1.5 Gaussian Elimination With Partial Pivoting.

In the previous section we discussed Gaussian elimination. In that discussion we used equation 1 to eliminate \( x_1 \) from equations 2 through \( n \). Then we used equation 2 to eliminate \( x_2 \) from equations 2 through \( n \) and so on. In each case we used equation \( j \) to eliminate \( x_j \) from equations \( j \) through \( n \). It turns out that in some cases roundoff errors can be significantly reduced if we make some other choice of the equation to use to eliminate a particular variable. The use of a certain equation to eliminate a variable from other equations is called a pivot and a rule we use to choose which equation to use is called a pivoting strategy. The resulting modified algorithm is called Gaussian elimination with partial pivoting.

1.5.1 The Algorithm.

We illustrate this method by means of an example.

**Example 1.**

\[
\begin{align*}
x_1 - x_2 + 3x_3 &= 13 \\
4x_1 - 2x_2 + x_3 &= 15 \\
-3x_1 - x_2 + 4x_3 &= 8
\end{align*}
\]

or \( Ax = b \) where \( A = \begin{pmatrix} 1 & -1 & 3 \\ 4 & -2 & 1 \\ -3 & -1 & 4 \end{pmatrix} \) and \( b = \begin{pmatrix} 13 \\ 15 \\ 8 \end{pmatrix} \). As with the previous version, we keep track of the multipliers in the matrix \( L \). We start out with \( L^{(0)} = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \).

In the previous version we subtracted multiples of equation 1 from each of the other equations to eliminate \( x_1 \) from the other equations 2, 3, \ldots, \( n \). Now we are going to allow for the possibility of using some equation other than the first to eliminate \( x_1 \) from the other equations. A widely used method is to use the equation with the coefficient of \( x_1 \) which is largest in absolute value. In this case that would be equation 2. It turns out that this strategy reduces roundoff errors in a number of situations, a point we shall discuss further later.

So we are going to use equation 2 to eliminate \( x_1 \) from the other two equations. In order to keep the rest of the algorithm the same as before, we interchange equations 1 and 2. In terms of the matrix \( A \) and vector \( b \), we just interchange rows 1 and 2. We make no change in \( L \) at this stage. The new equations and \( A \) and \( b \) are the following.

\[
\begin{align*}
4x_1 - 2x_2 + x_3 &= 15 \\
x_1 - x_2 + 3x_3 &= 13 \\
-3x_1 - x_2 + 4x_3 &= 8
\end{align*}
\]

\[
A^{(1)} = \begin{pmatrix} 4 & -2 & 1 \\ 1 & -1 & 3 \\ -3 & -1 & 4 \end{pmatrix}, \quad L^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b^{(1)} = \begin{pmatrix} 15 \\ 13 \\ 8 \end{pmatrix}
\]
Thus the equations are $A^{(1)}x = b^{(1)}$. To keep track of the row interchanges as we go along, we use a permutation vector $p$. This is just a vector with the integers $1, \ldots, n$ in some order. We start out with $p^{(0)} = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix}$. In Example 1, $p^{(0)} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. After we do the interchange of the equations and rows, we interchange the corresponding components of $p$. In this example we interchange components 1 and 2 giving $p^{(1)} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$. One interpretation of $p = p^{(1)}$ is that

- Row 1 of $A^{(1)} = \text{Row } p_1 = \text{Row } 2$ of $A$
- Row 2 of $A^{(1)} = \text{Row } p_2 = \text{Row } 1$ of $A$
- Row 3 of $A^{(1)} = \text{Row } p_3 = \text{Row } 3$ of $A$

There is another way of expressing this that involves the permutation matrix $P$ corresponding to the permutation vector $p$. The relationship between $P = P^{(1)}$ and the identity matrix $I$ is the same as the relationship between $A^{(1)}$ and $A$, i.e.

