A Small Time Approximation for Convolutions of Gamma Distributions

Frank Massey

In probability and statistics, a random variable $X$ has a gamma distribution with shape parameter $m > 0$ and scale parameter $\lambda > 0$ if its probability density function has the form

$$f(t) = f(t; m, \lambda) = \begin{cases} \frac{\lambda^m e^{-\lambda t}}{\Gamma(m)} & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$$

Here $\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} dy$ is the Gamma function. For non-negative integers $x$ one has $\Gamma(x) = (x-1)!$.

Let $T_1, \ldots, T_n$ be independent gamma random variables with $T_j$ having parameters $m_j$ and $\lambda_j$ and $S = T_1 + \cdots + T_n$ be their sum. The density function of the sum of independent random variables is the convolution of their density functions. So $S$ has density function

$$A(t) = A(t; m, \lambda) = f(t; m_1, \lambda_1) * f(t; m_2, \lambda_2) * \cdots * f(t; m_n, \lambda_n)$$

$$= \left( \frac{(\lambda_1)^{m_1} e^{-\lambda_1 t}}{\Gamma(m_1)} \right) * \left( \frac{(\lambda_2)^{m_2} e^{-\lambda_2 t}}{\Gamma(m_2)} \right) * \cdots * \left( \frac{(\lambda_n)^{m_n} e^{-\lambda_n t}}{\Gamma(m_n)} \right)$$

Here $*$ denotes convolution, i.e.

$$g(t) * h(t) = \int_0^t g(t-s) h(s) ds = t \int_0^1 f(t(1-u)) g(tu) du$$

for $t \geq 0$. Also, $m = (m_1, \ldots, m_n)$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ denote the vector of shape and scale parameters of the $T_j$. 
One can not express $A(t)$ in terms of elementary functions except if the $m_j$ are all integers or if the $\lambda_j$ are all equal. In fact, if $m = (m_1,\ldots,m_n)$ and all the $\lambda_j$ are equal, then it is well known that

$$
A(t; m, (\lambda,\ldots,\lambda)) = \frac{\lambda^m t^{|m|-1} e^{-\lambda t}}{\Gamma(|m|)}
$$

where

$$
|m| = m_1 + \cdots + m_n
$$
is the sum of the shape parameters.

So, it is of interest to have approximations to $A(t)$ that reveal its properties and are useful for numerical computations. This report establishes one such approximation, namely $A(t)$ is approximately proportional to a gamma distribution for relatively small values of $t$.

More precisely, one has

$$
A(t) \approx \frac{\lambda^m t^{|m|-1} e^{-\lambda t}}{\Gamma(|m|)}
$$

for relatively small values of $t$. Here

$$
\bar{\lambda} = \frac{m_1 \lambda_1 + \cdots + m_n \lambda_n}{|m|}
$$
is the weighted mean of the scale parameters $\lambda_j$ where we use the shape parameters $m_j$ as weights. Also,

$$
\lambda^m = (\lambda_1)^{m_1}(\lambda_2)^{m_2}\cdots(\lambda_n)^{m_n}
$$
For the purposes of derivations, it is convenient to introduce the following notation.

(9) \[ A(t;m,\lambda) = \lambda^m E(t;m,\lambda) \]

where

(10) \[ E(t) = E(t;m,\lambda) = \left( \frac{t^{m_1-1}e^{-\lambda_1 t}}{\Gamma(m_1)} \right) \ast \left( \frac{t^{m_2-1}e^{-\lambda_2 t}}{\Gamma(m_2)} \right) \ast \ldots \ast \left( \frac{t^{m_n-1}e^{-\lambda_n t}}{\Gamma(m_n)} \right) \]

Note that if the \( m_j \) are positive, then \( E(t;m,\lambda) \) is defined for all real \( \lambda_j \).

We establish the following error bound for the approximation (5).

**Theorem 1.** Suppose \( \lambda_1,\ldots,\lambda_n \) are real numbers, \( m_1,\ldots,m_n \) are positive real numbers, \( E(t) = E(t;m,\lambda) \) is defined by (10), \( \mu = \min\{\lambda_1,\ldots,\lambda_n\} \), \( \sigma^2 \) is the weighted variance of the \( \lambda_j \) where we use the \( m_j \) as weights, i.e.

