Announcements

1) HW 5 Q2 was incorrect, has been corrected (2b) was false)

2) Next time: read 6.4

3) Second Midterm Tuesday before Thanksgiving Break

4) Extra Credit: prove that $\sqrt{33}$ is irrational (5 pts), due next Tuesday

Definition: (cardinality)

Two sets S and T are said to have the same **cardinality** if $\exists$ bijection $f: S \to T$.

Characterization of Infinite Sets

A set S is infinite if and only if S admits a bijection from itself onto a proper subset of S.

Example 1 $\mathbb{N}$ and $\mathbb{Z}$

Show that $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality.

We need a bijection

$$f: \mathbb{N} \to \mathbb{Z}$$

Define f as

$$\boxed{f(n) = \begin{cases} \frac{n}{2}, & n \text{ even} \\ 0, & n = 1 \\ -\left(\frac{n-1}{2}\right), & n > 1,\ n \text{ odd} \end{cases}}$$

Need to check surjectivity and injectivity

Injectivity: Suppose $f(n) = f(m)$.

We want to show $n = m$

By cases:

i) both n and m even.

Then

$$\frac{n}{2} = f(n) = f(m) = \frac{m}{2}$$

Cancelling out the 2's,

$n = m$.

ii) n and m both odd, $n \neq 1 \neq m$

$$-\left(\frac{n-1}{2}\right) = f(n) = f(m) = -\left(\frac{m-1}{2}\right)$$

so

$$\frac{n-1}{2} = \frac{m-1}{2}$$

$$n-1 = m-1$$

$$n = m$$

iii) $n \neq 1$, n odd, m even.

$$-\left(\frac{n-1}{2}\right) = f(n) = f(m) = \frac{m}{2}$$

$\Rightarrow$ an positive number equals a negative number, contradiction

So $f(n) \neq f(m)$ in this case.

(v) $n=1$, $m=$ anything else

$$f(1)=0$$

$f(m)\neq 0$, so again,

$f(1)=f(m)$ is impossible.

This completes the proof that f is injective.

<u>Surjectivity:</u> Solve.

We need to show that if

$b \in \mathbb{Z}$, $\exists$ $n \in \mathbb{N}$, $f(n) = b$.

By cases

i) $b = 0$, $n = 1$

ii) $b > 0$, $n = 2b \in \mathbb{N}$

iii) $b < 0$, $n = -2b + 1 \in \mathbb{N}$

$$b = f(n) = -\left(\frac{n-1}{2}\right)$$

solve

This shows f is a bijection, and so $\mathbb{Z}$ and $\mathbb{N}$ have the same cardinality! In particular, since $\mathbb{N} \subset \mathbb{Z}$, we have shown that $\mathbb{Z}$ is infinite.

Some Cardinality Results

We denote $|S|$ as the cardinality of S.

1) $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$

2) $|\mathbb{R}| > |\mathbb{N}|$

3) $|\mathbb{R}^2| = |\mathbb{R}|$

4) $|[0,1]| = |\mathbb{R}|$

5) For any set S, $|\wp(S)| > |S|$

Set Operations

(Section 5.3)

Example 2: $(A^c)^c = A$

Show both $A \subseteq (A^c)^c$ and $(A^c)^c \subseteq A$.

1) Let $x \in A$, show $x \in (A^c)^c$.

But if $x \in A$, $x \notin A^c$

$\Rightarrow x \in (A^c)^c$

2) Let $x \in (A^c)^c$. Show $x \in A$.

But if $(A^c)^c$, then $x \notin A^c$.

Since any element of a set U with $A \subseteq U$ has $x \in A$ or $x \in A^c$

2) Let $x \in (A^c)^c$. Show $x \in A$.

But if $x \in (A^c)^c$, then $x \notin A^c$.

Since any element of a set U with $A \subseteq U$ has $x \in A$ or $x \in A^c$ and $A \cap A^c = \emptyset$, we must have $x \in A$.

$\square$

More on Functions

(Section 6.4)

Definition: ($\mathbb{P}_n$) The set $\mathbb{P}_n$ consists of all polynomials with coefficients in $\mathbb{R}$ whose degree is less than or equal to n.

$\mathbb{P}_0$ = constant functions

$\mathbb{P}_1$ = lines

$\mathbb{P}_2$ = quadratics and lines

$\mathbb{P}_3$ = cubics, quadratics, and lines

etc.