Announcements

1) HW 4 - due today by 4:30

2) HW 5 up now, due Thursday
next week

3) Latex template under "Files"
on Canvas

4) Next section: 6.3
reading quiz Thursday

Other Forms of Induction

(Section 4.2)

Extended Principle of Mathematical Induction

How to use: establish $P(1)$ is true.

Then assume $P(k)$ is true for all $1 \leq k \leq n$, prove $P(n+1)$ is true.

Second Principle of Induction

How to use: Show $\exists\ m \in \mathbb{N}$, $m \geq 1$, such that $P(m)$ is true. Then assume $P(n)$ is true for $n \geq m$, prove $P(n+1)$ is true.

i.e. Start your induction at a number that might be larger than one because the statement is false for all smaller values of n.

Example 1: Show that $\exists\ m \in \mathbb{N}$ such that $2^n > 5n+1\ \forall\ n \geq m$

$n=1$: $2 < 6$ false

$n=2$: $4 < 11$ false

$n=3$: $8 < 16$ false

$n=4$: $16 < 21$ false

$n=5$: $32 > 26$ true!

Start at $m=5$. Now assume

$2^n > 5n+1$ for some $n \geq 5$.

If $2^n > 5n+1$, we want to show $2^{n+1} > 5(n+1)+1 = 5n+6$.

$$2^{n+1} = 2^n \cdot 2$$

$$> (5n+1) \cdot 2 \quad \text{(by induction)}$$

$$= 10n+2$$

We need to show

$$10n+2 > 5(n+1)+1$$

$$= 5n+6$$

Equivalent to

$5n+2 > 6$, true for all $n \geq 1$.

This shows $2^{n+1} > 5n+6$ $\square$

Recursion

(Section 4.3)

For sequences, which are functions $f: \mathbb{N} \to S$ where S is some set.

We usually denote the value $f(n)$ by f_n.

A recursive sequence is a sequence whose n^{th} terms depends on the value of the previous terms.

Example 2:

Let $a_1 = 1$,

$$a_{n+1} = \frac{1}{1 + \frac{1}{5 + a_n}}$$

Show: i) $a_{n+1} \leq a_n \quad \forall \; n \in \mathbb{N}$

ii) $a_n \leq 1 \quad \forall \; n \in \mathbb{N}$

$$a_1 = 1, \quad a_2 = \frac{1}{1 + \frac{1}{6}} = \frac{1}{\frac{7}{6}} = \frac{6}{7}$$

$$a_3 = \frac{1}{1 + \frac{1}{5 + \frac{6}{7}}}$$

$$= \frac{1}{1 + \frac{1}{\left(\frac{41}{7}\right)}} = \frac{1}{1 + \frac{7}{41}} = \frac{41}{48}$$

i) $a_{n+1} \leq a_n$ true for $n=1$.

(checked on previous page)

Show $a_{n+1} = \dfrac{1}{1+\frac{1}{5+a_n}} \leq a_n$

if we assume

$a_n \leq a_{n-1}$. ← inductive step

Then $5+a_n \leq 5+a_{n-1}$

$$\frac{1}{5+a_n} \geq \frac{1}{5+a_{n-1}}$$

$$1+\frac{1}{5+a_n} \geq 1+\frac{1}{5+a_{n-1}}$$

$$a_{n+1} = \frac{1}{1+\frac{1}{5+a_n}} \leq \frac{1}{1+\frac{1}{5+a_{n-1}}} = a_n \checkmark$$

ii) $a_n \leq 1 \quad \forall\, n \in \mathbb{N}$.

Checked for $n=1$ two pages ago

Assume $a_n \leq 1$. ← inductive step

Show $a_{n+1} \leq 1$.

But if $a_n \leq 1$,

$$5 + a_n \leq 1 + 5 = 6$$

$$\frac{1}{5+a_n} \geq \frac{1}{6}$$

$$1 + \frac{1}{5+a_n} \geq 1 + \frac{1}{6}$$

$$a_{n+1} = \frac{1}{1+\frac{1}{5+a_n}} \leq \frac{1}{1+\frac{1}{6}} = \frac{6}{7} < 1$$

✓

Notes: 1) If you can prove the monotone convergence theorem for sequences (Math 451), you've just shown that $(a_n)_{n=1}^{\infty}$ has a limit!

2) $a_3 = \dfrac{1}{1+\frac{1}{5+a_2}} = \dfrac{1}{1+\frac{1}{5+\left(\frac{1}{1+\frac{1}{5+a_1}}\right)}}$

$$= \frac{1}{1+\frac{1}{5+\left(\frac{1}{1+\frac{1}{6}}\right)}}$$

The sequence $(a_n)_{n=1}^{\infty}$ determines a continued fraction.

Notes: 1) The first case is called the initial step.

2) Going from $P(n)$ to $P(n+1)$ is called the inductive step.

3) In 2), $P(n)$ is called the inductive hypothesis