) with

x and y interchanged

4) $x<0,\ y<0$

$$|x| = -x$$

$$|y| = -y$$

$$|x+y| = -(x+y)$$
$$= -x-y$$

and so we have equality.

This covers all cases, so

$$|x+y| \leq |x| + |y|$$

$\square$

Intro to Functions

(Section 6.1)

We will be talking about functions from sets to sets.

Definition: If S, T are sets, a **function** from S to T, denoted $f: S \to T$, is an assignment that for each $s \in S$ gives precisely one element in T

Definition: (domain, codomain, range)

If $f: S \to T$ is a function,

we call S the **domain** of f.

We call T the **codomain** of f

The **range** of f is the set

$$\boxed{f(S) = \{ t \in T \mid \exists\, s \in S,\ f(s) = t \} \subseteq T}$$

Example 2: Let $f: \mathbb{R} \to \mathbb{R}$,

$$f(x) = x^2.$$

The domain of f is: $\mathbb{R}$

The codomain of f: $\mathbb{R}$

The range of f:

$$f(\mathbb{R}) = \{ y \in \mathbb{R} \mid \exists\, x \in \mathbb{R},\ f(x) = y \}$$

$$= \{ y \in \mathbb{R} \mid \exists\, x \in \mathbb{R},\ x^2 = y \}$$

$$= [0, \infty)$$

Definition: (image, preimage)

Let $f: S \to T$, let $A \subseteq S$, $B \subseteq T$. The image of A, denoted by $f(A)$, is the set

$$f(A) = \{ t \in T \mid \exists\, a \in A,\ f(a) = t \} \subseteq T$$

The preimage of B, denoted $f^{-1}(B)$, is the set

$$f^{-1}(B) = \{ s \in S \mid \exists\, b \in B,\ f(s) = b \} \subseteq S$$

Warnings: 1) The image and preimage are <u>subsets</u>

2) The notation for preimage, f^{-1}, does not imply f is invertible!

Example 3: $f: \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$$

Domain of f: $\mathbb{R}$

Codomain of f: $\mathbb{R}$

Range of f: $\mathbb{Q}$

$$f(1/2) = 1/2$$

$$f(-5/3) = -5/3$$

$$f(\sqrt{2}) = 0$$

$$f(e) = 0$$

$$f(\pi + e) = ?$$

Let $A = [0,1]$.

$$f(A) = \{x \in [0,1] \mid x \in \mathbb{Q}\}$$

$$= [0,1] \cap \mathbb{Q}$$

$$A_1 = [2,3]$$

$$f(A_1) = ([2,3] \cap \mathbb{Q}) \cup \{0\}$$

Preimages:

$$f^{-1}(\{2\}) = \{2\}$$

$$f^{-1}(\{5/3, -11/9\}) = \{5/3, -11/9\}$$

$$f^{-1}(\{0\}) = \mathbb{R} \setminus \mathbb{Q} \cup \{0\}$$

$$f^{-1}(\{\sqrt{2}\}) = \emptyset$$

Since there are no real numbers x for which $f(x) = \sqrt{2}$

$$f^{-1}([0,1]) = (\mathbb{Q} \cap [0,1]) \cup \mathbb{R} \setminus \mathbb{Q}$$

$$f^{-1}((0,1]) = \mathbb{Q} \cap (0,1]$$