Projectile Motion

So far you have focused on motion in one dimension: \( x(t) \). In this lab, you will study motion in two dimensions: \( x(t) , y(t) \). This 2D motion, called “projectile motion”, consists of a ball projected with an initial velocity in the earth’s gravitational field.

Basic Principles

Consider launching a ball with an initial velocity \( v_o \) near the surface of the earth where the acceleration of gravity is \( g \).

The position of the ball is given by the coordinates \((x , y)\). The position of the ball depends on time \( t \). The motion of the ball is defined by the motion functions: \( x(t) , y(t) \). Note that at time \( t=0 \), the ball is launched from the point \((x , y) = (0 , y_o)\). The initial conditions are the conditions at time \( t = 0 \): the initial position \( x(0) = 0 \), \( y(0) = y_o \) and the initial velocity \( v(0) \) (magnitude \( v_o \) and direction \( \theta_o \)). The fundamental kinematic quest is:

**Given:** The Initial Conditions: \( y_o , v_o , \theta_o \).

**Find:** The Motion Functions: \( x(t) , y(t) \).

The quest to find how \( x(t) \) and \( y(t) \) depend on \( t \) is greatly simplified by the following facts, first discovered by Galileo:

*The horizontal \( x(t) \) and vertical \( y(t) \) motions are completely independent of each other.*

\[
x(t) = \text{constant-velocity motion.} \quad y(t) = \text{constant-acceleration motion.}
\]

Analyzing the whole complicated motion as a superposition of manageable parts is a paradigm of modern theoretical physics.
The theory of projectile motion goes as follows. The general motion functions for any kind of uniformly-accelerated motion in two dimensions are

\[
x(t) = x_o + v_{ox} t + \frac{1}{2} a_x t^2.
\]

\[
y(t) = y_o + v_{oy} t + \frac{1}{2} a_y t^2.
\]

Projectile motion is a special case of uniformly-accelerated motion. Near the surface of the earth, the acceleration of gravity points downward and has magnitude \(9.8 \, m/s^2\) and therefore \((a_x, a_y) = (0, -9.8 \, m/s^2)\). Substituting these special “earth gravity values” of \(a_x\) and \(a_y\) into the general functions \(x(t)\) and \(y(t)\), and also setting \(x_o\) equal to zero, gives the following motion functions for any object projected in the earth’s gravitational field:

**The Projectile Motion Equations**

\[
\begin{align*}
\text{x}(t) &= v_{ox} t \\
\text{y}(t) &= y_o + v_{oy} t - 4.9 \, t^2
\end{align*}
\]

These equations tell you everything about the motion of a projectile (neglecting air resistance). If you know the conditions \((y_o, v_{ox}, v_{oy})\) at \(t = 0\), then these equations tell you the position \((x(t), y(t))\) of the projectile for all future time \(t > 0\). Make sure you understand The Projectile Motion Equations. They will be used in all future parts of this lab.

Note: In terms of the initial launch angle \(\theta_o\), the components \((v_{ox}, v_{oy})\) of the initial velocity vector \(v_o\) are \(v_{ox} = v_o \cos \theta_o\) and \(v_{oy} = v_o \sin \theta_o\).

**Exercise**

The initial \((t = 0)\) launch parameters of a projectile are \(y_o = 3.6 \, m\), \(v_o = 8.9 \, m/s\), \(\theta_o = 54^\circ\). Where is the projectile at time \(t = 1.2\) seconds?

\[
\begin{align*}
\text{x} &= \underline{\quad} \, m. & \text{y} &= \underline{\quad} \, m.
\end{align*}
\]
**Projectile Motion** = **Inertial Motion** + **Falling Motion**

Note that $x(t)$ and $y(t)$ are the components of the position vector: $r(t) = x(t)i + y(t)j$. The two scalar equations $x(t)$ and $y(t)$ of Projectile Motion displayed above can be combined into one vector equation:

$$r(t) = r_o + v_o t + \frac{1}{2}gt^2.$$ 

Note that this vector equation expresses the actual displacement $r(t) - r_o$ of the projectile, as it moves from $r_o$ to $r(t)$ during the time $t$, as a combination (vector sum) of two “virtual” displacements: a constant-velocity displacement $v_o t$ combined with a constant-acceleration displacement $\frac{1}{2}gt^2$.

With no gravity, the projectile would move along the tangent straight-line path at the constant velocity $v_o$ by virtue of its “inertia” alone and cover the distance $v_o t$ in the time $t$. But because of gravity, the projectile continually falls beneath this imaginary inertial line with the acceleration $g$ and covers the vertical distance $\frac{1}{2}gt^2$ in the same time $t$. 

