Math 300 In-Class Worksheet 17: Cardinality

1) a) Let

$$S = \{x \in \mathbb{R} \mid \exists\ a, b \in \mathbb{Z},\ ax + b = 0\}.$$

Is S is countable? Justify your assertion with a proof.

b) Let

$$T = \{x \in \mathbb{R} \mid \exists\ a, b, c \in \mathbb{Z},\ ax^2 + bx + c = 0\}.$$

Is T is countable? Justify your assertion with a proof.

2) Let $A \subset B$ be infinite and suppose B is countable. Must A be countable?

3) (#5, Section 9.2) Prove Theorem 9.16:

If A is a countably infinite set and B is a finite set, then $A \cup B$ is a countably infinite set.

Hint: Let $\text{card}(B) = n$ and use a proof by induction on n. Theorem 9.15 is the base step.

4) A student has a challenge to create a program to list the first 100,000 prime numbers. Consider the code below which the student has created:

```
1    Define function is_prime(n) by:
2      for i ∈ ℕ with 2 ≤ i ≤ √n:
3        if i | n then:
4          return 'not_prime'
5      return 'prime'
6
7    primes_list = {2,3}
8
9    for k ∈ ℕ with 1 ≤ k ≤ 100,000:
10     if is_prime(6k-1) == 'prime' then:
11       add 6k-1 to end of primes_list
12     if is_prime(6k+1) == 'prime' then:
13       add 6k+1 to end of primes_list
14
15   print primes_list
```

(a) In line 2 of the code we notice that while checking for divisors i only ranges from 2 to the $\sqrt{n}$. Prove that this is mathematically valid. That is: if $n \in \mathbb{N}$ is composite then $\exists i \in \mathbb{N}$ with $2 \leq i \leq \sqrt{n}$ and $i \mid n$.

(b) Notice that the student is only checking numbers of the form $6k - 1$ and $6k + 1$ (see lines 11 and 13). Why would they want to do this? Will they miss any primes because of this?

  (i) If your answer is yes they will miss prime(s) then find one that they miss.

  (ii) If your answer is no then prove it. That is: all primes greater than 3 are of the form $6k - 1$ or $6k + 1$.

(c) Will this code succeed in finding a list of the first 100,000 prime numbers? Why or why not?

or

Show that any EVEN perfect number is of the form $(2^p - 1)(2^{p-1})$ where $2^p - 1$ is a Mersenne prime. (must define perfect)