Announcements

1) HW #7 due last day of classes

2) HW #8 up for study purposes, not for a grade

More on Cardinality

Recall definition: two sets have the same cardinality if $\exists$ a bijection φ mapping between them.

We showed $\text{card}(\mathbb{N}) = \text{card}(\mathbb{Z})$

Definition: A set S is said to be countably infinite if $\exists$ bijection $\varphi : S \to \mathbb{N}$, i.e., $\mathrm{card}(S) = \mathrm{card}(\mathbb{N})$.

We say a set S is countable if it is either finite or countably infinite.

Example 1: ($\mathbb{Z}$ and $2\mathbb{Z}$)

Define $\varphi: \mathbb{Z} \to 2\mathbb{Z}$

$$\varphi(n) = 2n.$$

Then φ is a bijection,

so $2\mathbb{Z}$ is countably infinite.

Theorem: Let S be countably infinite.

Then any subset of S is countable.

proof: By applying a bijection.

we may reduce to $S = \mathbb{N}$.

Let T be a subset of $\mathbb{N}$.

If $\text{card}(T) < \infty$, done.

If $\text{card}(T) = \infty$, then

T has a smallest element,

n_1.

Define $\varphi : \mathbb{N} \to T$ inductively by

1) $\varphi(1) = n_1$

2) $\varphi(2) =$ smallest element of

$$T \setminus \{n_1\} = n_2$$

3) $\varphi(3) =$ smallest element of

$$T \setminus \{n_1, n_2\} = n_3$$

In general, $\varphi(k) =$ smallest element of $T \setminus \{n_1, n_2, n_3, \ldots, n_{k-1}\}$.

Check this is a bijection!

$\square$

Theorem: $\mathbb{Q}$ is countable

proof: Pick two primes

p & q, $p \neq q$.

Note: $\{p^n \mid n \in \mathbb{N}\}$ is

infinite and a subset of $\mathbb{N}$,

hence countable.

Define $\varphi : \mathbb{Q} \to \mathbb{Z}$ by

if $\frac{a}{b}$ is in lowest terms,

$$\varphi\left(\frac{a}{b}\right) = p^a q^b \text{ if } \frac{a}{b} > 0$$

$$\varphi(0) = 0$$

$$\varphi\left(\frac{a}{b}\right) = -p^a q^b \text{ if } \frac{a}{b} < 0$$

With $b = 1$, $a \in \mathbb{N}$, we get

$$\{ p^n \mid n \in \mathbb{N} \} \subseteq \varphi(\mathbb{Q})$$

$\Rightarrow \varphi(\mathbb{Q})$ is infinite.

Why is φ injective?

Unique factorization!

$\square$

Theorem: $|\mathcal{P}(S)| > |S| \ \forall$ sets S

Theorem: $[0,1]$ is uncountable