

# Announcements

- 1) HW 5 Q2 was incorrect, has been corrected (2b) was false)
- 2) Next time: read 6.4
- 3) Second Midterm Tuesday before Thanksgiving Break
- 4) Extra Credit: prove that  $\sqrt{33}$  is irrational (5 pts), due next Tuesday

Definition: (cardinality)

Two sets  $S$  and  $T$  are said  
to have the same cardinality

if  $\exists$  bijection  $f: S \rightarrow T$ .

# Characterization of Infinite Sets

A set  $S$  is infinite if and only if  $S$  admits a bijection from itself onto a proper subset of  $S$ .

## Example 1 $\mathbb{N}$ and $\mathbb{Z}$

Show that  $\mathbb{N}$  and  $\mathbb{Z}$  have the same cardinality.

We need a bijection

$$f: \mathbb{N} \rightarrow \mathbb{Z}$$

Define  $f$  as

$$f(n) = \begin{cases} \frac{n}{2}, & n \text{ even} \\ 0, & n = 1 \\ -\left(\frac{n-1}{2}\right), & n > 1, n \text{ odd} \end{cases}$$

Need to check surjectivity and injectivity

Injectivity: Suppose  $f(n) = f(m)$ .

We want to show  $n = m$

By cases:

i) both  $n$  and  $m$  even.

Then

$$\frac{n}{2} = f(n) = f(m) = \frac{m}{2}$$

Cancelling out the 2's,

$$n = m.$$

ii)  $n$  and  $m$  both odd,  $n \neq m$

$$-\left(\frac{n-1}{2}\right) = f(n) = f(m) = -\left(\frac{m-1}{2}\right)$$

So

$$\frac{n-1}{2} = \frac{m-1}{2}$$

$$n-1 = m-1$$

$$n = m$$

iii)  $n \neq 1$ ,  $n$  odd,  $m$  even.

$$-\left(\frac{n-1}{2}\right) = f(n) = f(m) = \frac{m}{2}$$

$\Rightarrow$  a positive number equals a negative number, contradiction.

So  $f(n) \neq f(m)$  in this case.

(v)  $n=1$ ,  $m = \text{anything else}$

$$f(1) = 0$$

$$f(m) \neq 0 \quad , \text{ so again,}$$

$f(1) = f(m)$  is impossible.

This completes the proof that  
 $f$  is injective.

Surjectivity: Solve.

We need to show that if

$$b \in \mathbb{Z}, \exists n \in \mathbb{N}, f(n) = b.$$

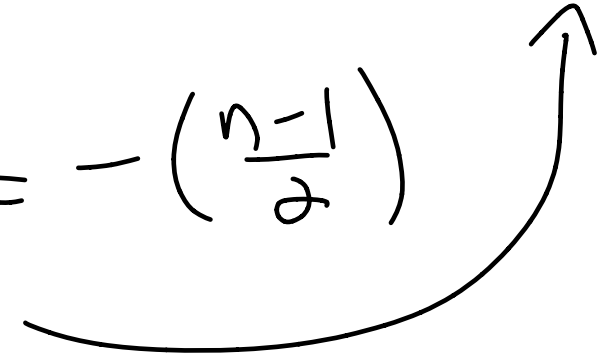
By cases

$$(i) \quad b = 0, \quad n = 1$$

$$(ii) \quad b > 0, \quad n = 2b \in \mathbb{N}$$

$$(iii) \quad b < 0, \quad n = -2b + 1 \in \mathbb{N}$$

$$b = f(n) = -\binom{n-1}{2}$$

solve 



This shows  $f$  is a bijection,  
and so  $\mathbb{Z}$  and  $\mathbb{N}$  have  
the same cardinality! In  
particular, since  $\mathbb{N} \subset \mathbb{Z}$ ,  
we have shown that  $\mathbb{Z}$  is  
infinite.

## Some Cardinality Results

We denote  $|S|$  as the cardinality of  $S$ .

$$1) |\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$$

$$2) |\mathbb{R}| > |\mathbb{N}|$$

$$3) |\mathbb{R}^2| = |\mathbb{R}|$$

$$4) |[0, 1]| = |\mathbb{R}|$$

$$5) \text{ For any set } S, |\mathcal{P}(S)| > |S|$$

# Set Operations



(Section 5.3)

Example 2:  $(A^c)^c = A$

Show both  $A \subseteq (A^c)^c$  and

$$(A^c)^c \subseteq A.$$

1) Let  $x \in A$ , show  $x \in (A^c)^c$ .

But if  $x \in A$ ,  $x \notin A^c$

$$\Rightarrow x \in (A^c)^c$$

2) Let  $x \in (A^c)^c$ . Show  $x \in A$ .

But if  $(A^c)^c$ , then  $x \notin A^c$ .

Since any element of a set  $U$  with  $A \subseteq U$  has  $x \in A$  or  $x \in A^c$

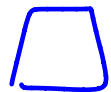
2) Let  $x \in (A^c)^c$ . Show  $x \in A$ .

But if  $x \in (A^c)^c$ , then  $x \notin A^c$ .

Since any element of a set  $U$  with  $A \subseteq U$  has  $x \in A$  or  $x \in A^c$

and  $A \cap A^c = \emptyset$ , we must

have  $x \in A$ .



# More on Functions

(Section 6.4)

Definition: ( $\mathbb{P}_n$ ) The set  $\mathbb{P}_n$  consists of all polynomials with coefficients in  $\mathbb{R}$  whose degree is less than or equal to  $n$ .

$\mathbb{P}_0 =$  constant functions

$\mathbb{P}_1 =$  lines

$\mathbb{P}_2 =$  quadratics and lines

$\mathbb{P}_3 =$  cubics, quadratics, and lines

etc.