Part I  Non Constant Force – A Case Study

So far in this course, all the forces you have studied have been constant forces – independent of position. Many forces in nature depend on position. The gravitational force on a planet depends on its distance from the sun. The magnetic force on a nail depends on its distance from the magnet. In this lab, you will encounter one of the most important position-dependent forces in physics. The force due to a spring. The mechanical system that you will study consists of a cart connected to two springs moving on an aluminum track:

The origin of the x-axis is chosen to be the point where the net force on the cart is zero. The cart will remain at rest at x = 0. At this so-called “equilibrium point”, the left spring pulls on the cart with a force that exactly balances the pull of the right spring. Thus F = 0 at x = 0. With your hand, pull the mass away from x = 0 to another position, such as x = 10 cm or x = −5 cm and feel the force. The force always restores the mass back toward x = 0:

As you pull the cart further away from x = 0, the restoring force back to x = 0 becomes stronger due to the increasing expansion and compression of the springs. Thus the magnitude of F is not constant: F depends on x.

Since the force F on the cart depends on the position x of the cart, this force quantity is a mathematical function F(x). Given any value of x, the function F(x) specifies the value of the force at that particular x. Based on our qualitative study so far, all we know about this function is that F(0) = 0 and that the magnitude of F(x) is an increasing function. Your goal in this lab is to...
find the precise form of this force function. Does \( F(x) \) have the form \( F = 3x \), or \( F = 5x^2 \), or \( F(x) = \ln(1+x) \)? You will find \( F(x) \) two different ways: (1) Dynamically, by observing the back-and-forth motion of the cart. (2) Statically, by using a spring scale.

1. Discovering the Law of Force

In physics, a force law is a mathematical rule that specifies how force depends on position. From our previous Force and Motion Lab, you learned how physicists discover the force laws of nature: observe the motion, deduce the force. More specifically, measure the change in velocity (acceleration \( a \)) of the object (mass \( m \)) due to the force in question (\( F \)) and then deduce the magnitude of \( F \) from the relation \( F = ma \).

For any constant force, the motion is described by a quadratic function of time: \( x(t) = \frac{1}{2}at^2 \). The acceleration is constant. For gravity (near the earth’s surface), the well known constant acceleration is \( a = 9.8 \, \text{m/s}^2 \). The trademark of constant-force motion is the parabolic shape of the \( x-t \) curve. In this lab we must address two basic questions: For the non-constant force due to springs, what is the shape of \( x(t) \)? How does the non-constant acceleration \( a(x) \) depend on \( x \)? In this lab, you will measure \( x(t) \) of the cart, determine \( a(x) \) from \( x(t) \), and find \( F(x) \) from \( F(x) = ma(x) \).

Sinusoidal Oscillations

Pull the cart from \( x = 0 \) to about \( x = 10 \, \text{cm} \) or \( 15 \, \text{cm} \) and release. Use the motion sensor to record the position \( x(t) \) of the cart as a function of time. [Open Logger Pro file Moving Along] Does your \( x-t \) curve look like a “sine curve”? A sine it is!

Note the connection between force and geometry:

<table>
<thead>
<tr>
<th>Gravitational Force</th>
<th>Elliptical Motion</th>
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</thead>
<tbody>
<tr>
<td>Spring Force</td>
<td>Sinusoidal Motion</td>
</tr>
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</table>

As the cart moves back and forth along the \( x \) axis, the motion repeats itself. The cart oscillates or vibrates. The position function \( x(t) \) is a periodic function of time \( t \). Oscillatory motion is everywhere in Nature: pendulum, tuning fork, guitar string, sound waves, light waves, radio waves, electron in antenna, atomic clock, molecular vibrations (heat), nuclear magnetic resonance (NMR), pulsating stars (pulsars), oscillating universe (big bang-big crunch).

The mathematical equation describing the \( x-t \) curve recorded by your motion sensor is

\[
x(t) = A \sin \left( \frac{2\pi t}{T} \right).
\]

The shape of a sine curve is uniquely defined by two parameters: the amplitude \( A \) and the period \( T \). Qualitatively speaking, \( A \) is the height and \( T \) is the width of the curve. More precisely, the amplitude \( A \) specifies the maximum displacement of the cart away from \( x = 0 \). The period \( T \) is the time for one oscillation or one cycle (one complete back-and-forth motion). Note that the sine function \( \sin(2\pi T) \) repeats itself whenever \( t \) changes by the amount \( T \), i.e. \( x(0) = x(T) \) since \( \sin(0) = \sin(2\pi) \). A plot of \( x(t) = A \sin(2\pi T) \) displays the graphical meaning of the motion (shape) parameters \( A \) and \( T \).
Measuring the Period