- Row 1 of $P^{(1)} = \text{Row } p_1$ of $I = \text{Row } 2$ of $I$
- Row 2 of $P^{(1)} = \text{Row } p_2$ of $I = \text{Row } 1$ of $I$
- Row 3 of $P^{(1)} = \text{Row } p_3$ of $I = \text{Row } 3$ of $I$

Thus

$$P^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Putting it another way, every time we interchange two rows of $A^{(1)}$ we also interchange the same two rows of $P^{(1)}$. Since the $j^{th}$ row of $P^{(1)}A$ is the linear combination of the rows of $A$ using the entries of the $j^{th}$ row of $P^{(1)}$ as coefficients, one has

$$A^{(1)} = P^{(1)}A \quad \quad b^{(1)} = P^{(1)}b$$

Now we resume the regular Gaussian elimination, i.e. we subtract multiples of equation 1 from each of the other equations to eliminate $x_1$. In particular, in the above example we

- Subtract $L_{21} = \frac{a_{21}}{a_{11}} = \frac{1}{4}$ times equation / row 1 from equation / row 2
- Subtract $L_{31} = \frac{a_{31}}{a_{11}} = \frac{3}{4}$ times equation / row 1 from equation / row 3

After doing this the equations have become...
\[
\begin{align*}
4x_1 - 2x_2 + x_3 &= 15 \\
- \frac{1}{2}x_2 + \frac{11}{4}x_3 &= \frac{37}{4} \\
- \frac{5}{2}x_2 + \frac{19}{4}x_3 &= \frac{77}{4}
\end{align*}
\]

or \(A^{(2)} x = b^{(2)}\) where \(A^{(2)} = \begin{pmatrix} 4 & -2 & 1 \\ 0 & \frac{1}{2} & \frac{11}{4} \\ 0 & -\frac{5}{2} & \frac{19}{4} \end{pmatrix}\) and \(b^{(2)} = \begin{pmatrix} 15 \\ \frac{37}{4} \\ \frac{77}{4} \end{pmatrix}\). One has \(L^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ -\frac{3}{4} & 0 & 1 \end{pmatrix}\), \(p\) and \(P\) don't change, i.e. \(p^{(2)} = p^{(1)} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}\) and \(P^{(2)} = P^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\). One has \(L^{(2)} A^{(2)} = A^{(2)} = P^{(2)} A\).

Now we have to eliminate \(x_2\) from one of the last two equations by subtracting a multiple of one equation from the other. We look for the equation which has the largest coefficient of \(x_2\) in absolute value and we interchange this with equation 2. We keep track of this interchange by interchanging the corresponding components of \(A\), \(b\), \(p\) and \(P\). In our example equation 3 has the largest coefficient of \(x_2\) in absolute value, so we interchange equations 2 and 3. We do the same with the corresponding components of \(A\), \(b\), \(p\) and \(P\). With \(L\) we interchange the part of rows 2 and 3 below the main diagonal. In a moment we shall explain why we do this. After doing all this we have

\[
\begin{align*}
4x_1 - 2x_2 + x_3 &= 15 \\
- \frac{5}{2}x_2 + \frac{19}{4}x_3 &= \frac{77}{4} \\
- \frac{1}{2}x_2 + \frac{11}{4}x_3 &= \frac{37}{4}
\end{align*}
\]

or \(A^{(3)} x = b^{(3)}\) where \(A^{(3)} = \begin{pmatrix} 4 & -2 & 1 \\ 0 & \frac{5}{2} & \frac{19}{4} \\ 0 & -\frac{1}{2} & \frac{11}{4} \end{pmatrix}\) and \(b^{(3)} = \begin{pmatrix} 15 \\ \frac{77}{4} \\ \frac{37}{4} \end{pmatrix}\). One has \(L^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ \frac{1}{4} & 0 & 1 \end{pmatrix}\), \(p^{(3)} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}\) and
\[
P^{(3)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
\]

We have the following relationship relating these matrices.

\[L^{(3)} A^{(3)} = p^{(3)} A\]

To see this, we multiply \(L^{(2)} A^{(2)} = P^{(2)} A\) by \(P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\) to get \(P_{23} L^{(2)} A^{(2)} = P_{23} P^{(2)} A\). Note that
\[
P_{23} P^{(2)} = P^{(3)}\] so \(P_{23} L^{(2)} A^{(2)} = P^{(3)} A\). We have \(P_{23} L^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 0 & 1 \\ \frac{1}{4} & 1 & 0 \end{pmatrix}\), i.e. rows 2 and 3 of \(L^{(2)}\) are interchanged.