![Diagram of projectile motion showing inertial and falling motion components](image)
Part I. An Illustration of the Independence of x(t) and y(t)

Roll the plastic ball off the edge of the table. At the instant the ball leaves the table, drop a coin from the edge of the table. Listen for the ball and the coin to hit the floor. Try rolling the ball faster off the table. Summarize your observations:

The picture below shows the position of the projected ball at five different times. Mark the position of the dropped coin at the same five times on the dashed vertical axis. Mark the position of the projected ball – if there were no gravity – at the same five times on the dashed horizontal axis. A ruler will help.

Exercise

The horizontal (inertial) and vertical (falling) displacements of the ball during a certain time interval are pictured below. How fast was the ball moving when it left the edge of the table at \( t = 0 \)?

\[ v_0 = \text{------------------} \, m/s . \]
Part II. The Ball Launcher. Finding the Initial Speed.

In this lab, you will use a “projectile machine” to give the ball an initial velocity, i.e. to project the ball in a certain direction with a certain speed. This ball launcher is a spring-loaded “cannon”. Note that you can change the angle $\theta_o$ of the launch (the direction of the $v_o$ vector) by tilting the launcher (loosen and tighten the nut). Note that the value of $\theta_o$ is measured relative to the horizontal. A protractor on the side of the launcher specifies the numerical value $\theta_o$. Tilt the launcher so that it is set for a $\theta_o = 30^\circ$ launch. Place the plastic ball inside the barrel of the launcher. Use the plunger rod to slowly push down on the ball until you hear and feel the first “click”. At this first setting, the spring is locked into a state of minimum compression.

CAUTION: DO NOT compress the spring beyond the first setting.

CAUTION: Always make sure that the launcher is aimed in a SAFE direction, away from people and objects.

When it is safe, pull the lever that launches the ball. Observe the projectile motion.

What is the Initial Speed of the Projectile?

Here you will find the value of $v_o = \text{the speed of the ball as it exits the launcher}$. This is known as the “muzzle velocity” of the cannon. First note that the point at which the ball exits the launcher (leaves the spring) is marked as a dot at the center of the small circle that appears on the side of the launcher. This dot represents the “center of mass” of the ball (circle). Since the speed of the ball is mostly determined by the force of the spring, the value of $v_o$ is approximately constant, independent of the tilt of the launcher. Tilt the launcher so that it points upward in the vertical direction ($\theta_o = 90^\circ$). Launch the ball and observe how high the ball rises. By measuring this maximum height, you can deduce the launch speed.

Here is an experimental technique to determine the maximum height reached by the ball. On your table is a vertical rod assembly with a small metal plate attached to the rod. Position the plate directly above the launcher near the point where the ball reaches its maximum height. Adjust the plate up or down so that the launched ball barely hits or barely misses the plate. Launch the ball several times, each time “fine tuning” the vertical position of the plate (slightly up/down) until you are confident ($\pm 1 \text{ cm}$) in the location of the maximum height. Measure the distance from the center of the ball at its launch point to the center of the ball at its maximum-height point ... or equivalently from the top of the ball to the bottom of the plate.
Maximum Height: \( H = \) \( \underline{\text{______________}} \) \( m \).

From this measured value of \( H \), compute the initial speed \( v_o \) of the ball. Hint: Use one of the kinematic equations for uniformly-accelerated motion – the one that does not contain the time variable. Note that the final speed of the ball at its maximum height is equal to zero. Show your calculation.

Initial Speed: \( v_o = \) \( \underline{\text{______________}} \) \( \text{m/s} \).

**Part III. Discovering the Parabola**

Geometric trademark of projectile motion: *Projectile Path = Parabolic Curve.*

Set the launcher for a \( \theta_o = 60^\circ \) launch. Launch the ball and observe the curved path traced out by the ball as it moves through space. It is difficult to map out the exact shape of the path when you only have about *one second* to make the observation! The path is definitely not straight, but how do you know it is a parabola? Why couldn’t it be a semi-circular arc, or an oval curve, or a cubic curve, or an exponential curve, or a piece of a sine curve?

Let’s explore the special case of a *horizontal* launch. Set the launch angle to be \( \theta_o = 0^\circ \).

