Announcements

1) Guide for Induction under "Files" on Canvas

2) For next Tuesday: read Section 5.3

Injections, Surjections, and Bijections

(Section 6.3)

Motivation: cardinality for infinite sets

Two finite sets have the same cardinality if they have the same number of elements.

For infinite sets, we need to know what "same number of elements" means

Definition: (injective function)

Also called a "one-to-one" function or a "monomorphism".

Let $f: S \to T$ where S and T are sets. We say f is injective (or is an injection) if $\forall x, y \in S$,

$$\boxed{\text{if } f(x) = f(y), \text{ then } x = y.}$$

Example 1: Let $f: \mathbb{N} \to \mathbb{Z}$

$$f(n) = 2n-5$$

Prove f is injective

Suppose that $m, n \in \mathbb{Z}$, and

$$f(m) = f(n)$$

Show: $m = n$

$$2m-5 = f(m) = f(n) = 2n-5$$

simplify

$2m = 2n$ and

$m = n$, so f is injective.

$\square$

Example 2: $\quad f : \mathbb{N} \to \mathbb{Q}$

$$f(n) = \frac{(n-2)^2}{2}$$

Show f is not injective

Negate the definition:

$\exists\ m, n \in \mathbb{N}$ with $f(n) = f(m)$

but $m \neq n$.

$$n = 1, \quad m = 3$$

$$f(n) = f(1) = \frac{(1-2)^3}{2} = \frac{1}{2}$$

$$f(m) = f(3) = \frac{(3-2)^2}{2} = \frac{1}{2}$$

so f is not injective. $\square$

Definition: (surjective function)

Also called "onto" or an "epimorphism".

Let S, T be sets, $f: S \to T$.

We say f is surjective if

$$\boxed{\forall\ y \in T,\ \exists\ x \in S,\quad f(x) = y.}$$

This says the range and codomain of f are equal.

Example 3: $f: \mathbb{C} \to \mathbb{R}$

$$f(a+bi) = a$$

Show f is surjective

Let $y \in \mathbb{R}$. We need $x \in \mathbb{C}$, $f(x) = y$.

We can let $x = y$.

Then $f(y) = y$,

so f is surjective.

$\square$

Example 4: $f: \mathbb{H} \longrightarrow \mathbb{N}$

Hamiltonian quaternions
(look it up)

$$f(a+bi+cj+dk) = 2$$

$a,b,c,d \in \mathbb{R}$

Show f is not surjective

Negation is "$\exists\ n \in \mathbb{N}$ such that $\forall\ \alpha \in \mathbb{H},\ f(\alpha) \neq n$."

Choose $n=5$. Then $\forall\ \alpha \in \mathbb{H}$, $f(\alpha) = 2 \neq 5$, so f is not surjective.

Definition: (bijective)

Let S, T be sets, $f: S \to T$.

We say f is **bijective** (or is a bijection) if f is **both** injective and surjective.

Back to examples 1 & 3

1) $f : \mathbb{N} \to \mathbb{Z}$

$f(n) = 2n - 5$. We showed

f is injective. Is f surjective?

No. Suppose $f(n) = -4$.

Then $-4 = 2n - 5$

$$1 = 2n$$

$$n = 1/2 \notin \mathbb{N}$$

So f is not surjective.

Therefore, f is not a bijection.

3) $f: \mathbb{C} \to \mathbb{R}$

$f(a+bi) = a$. We showed f is surjective.

Is f injective?

No.

$$f(6) = 6$$

$$f(6-i) = 6$$

but $6 \neq 6-i$, so f is not injective, and hence not a bijection.

Warning: (domain/codomain dependence)

Injectivity and surjectivity depend heavily on the choice of domain and codomain.

For example,

$f(x) = x^2$ is not injective as a map from $\mathbb{R}$ to $\mathbb{R}$, but it is when considered as a map from $[0, \infty)$ to $\mathbb{R}$.

Definition: (cardinality)

Two sets S and T are said to have the same **cardinality** if $\exists$ bijection $f: S \to T$.