Now consider \(L^{(3)} P_{23}\). You will recall that one way of computing \(BC\) is to take linear combinations of the rows of \(C\) using the elements of the rows of \(B\) as multipliers. Another way to compute \(BC\) is to take linear combination of the columns of \(B\) using the elements of the columns of \(C\) as multipliers. More precisely the \(k^{th}\) column of \(BC\) is the linear combination of the columns of \(B\) using the elements of column \(k\) of \(C\) as the
multipliers. In particular $L^{(3)}P_{23}$ is $L^{(3)}$ with columns 2 and 3 interchanged, i.e. $egin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ \frac{1}{4} & 0 & 1 \end{pmatrix}$. Thus $L^{(3)}P_{23} = P_{23}L^{(2)}$. Combining with the above gives $L^{(3)}P_{23}A^{(2)} = P^{(3)}A$. However, $P_{23}A^{(2)} = A^{(3)}$, so we get $L^{(3)}A^{(3)} = P^{(3)}A$.

Now we eliminate $x_2$ from equation 3, i.e. we subtract $L_{32} = \frac{a_{32}}{a_{22}} = \frac{-\frac{3}{4}}{\frac{1}{4}} = -3$ times equation / row 2 from equation / row 3

After doing this the equations have become

\[
\begin{align*}
4x_1 - 2x_2 + x_3 &= 15 \\
-\frac{5}{2}x_2 + \frac{19}{4}x_3 &= \frac{77}{4} \\
\frac{9}{5}x_3 &= \frac{27}{5}
\end{align*}
\]

or $A^{(4)}x = b^{(4)}$ where $A^{(4)} = \begin{pmatrix} 4 & -2 & 1 \\ 0 & \frac{5}{2} & \frac{19}{4} \\ 0 & 0 & \frac{9}{5} \end{pmatrix}$ and $b^{(4)} = \begin{pmatrix} 15 \\ \frac{37}{4} \\ \frac{27}{5} \end{pmatrix}$. One has $L^{(4)} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ \frac{1}{4} & \frac{1}{5} & 1 \end{pmatrix}$. $p$ and $P$ don’t change, i.e. $p^{(4)} = p^{(3)} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and $P^{(4)} = P^{(3)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. One has $L^{(4)}A^{(4)} = A^{(3)} = P^{(4)}A$. At this point we are done with the elimination phase of the algorithm. Let’s call $L = L^{(4)}$, $U = A^{(4)}$, $p = p^{(4)}$ and $P = P^{(4)}$. Then we can summarize the elimination phase of the algorithm as $LU = PA$. Note that

Row 1 of $U$ = Row $p_1$ of $A$ = Row 2 of $A$
Row 2 of $U$ = Row $p_2$ of $A$ = Row 3 of $A$
Row 3 of $U$ = Row $p_3$ of $A$ = Row 1 of $A$

In our example we have been doing the same operations on $b$ as we have done on $A$ so we are ready to do the backsubstitution phase of the algorithm. This is the same as section 1.4. However, if we had not been doing the operations on $b$, or if we were given a new $b$, then we would have to do these operations on $b$. However, there is one wrinkle. Instead of doing them in the same order as we did them on $A$, we do all the interchanges first and all at once. The key to this is to start with the original equation $Ax = b$ and multiply by $P$ to get $PAX = Pb$. Note that

Component 1 of $Pb$ = Component $p_1$ of $b$ = Component 2 of $b$ = 15
Component 2 of $Pb$ = Component $p_2$ of $b$ = Component 3 of $b$ = 8
Component 3 of $Pb$ = Component $p_3$ of $b$ = Component 1 of $b$ = 13
Thus \( Pb = \begin{pmatrix} 15 \\ 8 \\ 13 \end{pmatrix} \). So we have \( PAx = \begin{pmatrix} 15 \\ 8 \\ 13 \end{pmatrix} \). Since \( PA = LU \) we have \( LUx = g \) where \( g = \begin{pmatrix} 15 \\ 8 \\ 13 \end{pmatrix} \). Now we are in the same position as in section 1.4. We use the multipliers in \( L \) to do the same row operations on \( g \) as was done on \( A \) to transform it to \( U \). First