![Diagram of projectile motion](image)

*The Initial Conditions*

Measure the initial height \( y_o \) of the ball: measure from the table surface to the *bottom* of the ball (see picture). Record your previously measured value of the initial speed \( v_o \) of the ball.

\[
\begin{align*}
y_o &= \underline{\text{______________}} \text{ m} \\
v_o &= \underline{\text{______________}} \text{ m/s}
\end{align*}
\]

These initial parameters, which specify how you *start* the projectile motion, uniquely determine the *shape* of the projectile *path.*
Mapping Out the Parabola

Use your values of $v_0$ and $v_o$ to “map out” the path of your projectile. First find the motion functions $x(t)$ and $y(t)$ that specify the position of the ball at any time $t$ between the initial launch time $t = 0$ and the final hit-the-table time. *Hint:* Substitute the numerical values of your measured initial parameters $v_0$ and $v_o$ into the Projectile Motion Equations.

$$x(t) = \left( \frac{m}{s} \right) t, \quad y(t) = ( m ) - (4.9 \text{ m/s}^2) t^2.$$

Find $x(t)$ at $t = 0$, $0.05 \text{ s}$, $0.10 \text{ s}$, $0.15 \text{ s}$, $0.20 \text{ s}$ and mark these five values of $x$ by placing five dots at the appropriate locations on the $x(m)$ axis shown below. Find $y(t)$ at $t = 0$, $0.05 \text{ s}$, $0.10 \text{ s}$, $0.15 \text{ s}$, $0.20 \text{ s}$ and mark these five values of $y$ on the $y(m)$ axis below.

Place five dots within the $xy$ plane pictured above that represent the actual five positions of the projectile at the five times considered above. Use these five dots as a guide to draw the entire smooth actual path of the projectile as it flies through the air.

Find the equation $y(x)$ that describes this curved path. *Hint:* Eliminate $t$ from your $x(t)$ and $y(t)$ equations by solving the $x$ equation for $t$ and then substituting this $t$ expression into the $y$ equation. You will be left with “$y$ as a function of $x$” which should be a quadratic function of the form $y(x) = A + Bx^2$. Show your derivation of $y(x)$ in the space below. The underlined coefficients that you fill in for the $y(x)$ equation are the special “parabolic parameters” of your projectile motion.

$$y(x) = \underline{\quad} + \underline{\quad} x^2.$$
Where Does the Projectile Land?

In theory, the landing point is defined by the coordinate point \((x, y) = (L, 0)\).

Use your parabola equation \(y(x)\) to compute the landing distance \(L\) of your projectile. Show your calculation.

\[
L \text{ (theory)} = \underline{\text{______________}} \text{ m}.
\]

Launch the ball five times. Arrange for the ball to land on a piece of carbon paper, which is placed on top of copy paper taped to the table. The scatter of landing points (dots) recorded on the paper provides a nice visual display of the uncertainty in \(L\). Find the average value of \(L\) and the uncertainty in \(L\) (half-width spread around the average).

\[
\begin{array}{c|c|c|c|}
L (m) & & & \\
\hline
\end{array}
\]

\[
L \text{ (experiment)} = \underline{\text{______________}} \pm \underline{\text{______________}} \text{ m}.
\]

Does your value of \(L\) (theory) fall within the range of values of \(L\) (experiment)?
Part IV.  Range, Altitude, Flight Time

Here you will explore how the properties of the motion depend on the angle of the launch. There are three important properties of projectile motion:

\[
\begin{align*}
\text{Range} & \quad R = \text{Maximum horizontal distance.} \\
\text{Maximum Height} & \quad H = \text{Maximum vertical distance.} \\
\text{Time of Flight} & \quad t_f = \text{Time in air between launch and land.}
\end{align*}
\]

Note that for this projectile motion, the projectile lands at the same level from where it was launched.

Theory

Find the motion functions \(x(t)\) and \(y(t)\) that describe the motion of the ball projected at an angle of \(\theta_o = 70^\circ\) from your launcher. To find these functions, use your measured value of \(v_o\) (along with the relations \(v_{ox} = v_o \cos \theta_o\) and \(v_{oy} = v_o \sin \theta_o\)) as the initial velocity parameter in the general Projectile Motion Equations. Choose the origin of the xy coordinate system to be the launch point (see picture above). So as not to “clutter” the equations, do not include the units (\(m\) and \(m/s\)) of the numbers that you write in the blanks below.

\[
\begin{align*}
x(t) &= \text{_______} \quad t \\
y(t) &= \text{_______} \quad t - 4.9 \quad t^2
\end{align*}
\]

Finding \(t_f\), \(H\), and \(R\) from your \(x(t)\) and \(y(t)\)

Look at the picture of the parabolic path above and note that the vertical position \(y(t)\) of the ball starts at \(y = 0\) at \(t = 0\), increases to the maximum value \(y = H\) at \(t = t_f/2\), and then decreases back to \(y = 0\) at \(t = t_f\). The horizontal position \(x(t)\) of the ball increases from \(x = 0\) at \(t = 0\) to the
maximum value \( x = R \) at \( t = t_f \). This means that you can compute the projectile properties \( t_f, H, \) and \( R \) from your motion functions \( x(t) \) and \( y(t) \) by using the following conditions imposed on \( x(t) \) and \( y(t) \):

*Time of Flight:* \( y(t) = 0 \) when \( t = t_f \).