Find the numerical value of the period $T$ two ways: 1. Read $T$ directly off the $x(t)$ curve recorded by your motion sensor. 2. Measure $T$ by timing the oscillating cart with a stopwatch. In both cases, find the total time for five oscillations and then divide by five to get the period. This method of measuring longer time intervals is much easier and more accurate than trying to measure the short time associated with one period. List your two values of $T$. If they are not “identical”, then report the average.

$$T = \text{_______________ seconds.}$$

Relation Between Period and Acceleration

Why should the period be related to the acceleration? Here is a qualitative answer. If the cart moves back-and-forth rapidly, then the velocity changes rapidly. This means that a small period corresponds to a large acceleration. We are going to use some cool calculus to find the precise relationship between $a$ and $T$.

Given the function $x = A\sin(2\pi t/T)$, we can calculate the acceleration from the physics relations, $a = dv/dt$ and $v = dx/dt$, or equivalently $a = d^2x/dt^2$. We only need two calculus facts: $d\sin\theta/d\theta = \cos\theta$ and $d\cos\theta/d\theta = -\sin\theta$. It follows that $a = -(2\pi/T)^2 A\sin(2\pi t/T)$. But since $A\sin(2\pi t/T)$ is equal to $x$, we have $a = -(2\pi/T)^2 x$. Thus the vital motion quantity “ma” has the form

$$ma = -m (2\pi/T)^2 x.$$  

Note that the acceleration is a linear function of $x$ and is proportional to $1/T^2$. Since $a = d^2x/dt^2$, and thus roughly speaking, $a \sim 1/(dt)^2$, it is not surprising that for oscillatory motion, $a \sim 1/T^2$. This inverse-squared relation means that a small period (rapid oscillation) is associated with a large acceleration (quick changes in velocity).

Now that we have discovered the motion function $ma(x)$, it is easy to find the force function $F(x)$. Given Newton’s Law of Motion, $F = ma$, we can conclude that the force law for this mechanical system (cart connected to springs) is
\[ F(x) = -m \left( \frac{2\pi}{T} \right)^2 x. \]

Note that \( F(x) \) is a linear function of \( x \). The force \( F \) increases in direct proportion to the displacement \( x \). The minus sign makes perfect sense. If the cart is displaced to the right (left) of \( x=0 \), then the force on the cart is directed to the left (right). Physicists write this force law as

\[ F(x) = -kx, \]

where the force constant \( k \) is

\[ k = m \left( \frac{2\pi}{T} \right)^2. \]

Note that \( k \) is the proportionality constant between \( F \) and \( x \). If you know the value of \( k \), then you know everything about the spring force in your system. Different spring systems have different values of \( k \). Stiff springs (car suspensions) have large values of \( k \). Weak springs (toy slinkys) have small values. To find the value of the force constant, you need the values of \( m \) and \( T \).

Measure the mass of the cart. 

\[ m = \text{______________} \text{ kilograms}. \]

Insert your measured values of \( T \) and \( m \) into the analytical expression for \( k \). Report the numerical value of the force constant for your mechanical system.

\[ k = \text{______________} \text{ N/m}. \]

2. **Measuring \( F(x) \) with a Spring Scale**

Now that you have deduced the force function \( F(x) \) from the motion data \( x(t) \), you can check your result by directly measuring \( F(x) \) using a spring scale. This may seem like an easy method to find the force, but remember, someone had to calibrate the force meter to convert a length measurement (stretch of spring) into a force quantity (Newtons). Finding force from \( F = ma \) is the gold standard for measuring force.

First make sure the spring scale is “zeroed”. When the scale is held in a horizontal position and nothing is connected to the hook, the force should read “zero”. You may have to add a correction factor to the scale reading in order to compensate for the shift in the zero point.

Attach the spring scale to the cart. Pull on the scale so that the cart moves from \( x = 0 \) to \( x = 0.02 \) m. Hold the cart at 0.02 m and record the scale reading. Next pull the cart to 0.04 m, hold, and record the reading, etc. Note that since the cart is at rest at each location, the force exerted on the cart by the spring scale is equal and opposite to the force exerted on the cart by the two attached springs. Fill in the following \( F_s \) versus \( x \) table. \( F_s \) denotes the force measured with the force sensor (spring scale).

