Set Theory

(Chapter 5)

What is a set?

Unfortunately, there is no precise definition! However, set theory is basic enough to allow one to "construct" the natural numbers.

Examples of sets:

1) $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$

2) all functions from $\mathbb{R}$ to $\mathbb{R}$

3) all stars in the universe

Almost everything you can think of is a set! However, some of us have very active imaginations.

The members of a set S will be called the elements of S. If x is an element of S, we denote this using the symbol "$x \in S$". If y is not an element of S, we write "$y \notin S$".

Example 1: $1/2 \in \mathbb{Q}$, but $1/2 \notin \mathbb{N}$.

$\pi \in \mathbb{R}$, but $\pi \notin \mathbb{Q}$

Notation: We may list a finite set via its elements, such as

$$S = \{x_1, x_2, x_3\}.$$

We may also write a set using **set builder notation**:

$$S = \{x_i \mid 1 \leq i \leq 3\}$$

reads "S is the set of all x_i's such that $1 \leq i \leq 3$"

Example 2:

$$S = \{ n \in \mathbb{Z} \mid |n| < 5 \}$$

$$= \{ -4, -3, -2, -1, 0, 1, 2, 3, 4 \}$$

$$T = \{ \sqrt{2}, \{5, 8\} \}$$

$$\sqrt{2} \in T$$

$$\{5, 8\} \in T$$

$$\{\sqrt{2}\} \subseteq T$$

Not a set: (Bertrand Russell)

Let S be the set of all sets that do not contain itself as a member.

Is $S \in S$?

More notation: If every element of a set S is also an element of another set T, we write "$S \subseteq T$" for "S is contained in T".

If in addition $S \neq T$, we write "$S \subset T$" or "$S \subsetneq T$" and say S is properly contained in T.

Example 3: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

The fact that the last containment is strict is the only part that requires proof

The empty set

The empty set is the set with no elements, denoted "$\emptyset$".

You can use

$$\emptyset = \{x \mid x \neq x\}$$

as a definition. By convention,

$\emptyset \subseteq S$ for all sets S.

Operations with Sets: Let S and T be sets, and suppose $S, T \subseteq U$ where U is called a "universal" set. We define

1) The union of S and T, denoted by "$S \cup T$", as

$$\boxed{S \cup T = \{x \mid x \in S \text{ or } x \in T\}}$$

2) The intersection of S and T, denoted by "$S \cap T$", as

$$\boxed{S \cap T = \{x \mid x \in S \text{ and } x \in T\}}$$

3) The complement of S, denoted "S^c", by

$$\boxed{S^c = \{x \in U \mid x \notin S\}}$$

4) The relative complement of S with respect to T, denoted "$T \setminus S$" or "$T - S$", by

$$\boxed{T \setminus S = \{x \in T \mid x \notin S\}}$$

Note $T \setminus S = T \cap S^c$

Example 4: Let

$$U = \{ n \in \mathbb{Z} \mid |n| \leq 10 \}$$

$$A = \{-5, 2, 3, 9\}$$

$$B = \{-7, -1, 0, 2, 9, 10\}$$

$$A \cup B = \{-7, -5, -1, 0, 2, 3, 9, 10\}$$

$$A \cap B = \{2, 9\}$$

$$B^c = \{-10, -9, -8, -6, -5, -4, -3, -2, 1, 3, 4, 5, 6, 7, 8\}$$

$$A \setminus B = \{-5, 3\}$$

Cardinality and Power Sets

For a finite set S, we define the cardinality of S, denoted "$\operatorname{card}(S)$" (or $|S|$ or $\#S$) to be the number of elements in S.

If T is an arbitrary set, we define the power set of T, denoted by $\wp(T)$, as

$$\boxed{\wp(T) = \{R \mid R \subseteq T\}}.$$

Warning & Observation: The subsets of T are the elements of $\mathcal{P}(T)$. For example, $\mathbb{Z} \subseteq \mathbb{R}$, but $\mathbb{Z} \in \mathcal{P}(\mathbb{R})$.