(11) \[ \sigma^2 = \frac{1}{|m|} \sum_{j=1}^{n} m_j (\lambda_j - \tilde{\lambda})^2 \]

Let \( \eta(t) = \sigma^2 t^2 e^{(\tilde{\lambda} - \mu t)/(2(|m|+1))} \). Then for \( t \geq 0 \) one has

(12) \[ \frac{t^{|m|-1}e^{-\tilde{\lambda} t}}{\Gamma(|m|)} \leq E(t) \leq (1 + \eta(t)) \frac{t^{|m|-1}e^{-\tilde{\lambda} t}}{\Gamma(|m|)} \]

If the \( \lambda_j \) are all positive, then

(13) \[ \frac{\lambda^m t^{|m|-1}e^{-\tilde{\lambda} t}}{\Gamma(|m|)} \leq A(t) \leq (1 + \eta(t)) \frac{\lambda^m t^{|m|-1}e^{-\tilde{\lambda} t}}{\Gamma(|m|)} \]
The proof of this theorem requires some preliminary results which we discuss now before the proof of Theorem 1. The first are the following formulas for multiple convolutions. If \( f_1(t), \ldots, f_n(t) \) are defined for non-negative \( t \) and integrable on bounded intervals and \( \alpha \) is a real number, then

\[
\begin{align*}
[ e^{\alpha t} f_1(t) ] \ast \cdots \ast [ e^{\alpha t} f_n(t) ] &= e^{\alpha t} [ f_1(t) \ast \cdots \ast f_n(t) ] \\
 f_1(t) \ast \cdots \ast f_n(t) &= t^{n-1} \int_{\Omega} f_1(t(1-u_1)) f_2(t(u_2-u_1)) \cdots f_n(t(u_n-u_{n-1})) f_n(u_{n-1}) \, du
\end{align*}
\]

where \( ds = ds_1 \ldots ds_{n-1} \) and

\[
\Omega = \{ (u_1, \ldots, u_{n-1}) : 0 \leq u_1 \leq 1, \ 0 \leq u_2 \leq u_1, \ldots, 0 \leq u_{n-1} \leq u_{n-2} \}
\]

Both (14) and (15) are easily proved by induction. The case \( n = 2 \) for (15) is just the second integral in (3).

One key to the proof of Theorem 1 is the following representation of \( E(t) \).

**Theorem 2.**

\[
E(t) = t^{\left\lfloor m + 1 \right\rfloor} H(t)
\]

where

\[
\begin{align*}
H(t) &= H(t; m, \lambda) = \int_{\Omega} h(u) e^{-\lambda \# u} \, du \\
h(u) &= h(u; m) = \frac{(u_0 - u_1)^{m_1-1}(u_1 - u_2)^{m_2-1} \cdots (u_{n-2} - u_{n-1})^{m_{n-1}-1}}{\Gamma(m_1)\Gamma(m_2) \cdots \Gamma(m_n)} \\
\lambda \# u &= \lambda_1(u_0-u_1) + \lambda_2(u_1-u_2) + \lambda_3(u_2-u_3) + \cdots + \lambda_{n-1}(u_{n-2}-u_{n-1}) + \lambda_n(u_{n-1}-u_n)
\end{align*}
\]
Proof. Applying (14) with \( f_j(t) = \frac{t^{m_j-1} e^{-\lambda_j t}}{\Gamma(m_j)} \) gives

\[
E(t) = t^{n-1} \int_{\Omega} \frac{(t(u_0 - u_1))^{m_1-1} e^{-\lambda_1(t(u_1 - u_2))} \cdots (t(u_{n-1} - u_n))^{m_n-1} e^{-\lambda_n(u_n - u_0)}}{\Gamma(m_1) \Gamma(m_2) \cdots \Gamma(m_n)} \, du
\]

Formula (17) with the first formula for \( H(t) \) follows from this. \( \square \)

Note that (17) can be restated as

\[
H(t;m,\lambda) = \frac{E(t)}{t^{m-1}}
\]

Corollary 3. If \( \lambda = (\lambda,...,\lambda) \) is a vector of identical real numbers then

\[
H(t;m,(\lambda,...,\lambda)) = \frac{e^{-\lambda t}}{\Gamma(|m|)}
\]

Proof. This follows from (4), (9) and (17). \( \square \)

Another key to the proof of Theorem 1 is the following.