Subtract \( L_{21} = \frac{3}{4} \) times \( g_1 \) from \( g_2 \): \[ 8 - \left( \frac{3}{4} \right) (15) = \frac{77}{4} \]

Subtract \( L_{31} = \frac{1}{4} \) times \( g_1 \) from \( g_3 \): \[ 13 - \left( \frac{1}{4} \right) (15) = \frac{37}{4} \]

After doing this \( g \) has been transformed to \( g = g^{(1)} = \begin{pmatrix} 15 \\ \frac{77}{4} \\ \frac{37}{4} \end{pmatrix} \). Next

Subtract \( L_{32} = \frac{1}{5} \) times \( g_2 \) from \( g_3 \): \[ \frac{37}{4} - \left( \frac{1}{5} \right) (\frac{77}{4}) = \frac{27}{5} \]

After doing this \( g \) has been transformed to \( g^{(2)} = \begin{pmatrix} 15 \\ \frac{77}{4} \\ \frac{27}{5} \end{pmatrix} \).

The original equations have been transformed to \( Ux = g^{(2)} \), i.e.

\[
\begin{align*}
4x_1 - 2x_2 + x_3 &= 15 \\
-\frac{5}{2}x_2 + \frac{19}{4}x_3 &= \frac{77}{4} \\
\frac{9}{5}x_3 &= \frac{27}{5}
\end{align*}
\]

Solving the last equation for \( x_3 \) one obtains \( x_3 = \frac{27/5}{9/5} = 3 \). Substituting this into the previous equation and solving for \( x_2 \) gives \( x_2 = \frac{\frac{77}{4} - (3)\left( \frac{19}{4} \right)}{-\frac{5}{2}} = \frac{\frac{77}{4} - \frac{57}{4}}{-\frac{5}{2}} = \frac{-8}{5} = -2 \). Substituting this into the first equation and solving for \( x_1 \) gives \( x_1 = \frac{15 + (2)(-2) - 3}{4} = 2 \). (This part is the same as in the section without the pivoting.)

Let's see how our previous algorithms for \( LU \) decompositions and back-substitution need to be modified to take care of pivoting. Our previous algorithm for doing the \( LU \) decomposition of a matrix \( A \) was

\[
L = I \\
\text{for } k = 1 \text{ to } n-1 \\
\quad \text{for } j = k+1 \text{ to } n \\
\quad \quad m = a_{jk}/a_{kk} \\
\quad \quad L_{jk} = m \\
\quad \text{for } p = k+1 \text{ to } n \\
\quad \quad a_{jp} = a_{jp} - ma_{kp}
\]
Before starting the inner loop, we have to insert the code to do the row interchange. The first thing we need to do is find the row among \( j = k, k+1, \ldots, n \) that has the largest coefficient of \( a_{jk} \). Let's call this row \( r \). The code to do this would be

\[
\begin{align*}
  r &= k \\
  \text{for } j = k+1 \text{ to } n \\
  &\quad \text{if } |a_{jk}| > |a_{rk}| \text{ then } r = k 
\end{align*}
\]

Next we need to swap rows \( r \) and \( k \) of \( A \) and components \( r \) and \( k \) of \( p \). Note that the first \( k-1 \) components of these rows are zero, so we don't need to swap them. In many programming languages we need three statements to swap the values of two variables. For example, the code to swap components \( r \) and \( k \) of \( p \) would be

\[
\begin{align*}
  \text{temp} &= p_k \\
  p_k &= p_r \\
  p_r &= \text{temp} 
\end{align*}
\]

(2)

In our algorithms we shall appreciate this process by the single statement

\[
\text{swap } p_k \text{ and } p_r
\]

with the understanding that when we go to program the algorithm in an actual programming language we may have to expand this into the previous three statements (2). To swap the last \( n - r + 1 \) components of rows \( r \) and \( k \) of \( A \) we would do the following.

\[
\begin{align*}
  \text{for } q = k \text{ to } n \\
  &\quad \text{swap } a_{kq} \text{ and } a_{rq} 
\end{align*}
\]