*Maximum Height:* \( y(t) = H \) when \( t = t_f/2 \).

*Range:* \( x(t) = R \) when \( t = t_f \).

Show your calculations of \( t_f, H, \) and \( R \) in the space below.

\[
t_f = \underline{\text{___________ \( s \)}}. \quad H = \underline{\text{___________ \( m \)}}. \quad R = \underline{\text{___________ \( m \)}}.
\]

**Experiment**

Set the launcher for a \( \theta_o = 70^\circ \) launch. Position the large wooden box so that the ball lands on top of the box. See the schematic below. Note that the height of the box matches the initial height of ball. This insures that the projectile launches and lands at the same level.
Launch the ball and measure $t_f$, $H$, and $R$. One person should measure $t_f$ – the “hang time” of the ball through the air – with a stopwatch. Another person should measure $H$ – the maximum altitude – with a meter stick and a simple visible inspection of where the trajectory of the ball peaks. This measurement of $H$ may be a bit tricky (do not strive for high precision). Note: $H$ is measured from the x-axis level, not the table surface level (see the schematic above). Use carbon paper on top of copy paper to record the landing point. Measure the range $R$ – the horizontal distance between the launch point and the landing point – with a meter stick.

Repeat this 70° launch four more times. Find the average values of $t_f$, $H$, and $R$. Find the corresponding uncertainties (half-width spread around the average).

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<thead>
<tr>
<th></th>
<th>Average</th>
<th>±</th>
<th>Uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_f$ (s)</td>
<td></td>
<td></td>
<td>±</td>
</tr>
<tr>
<td>$H$ (m)</td>
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<td></td>
<td>±</td>
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<tr>
<td>$R$ (m)</td>
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</tbody>
</table>

**Compare Theory and Experiment**

In order for you to say “Our experimental results support the theoretical predictions”, you must show that your calculated values of $t_f$, $H$, $R$ fall within the range of your measured values of $t_f$, $H$, $R$. Summarize your theoretical and experimental results by completing the following sentences (fill in the blanks and circle “does” or “does not”)

The calculated $t_f = \underline{\text{________}}$ s does does not fall within the measured range $\underline{\text{________}} < t_f < \underline{\text{________}}$ s.

The calculated $H = \underline{\text{________}}$ m does does not fall within the measured range $\underline{\text{________}} < H < \underline{\text{________}}$ m.

The calculated $R = \underline{\text{________}}$ m does does not fall within the measured range $\underline{\text{________}} < R < \underline{\text{________}}$ m.

Conclusion:
Part V. Galileo’s 45° Rules

What angle should you launch the ball so that it goes the farthest? Galileo proved that the horizontal range is maximum when the angle is 45°. He also proved that two projectiles that are launched with the same speed but at different angles, one greater than 45° and the other less than 45° by the same amount, will travel the same horizontal distance. For example, a projectile launched at 45° + 5° = 50° has the same range as a projectile launched at 45° − 5° = 40°. In other words, any pair of complementary launch angles, such as (10°, 80°), (25°, 65°), or (38°, 52°), are associated with the same range.

Galileo’s Rules (Theorems) are theoretically significant and are the basis for modern-day applications (sports, military) of projectile motion. Here is a concise statement of Galileo’s famous “45° rules” in terms of the range function \( R(\theta_0) \) which specifies how \( R \) depend on \( \theta_0 \).

1. Maximum Rule: \( R(45°) = \) maximum.

2. Symmetry Rule: \( R(45° + \alpha) = R(45° - \alpha) \).

Note: These rules are valid for all projectile motion without air resistance as long as the projectile lands at the same level from where it was launched.

Testing The 45° Rules

Use your launcher to project the ball at the angles \( \theta_0 \) listed in the table below. Measure the range for each angle. One launch for each angle is fine. If you feel the need to check your result for a certain angle, then repeat the launch. The goal here is for you to discover the overall pattern of how \( R \) depends on \( \theta_0 \) and not to derive the “exact” function \( R(\theta_0) \).

<table>
<thead>
<tr>
<th>( \theta_0 )</th>
<th>10°</th>
<th>20°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>70°</th>
<th>80°</th>
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</thead>
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<tr>
<td>( R ) (m)</td>
<td></td>
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<td>max</td>
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**Landing Point Diagram**

Mark the landing points of the ball on the following Range (R) axis. Label each landing point with the value of the corresponding launch angle ($\theta_o$).

![Diagram showing a range R axis with launch points marked at different intervals.](image)

**Conclusion**

Based on your experiment of projecting the ball at different angles and observing the motion,

1. Which of the angles that you studied gave the maximum range?

2. Which angles gave the same range (within experimental error)?

3. Which angle gave the longest hang time and largest altitude?