Proposition 4. If \( h(u) \) is given by (19) then \( h(u) > 0 \) for \( u \in \Omega \) and

\[
\int_{\Omega} h(u) \, du = \frac{1}{\Gamma(|m|)}
\]

\[
\int_{\Omega} h(u) \left( \lambda^\#u \right) e^{-t(\lambda^\#u)} \, du = \sum_{j=1}^{n} m_j \lambda_j H(t; m+1_j, \lambda)
\]
where \( \mathbf{1}_j = (0, \ldots, 0, 1, 0, \ldots, 0) \) is the vector with a 1 in the \( j \)th coordinate and 0 elsewhere.

**Proof.** Take \( \lambda_j = 0 \) for \( 1 \leq j \leq n \) in (18) to get

\[
H(t; m, (0, \ldots, 0)) = \int h(u) \, du
\]

By Corollary 3 one has

\[
H(t; m, (0, \ldots, 0)) = \frac{1}{\Gamma(m)}
\]

Combining we get (23). To prove (24), note that

\[
\int h(u) \left( u_{j-1} - u_j \right) e^{-t(\lambda \# u)} \, du = \int h(u; m + \mathbf{1}_j, \lambda) \frac{\Gamma(m+1)}{\Gamma(m_j)} e^{-t(\lambda \# u)} \, du
\]

\[
= m_j \int h(u; m + \mathbf{1}_j, \lambda) e^{-t(\lambda \# u)} \, du
\]

So

\[
\int h(u) \left( u_{j-1} - u_j \right) e^{-t(\lambda \# u)} \, du = m_j H(t; m+\mathbf{1}_j, \lambda)
\]

Multiplying by \( \lambda_j \), summing on \( j \) proves (24). To prove (24) take \( \lambda_j = 0 \) for \( 1 \leq j \leq n \) in (27) and use (22) to get
Multiplying by $\lambda_j$, summing on $j$ and using $\Gamma(|m|+1) = |m|!\Gamma(|m|)$ proves (25). To prove (26) note that if $j \neq k$ one has

$$\int \Omega h(u)(u_{j-1} - u_j) e^{-r(\lambda \# u)} du = \int \Omega h(u; m+1_j, \lambda) e^{-r(\lambda \# u)} du$$

$$= m_j m_k \int \Omega h(u; m+1_j + 1_k, \lambda) e^{-r(\lambda \# u)} du = m_j m_k H(t; m+1_j + 1_k, \lambda)$$

and

$$\int \Omega \langle h(u)(u_{j-1} - u_j)^2 e^{-r(\lambda \# u)} du = \int \Omega h(u; m+1_j, \lambda) e^{-r(\lambda \# u)} du$$

$$= m_j(m_j+1) \int \Omega h(u; m+21_j, \lambda) e^{-r(\lambda \# u)} du = m_j(m_j+1) H(t; m+21_j, \lambda)$$

Combining with $\int \Omega h(u)(\lambda \# u)^2 e^{-r(\lambda \# u)} du = \sum_{j=1}^n \lambda_j^2 \int \Omega h(u)(u_{j-1} - u_j)^2 e^{-r(\lambda \# u)} du + 2 \sum_{j=1}^n \sum_{k=1}^{j-1} \lambda_j \lambda_k \int \Omega h(u)(u_{j-1} - u_j)(u_{k-1} - u_k) e^{-r(\lambda \# u)} du$, one obtains

$$\int \Omega h(u)(\lambda \# u)^2 e^{-r(\lambda \# u)} du = \sum_{j=1}^n m_j(m_j+1) \lambda_j^2 H(t; m+21_j, \lambda)$$

$$+ 2 \sum_{j=1}^n \sum_{k=1}^{j-1} m_j \lambda_j m_k \lambda_k H(t; m+1_j + 1_k, \lambda).$$