Finally, we need to swap the first \( k - 1 \) elements of rows \( r \) and \( k \) of \( L \). The code to do this would be

\[
\begin{align*}
  \text{for } q = 1 \text{ to } k-1 \\
  &\quad \text{swap } L_{kq} \text{ and } L_{rq} 
\end{align*}
\]

Inserting this in the previous algorithm we get the following.

\[
\begin{align*}
  L &= I \\
  \text{for } j = 1 \text{ to } n \\
  &\quad p_j = j \\
  \text{for } k = 1 \text{ to } n-1 \\
  &\quad r = k \\
  &\quad \text{for } j = k+1 \text{ to } n \\
  &\quad \quad \text{if } |a_{jk}| > |a_{rk}| \text{ then } r = k \\
  &\quad \text{swap } p_k \text{ and } p_r 
\end{align*}
\]
for \( q = k \) to \( n \)
  
  swap \( a_{kj} \) and \( a_{rq} \)

for \( q = 1 \) to \( k-1 \)
  
  swap \( L_{kj} \) and \( L_{rq} \)

for \( j = k+1 \) to \( n \)
  
  \( m = a_{jk}a_{kk} \)
  
  \( L_{jk} = m \)
  
  for \( p = k+1 \) to \( n \)
    
    \( a_{jp} = a_{jp} - ma_{kp} \)

Even though we added quite a few statements, it is not hard to see that the total number of flops is on the order of a constant times \( n^2 \). So this the total number of flops for the new algorithm is still approximately \( \frac{2}{3} n^3 \) for large \( n \).

Now lets look at the modifications to the back substitution algorithm. The previous algorithm was

for \( k = 1 \) to \( n-1 \)
  
  for \( j = k+1 \) to \( n \)
    
    \( b_j = b_j - L_{jk}b_k \)

for \( j = n \) down to 1
  
  for \( p = j+1 \) to \( n \)
    
    \( b_j = b_j - a_{jp}x_p \)
  
  \( x_j = b_j/a_{jj} \)

We need to insert at the beginning the steps to form \( g = Pu \). These would be

for \( j = 1 \) to \( n \)
  
  \( g_j = b_{pj} \)

Putting this at the beginning of the previous algorithm and changing \( b \) to \( g \) in the previous algorithm gives
for \( j = 1 \) to \( n \)
\[ g_j = b_{pj} \]

for \( k = 1 \) to \( n-1 \)
\[ \begin{align*}
&\text{for } j = k+1 \text{ to } n \\
&\quad g_j = g_j - L_{jk}g_k
\end{align*} \]

for \( j = n \) down to 1
\[ \begin{align*}
&\text{for } p = j + 1 \text{ to } n \\
&\quad g_j = g_j - a_{jp}x_p \\
&\quad x_j = g_j/a_{jj}
\end{align*} \]

Here is another example illustrating solving a set of linear equations by first doing a \( LU \) decomposition of the coefficient matrix and then using it to solve the equations.

**Example 2.**

\[
\begin{align*}
2x_1 + x_2 + x_3 &= 4 \\
4x_1 + 3x_2 + 3x_3 + x_4 &= 6 \\
8x_1 + 7x_2 + 9x_3 + 5x_4 &= 8 \\
6x_1 + 7x_2 + 9x_3 + 8x_4 &= -2
\end{align*}
\]

or \( Ax = b \) where \( A = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{pmatrix} \), \( x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \) and \( b = \begin{pmatrix} 4 \\ 6 \\ 8 \\ -2 \end{pmatrix} \). Start with \( L \) equal to the identity matrix and \( p = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \).

**Part 1. Find the \( LU \) decomposition.**

**Step 1.** Interchange row 1 of \( A \) with the row with the largest entry in column 1 of absolute value. This is row 3.

Do the same with \( p \). After doing this one has \( A^{(1)} = \begin{pmatrix} 8 & 7 & 9 & 5 \\ 4 & 3 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 6 & 7 & 9 & 8 \end{pmatrix} \) and \( p^{(1)} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 4 \end{pmatrix} \). We don't need to change \( L \) since there are no columns before column 1. So \( L^{(1)} = I \).

**Step 2.** Subtract multiples of row 1 of \( A^{(1)} \) from rows 2, 3 and 4 to get zeros in column 1 in rows 2 – 4 below the diagonal.

