More on Functions

(6.4 + 6.5)

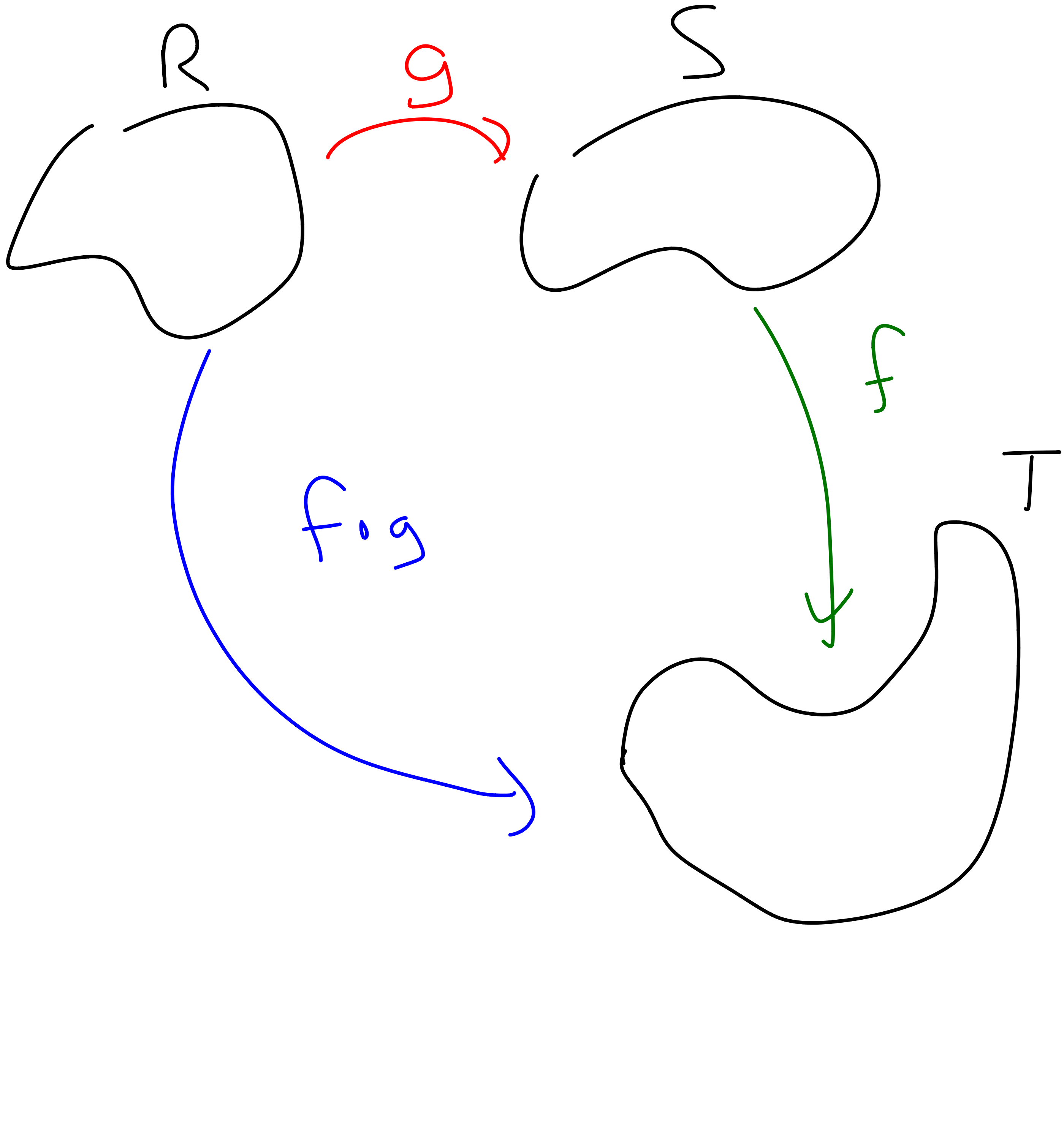
Function composition

Given $f: S \to T$ and $g: R \to S$ where S, T, R are sets, we define the composition $f \circ g$ as the function

$f \circ g: R \to T$,

$$\boxed{(f \circ g)(x) = f(g(x)) \quad \forall x \in R}$$

Picture



Example 1: $f : M_2(\mathbb{R}) \longrightarrow \mathbb{R}$

$M_2(\mathbb{R})$ = 2x2 matrices with real entries

$g : \mathbb{Z} \rightarrow M_2(\mathbb{R})$

$$g(n) = (-1)^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$f(A) = Tr(A)$

$Tr(A)$ = trace of A

$$Tr\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11} + a_{22}$$

$$g(1) = (-1)^1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$g(2) = (-1)^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Range of $g = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$

$$(f \circ g)(n) = f(g(n))$$

$$= \begin{cases} f\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right), & n \text{ even} \\ f\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right), & n \text{ odd} \end{cases}$$

$$f(g(n)) = \begin{cases} f\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right), & n \text{ even} \\ f\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right), & n \text{ odd} \end{cases}$$

$$= \begin{cases} \mathrm{Tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right), & n \text{ even} \\ \mathrm{Tr}\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right), & n \text{ odd} \end{cases}$$

$$= \begin{cases} 2, & n \text{ even} \\ -2, & n \text{ odd} \end{cases}$$

Warnings

1) function composition is non commutative in general. In particular, if $f \circ g$ is defined, $g \circ f$ may not be. (See previous example)

2) Function composition is **not** multiplication!

Composition and Injectivity & Surjectivity

Suppose $f: S \to T$, $g: R \to S$.

Then

1) If $f \circ g$ is injective, then g is injective.

2) If $f \circ g$ is surjective, then f is surjective.

<u>proof:</u> 1) By contradiction.

Suppose $f \circ g$ is injective but g is not injective.

Then $\exists\ r_1, r_2 \in R$, $r_1 \neq r_2$,

$$g(r_1) = g(r_2) = s.$$

Then

$$(f \circ g)(r_1) = f(g(r_1)) = f(s)$$
$$(f \circ g)(r_2) = f(g(r_2)) = f(s)$$

and so $f \circ g$ is not injective, contradiction. Therefore, g must be injective.

2) Similar, by contradiction.

$\square$

Invertibility

Let $f: S \to T$. We say f is invertible if $\exists$ $g: T \to S$ such that

1) $(f \circ g)(t) = t \quad \forall \, t \in T$

2) $(g \circ f)(s) = s \quad \forall \, s \in S$

Example 2: $f: \mathbb{Z} \to \mathbb{Z}$

$$f(n) = -n+1$$

Show f is invertible

Find a $g: \mathbb{Z} \to \mathbb{Z}$,

$$(g \circ f)(n) = (f \circ g)(n) = n.$$

$$g(n) = -(n-1)$$
$$= -n+1 = f(n)$$

$$(f \circ g)(n) = f(g(n)) = f(-n+1)$$
$$= -(-n+1)+1$$
$$= n-1+1 = n \checkmark$$

Example 3: $f: \mathbb{R} \to \mathbb{C}$

$$f(x) = 1$$

f is not invertible since

if $g: \mathbb{C} \to \mathbb{R}$,

$(f \circ g)(z) = z \quad \forall \; z \in \mathbb{C}$

$(g \circ f)(x) = x$, then

$$(g \circ f)(x) = g(f(x))$$
$$= g(1)$$

For g to be the inverse function, we'd need $g(1) = x \;\; \forall \; x \in \mathbb{R}$

Then g would not even be a function.

Checking for Invertibility

A function $f: S \to T$ is invertible if and only if f is bijective.