Take $\lambda_j = 0$ for $1 \leq j \leq n$ in (28) and (29) and use (22) to get
\[
\int_\Omega h(u)(u_{j-1} - u_j)(u_{k-1} - u_k) du = \frac{m_jm_k}{\Gamma(|m| + 2)}
\]

\[
\int_\Omega h(u)(u_{j-1} - u_j)^2 du = \frac{m_j(m_j+1)}{\Gamma(|m| + 2)}
\]

Combining with \( \int_\Omega h(u)(\lambda\#u)^2 du = \sum_{j=1}^n \lambda_j^2 \int_\Omega h(u)(u_{j-1} - u_j)^2 du + 2 \sum_{j=1}^n \sum_{k=1}^{j-1} \lambda_j \lambda_k \int_\Omega h(u)(u_{j-1} - u_j)(u_{k-1} - u_k) du \), one obtains

\[
\int_\Omega h(u)(\lambda\#u)^2 du = \frac{1}{\Gamma(|m| + 2)} \sum_{j=1}^n m_j(m_j+1)\lambda_j^2 + \frac{2}{\Gamma(|m| + 2)} \sum_{j=1}^n \sum_{k=1}^{j-1} m_j \lambda_j m_k \lambda_k.
\]

Since \(|m|^2\lambda^2 = \sum_{j=1}^n m_j^2 \lambda_i^2 + 2 \sum_{j=1}^n \sum_{k=1}^{j-1} m_j \lambda_j m_k \lambda_k\), it follows that

\[
\int_\Omega h(u)(\lambda\#u)^2 du = \frac{1}{\Gamma(|m| + 2)} |m|^2\lambda^2 + \frac{1}{\Gamma(|m| + 2)} \sum_{j=1}^n m_j\lambda_j^2
\]

Since \( \sum_{j=1}^n m_j \lambda_j^2 = |m|\sigma^2 + |m|\lambda^2\), it follows that, it follows that

\[
\int_\Omega h(u)(\lambda\#u)^2 du = \frac{1}{\Gamma(|m| + 2)} |m|^2\lambda^2 + \frac{1}{\Gamma(|m| + 2)} |m|\sigma^2 + |m|\lambda^2
\]

\[
= \frac{1}{\Gamma(|m| + 2)} |m|(|m| + 1)^2\lambda^2 + \frac{1}{\Gamma(|m| + 2)} |m|\sigma^2
\]

\[
= \frac{1}{\Gamma(|m|)} \left( \lambda^2 + \frac{\sigma^2}{|m| + 1} \right)
\]

which proves (26). ∎
We now have all the ingredients for the proof of Theorem 1.

**Proof of Theorem 1.** Using (14) it is not hard to show that

\[ E_n(t;m,\lambda) = e^{-\lambda t} E_n(t;m,\lambda_1-\lambda,\ldots,\lambda_n-\lambda) \].

Using this one can reduce (13) to the case that \( \lambda = 0 \) which we shall now assume. The rest of the proof uses Theorem 2 and Proposition 4. By Taylor's formula one has \( e^{-x} = 1 - x + x^2 e^{-\xi}/2 \) where \( \xi \) is between 0 and \( x \). So

\[
1 - x \leq e^{-x} \leq 1 - x + (x^2/2) \max\{1,e^{-x}\}.
\]

Applying this with \( x = \lambda u \) and using the fact that \( \max\{1, e^{i(\lambda u)}\} \leq e^{i\mu} \) for \( u \) in \( \Omega \), one obtains

\[
1 - \lambda u \leq e^{i(\lambda u)} \leq 1 - \lambda u + (\lambda u)^2 e^{i\mu}/2.
\]

Integrating over \( \Omega \), multiplying by \( t^{m|-1} \) and using Theorem 2 and Proposition 4 one obtains

\[
(1 - \lambda t)^{m|-1}/\Gamma(|m|) \leq E(t) \leq (1 - \lambda t + (\lambda^2 + \sigma^2/|m|+1)) t^{m|-1}/\Gamma(|m|).
\]

Since \( \lambda = 0 \), this proves (13). \( \square \)