- Subtract \( L_{21} = 4/8 = 1/2 \) times row 1 from row 2
- Subtract \( L_{31} = 2/8 = 1/4 \) times row 1 from row 3
- Subtract \( L_{41} = 6/8 = 3/4 \) times row 1 from row 4
After doing this $A$ has been transformed to $A^{(2)} = \begin{pmatrix} 8 & 7 & 9 & 5 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \\ 0 & 3 & -\frac{3}{2} & -\frac{5}{2} \\ 0 & 7 & 9 & 17/4 \end{pmatrix}$. Store the multipliers in $L$. It is now

$$L^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/4 & 0 & 1 & 0 \\ 3/4 & 0 & 0 & 1 \end{pmatrix}.$$ $p$ is not changed; $p^{(2)} = p^{(1)} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 4 \end{pmatrix}$.

**Step 3.** Interchange row 2 of $A^{(2)}$ with the row with the largest entry on or below the main diagonal in column 2 of absolute value. This is row 4. Do the same with $p^{(2)}$. After doing this one has $A^{(3)} = \begin{pmatrix} 8 & 7 & 9 & 5 \\ 0 & 3 & -\frac{3}{2} & -\frac{5}{2} \\ 0 & -\frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \\ 0 & 7 & 9 & 17/4 \end{pmatrix}$, and

$$p^{(3)} = \begin{pmatrix} 3 \\ 4 \\ 1 \\ 2 \end{pmatrix}.$$ Interchange the first entry in rows 2 and 4 of $L$. So $L^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3/4 & 1 & 0 & 0 \\ 1/4 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 1 \end{pmatrix}$.

**Step 4.** Subtract multiples of row 2 of $A^{(3)}$ from rows 3 and 4 to get zeros in column 2 in rows 3 and 4 below the diagonal.

Subtract $L_{32} = \frac{3}{4} = \frac{-3}{7}$ times row 2 from row 3

Subtract $L_{42} = \frac{-1}{2} = \frac{-2}{7}$ times row 2 from row 4

After doing this $A$ has been transformed to $A^{(4)} = \begin{pmatrix} 8 & 7 & 9 & 5 \\ 0 & 3 & -\frac{3}{2} & -\frac{5}{2} \\ 0 & 0 & -\frac{2}{7} & -\frac{4}{7} \\ 0 & 0 & -\frac{6}{7} & -\frac{2}{7} \end{pmatrix}$. Store the multipliers in $L$. It is now

$$L^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3/4 & 1 & 0 & 0 \\ 1/4 & 3 & 1 & 0 \\ 1/2 & 2 & 0 & 1 \end{pmatrix}.$$ $p$ is not changed; $p^{(4)} = p^{(3)} = \begin{pmatrix} 3 \\ 4 \\ 1 \\ 2 \end{pmatrix}$.
**Step 5.** Interchange row 3 of $A^{(4)}$ with the row with the largest entry on or below the main diagonal in column 3 of absolute value. This is row 4. Do the same with $p^{(4)}$. After doing this one has $A^{(5)} = \begin{bmatrix} 8 & 7 & 9 & 5 \\ 0 & 7/4 & 9/4 & 17/4 \\ 0 & 0 & 6/7 & 2/7 \\ 0 & 0 & -2/7 & 4/7 \end{bmatrix}$, and

$$p^{(5)} = \begin{bmatrix} 3/4 \\ 2 \\ 1 \end{bmatrix}.$$ Interchange the first two entries in rows 3 and 4 of $L$. So $L^{(5)} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 0 \\ 1/4 & 1/2 & 3/7 & 0 \end{bmatrix}$.  $p$ is not changed: $p^{(6)} = p^{(5)} = \begin{bmatrix} 3/4 \\ 2/3 \end{bmatrix}$.

**Step 6.** Subtract multiples of row 3 of $A^{(5)}$ from row 4 to get zeros in column 3 in row 4 below the diagonal.

Subtract $L^{(5)}_{43} = -2/7 = 1/3$ times row 3 from row 4

$$L^{(6)} = L = \begin{bmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 1/2 & 1/2 & 1/2 & 0 \\ 1/4 & 1/2 & 3/7 & 0 \end{bmatrix}.$$  $p^{(6)} = p^{(5)} = \begin{bmatrix} 3/4 \\ 2/3 \end{bmatrix}$.

