

Announcements

1) HW #7 due last day of classes

2) HW #8 up for study purposes, not for a grade

More on Cardinality

Recall definition: two sets have the same cardinality if \exists a bijection ϕ mapping between them.

We showed $\text{card}(\mathbb{N}) = \text{card}(\mathbb{Z})$

Definition: A set S is said to be countably infinite if

\exists bijection $\varphi: S \rightarrow \mathbb{N}$, i.e.,

$$\text{card}(S) = \text{card}(\mathbb{N}).$$

We say a set S is countable

if it is either finite or

countably infinite.

Example 1: (\mathbb{Z} and $2\mathbb{Z}$)

Define $\varphi: \mathbb{Z} \rightarrow 2\mathbb{Z}$

$$\varphi(n) = 2n.$$

Then φ is a bijection,

so $2\mathbb{Z}$ is countably infinite.

Theorem: Let S be countably infinite.

Then any subset of S is countable.

Proof: By applying a bijection.
we may reduce to $S = \mathbb{N}$.

Let T be a subset of \mathbb{N} .

If $\text{card}(T) < \infty$, done.

If $\text{card}(T) = \infty$, then

T has a smallest element,

n_1 .

Define $\varphi: \mathbb{N} \rightarrow T$ inductively by

$$1) \varphi(1) = n_1$$

$$2) \varphi(2) = \text{smallest element of}$$

$$T \setminus \{n_1\} = n_2$$

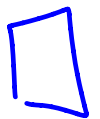
$$3) \varphi(3) = \text{smallest element of}$$

$$T \setminus \{n_1, n_2\} = n_3$$

In general, $\varphi(k) = \text{smallest element}$

of $T \setminus \{n_1, n_2, n_3, \dots, n_{k-1}\}$.

Check this is a bijection!



Theorem: \mathbb{Q} is countable

Proof: Pick two primes

$$p \neq q, \quad p \neq q.$$

Note: $\{p^n \mid n \in \mathbb{N}\}$ is

infinite and a subset of \mathbb{N} ,

hence countable.

Define $\varphi: \mathbb{Q} \rightarrow \mathbb{Z}$ by

if $\frac{a}{b}$ is in lowest terms,

$$\varphi\left(\frac{a}{b}\right) = p^a q^b \quad \text{if } \frac{a}{b} > 0$$

$$\varphi(0) = 0$$

$$\varphi\left(\frac{a}{b}\right) = -p^a q^b \quad \text{if } \frac{a}{b} < 0$$

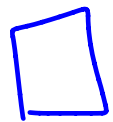
With $b=1$, $a \in \mathbb{N}$, we get

$$\{p^n \mid n \in \mathbb{N}\} \subseteq \varphi(\mathbb{Q})$$

$\Rightarrow \varphi(\mathbb{Q})$ is infinite.

Why is φ injective?

Unique factorization!



Theorem: $|P(S)| > |S| \quad \forall \text{ sets } S$

Theorem: $[0,1]$ is uncountable