**Part 2. Use the LU decomposition to solve the original equations.**

**Step 1.** Use $p$ to rearrange the entries of $b$ in the same fashion as the rows of $A$. Let $g$ be the resulting vector. Thus

Component 1 of $g$ = Component $p_1$ of $b$ = Component 3 of $b$ = 8
Component 2 of $g$ = Component $p_2$ of $b$ = Component 4 of $b$ = -2
Component 3 of $g$ = Component $p_3$ of $b$ = Component 2 of $b$ = 6
Component 4 of $g$ = Component $p_4$ of $b$ = Component 1 of $b$ = 4

So $g = \begin{bmatrix} 8 \\ -2 \\ 6 \\ 4 \end{bmatrix}$

**Step 2.** Use $L$ to do the same row operations on $g$ as was done on $A$ to transform it to $U$. 

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Subtract $L_{21} = \frac{3}{4}$ times $g_1$ from $g_2$: $-2 - \left(\frac{3}{4}\right)(8) = -8$

Subtract $L_{31} = \frac{1}{2}$ times $g_1$ from $g_3$: $6 - \left(\frac{1}{2}\right)(8) = 2$

Subtract $L_{41} = \frac{1}{4}$ times $g_1$ from $g_4$: $4 - \left(\frac{1}{4}\right)(8) = 2$

After doing this $g$ has been transformed to $g^{(1)} = \begin{pmatrix} 8 \\ -8 \\ 2 \\ 2 \end{pmatrix}$.

Subtract $L_{32} = -\frac{2}{7}$ times $g_2$ from $g_3$: $2 - (-\frac{2}{7})(-8) = \frac{2}{7}$

Subtract $L_{42} = -\frac{3}{7}$ times $g_2$ from $g_4$: $2 - (-\frac{3}{7})(-8) = \frac{10}{7}$

After doing this $g$ has been transformed to $g^{(2)} = \begin{pmatrix} 8 \\ -8 \\ \frac{2}{7} \\ \frac{10}{7} \end{pmatrix}$.

Subtract $L_{43} = \frac{1}{3}$ times $g_3$ from $g_4$: $-\frac{10}{7} - \left(\frac{1}{3}\right)(-\frac{2}{7}) = -\frac{4}{3}$

After doing this $g$ has been transformed to $g^{(3)} = \begin{pmatrix} 8 \\ -8 \\ \frac{2}{7} \\ -\frac{4}{3} \end{pmatrix}$. The original equations have been transformed to

$Ux = g^{(3)}$, i.e.

\[
\begin{align*}
8x_1 - 7x_2 + 9x_3 + 5x_4 &= 8 \\
\frac{7}{4}x_2 + \frac{9}{4}x_3 + \frac{17}{4}x_4 &= -8 \\
-\frac{6}{7}x_3 - \frac{2}{7}x_4 &= -\frac{2}{7} \\
\frac{2}{3}x_4 &= \frac{4}{3}
\end{align*}
\]

**Step 3 – Back Substitution.** Solve the transformed equations by back substitution.

\[
x_4 = \frac{-\frac{4}{3}}{\frac{2}{3}} = -2
\]

\[
x_3 = \frac{-\frac{2}{7}(-\frac{2}{7})(-2)}{\frac{6}{7}} = 1
\]

After doing this $g$ has been transformed to $g^{(3)} = \begin{pmatrix} 8 \\ -8 \\ \frac{2}{7} \\ -\frac{4}{3} \end{pmatrix}$. The original equations have been transformed to

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\begin{align*}
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**Step 3 – Back Substitution.** Solve the transformed equations by back substitution.

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\]
\[ x_2 = \frac{-8 \cdot \left(\frac{9}{4}\right)(1) - \frac{17}{4}(c - 2)}{7} = \frac{-32 - 9 + 34}{7} = -1 \]

\[ x_1 = \frac{8 \cdot (7)(c - 1) - (9)(c - 1) - (5)(c - 2)}{8} = \frac{8 + 7 - 9 + 10}{8} = 2 \]