<table>
<thead>
<tr>
<th>( x ) (m)</th>
<th>0</th>
<th>0.02</th>
<th>0.04</th>
<th>0.06</th>
<th>0.08</th>
<th>0.10</th>
<th>0.12</th>
<th>0.14</th>
<th>0.16</th>
<th>0.18</th>
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<tbody>
<tr>
<td>( F_s ) (N)</td>
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Use the program *Graphical Analysis for Windows* to plot $F_s$ versus $x$ ($F_s$ on the y axis, $x$ on the x axis). Your data points should fall on a line. Analyze the plot to find the best-fit line through the points. The equation of the line is $F_s = k_s x$. Report the value of the slope:

$$k_s = \quad \text{____________} \quad \text{N/m}.$$ 

On the same graph, plot the force function $F = kx$ (inserting your measured value of $k$) that you discovered from your analysis of the oscillatory motion. Since we are only interested in the magnitude of the force (not the direction) the minus sign in the force law $F = -kx$ has been omitted. On your printed graph, label the $F = kx$ line with the name “$F$ deduced from the Motion”. Label the $F_s = k_s x$ line with the name “$F_s$ read from the Force Sensor”. Hand in the graph with your report.

**Compare Your Two Measurements of $F(x)$**

The force function $F(x) = kx$ deduced by analyzing the oscillatory motion is

$$F(x) =$$

The force function $F_s(x) = k_s x$ obtained by directly reading the force sensor is

$$F_s(x) =$$

The percent difference between the force constants $k$ and $k_s$ is ________.

**Part II. Exploring Friction**

The force of kinetic friction acts on any object sliding across a surface. This friction force $f$ obeys the empirical law:

$$f = \mu N,$$

where $N$ is the normal force and $\mu$ is the coefficient of kinetic friction. The coefficient $\mu$ depends on the nature of the two surfaces in contact. For steel on steel, $\mu = 0.5$. For Teflon on steel, $\mu = 0.04$. For rope on wood, $\mu = 0.3$. For shoes on ice, $\mu = 0.05$. For rubber on dry concrete, $\mu = 0.8$. For rubber on wet concrete, $\mu = 0.3$.

**Measuring the Coefficient of Friction**

You goal is to measure the value of $\mu$ for felt on aluminum. Attach the spring scale to the wooden block covered in felt. Practice pulling the block across the aluminum track at a constant speed. Judge the speed (qualitatively) by covering a constant amount of distance every second of time. Try a speed of about 5 cm/s. Then try a speed of about 10 cm/s. To a very good approximation, the force of friction is independent of speed.
Why pull at constant speed? If the speed of the block is constant, then its acceleration is zero, which means that the net force on the block is zero, which means that the scale reading (spring force) equals the magnitude of the friction force. See the force diagram below.

![Force Diagram](image)

Record the value of the friction force $f$ for five different values of the normal force $N$. There is an assortment of brass weights that can be used to load the block in order to change $N$.

<table>
<thead>
<tr>
<th>$N$ (Newtons)</th>
<th>$f$ (Newtons)</th>
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Use the program *Graphical Analysis* to graph $f$ versus $N$ ($f$ on y axis, $N$ on x axis). Print this graph and include it with your report. Find the best-fit line through your five data points. If you compare the mathematical equation of a line, $y = mx+b$, with the physical equation of friction, written in the form $f = \mu N + 0$, it follows that the coefficient of friction $\mu$ is equal to the slope ($m$) and the y-intercept ($b$) is zero. Indeed, $\mu = f/N$ is the rise over the run of the line. Report your measured value of the coefficient of friction:

$$\mu = \underline{\quad} \quad \text{ (felt on aluminum).}$$

Since one measurement of $\mu$ is subject to uncertainty, let's average the results of many experiments to obtain an accurate value of $\mu$. Write your value of $\mu$ on the chalkboard and compute the class average:

Average $\mu = \underline{\quad} \quad \text{ (felt on aluminum).}$

**Part III. Experimental Design: Lowering a Weight at g/30**

Your goal is to lower a “heavy” weight (100 gram brass cylinder) over a pulley so that the weight falls to the ground with an acceleration that is 1/30 that of free fall. Here is the schematic of the mechanical system:

![Mechanical System Diagram](image)
Theory

Work out the theory in the space below. Draw free-body diagrams and set up Newton’s equations of motion. Derive the formula that gives the acceleration $a$ of the system as a function of the system parameters: $m_L$, $m_B$, $m_H$, $\mu$, and $g$.

\[
 a = \quad \text{(to be derived)}
\]

Into this theoretical equation, plug your measured values of $m_B$ and $\mu$, the well known value of $g$, and the design specifications for $m_H$ and $a$. Solve the resulting equation for the predicted value of the load mass $m_L$.

\[
 m_L = \quad \text{(to be calculated)} \text{ kg}.
\]

Experiment
Add the predicted amount of load $m_l$ to the block. Release the system from rest and measure the acceleration. Report your measured value of $a$. Include the *uncertainty* in $a$. Describe how you made the measurement and estimated the uncertainty. Provide any data and/or graphs that you used to find $a$.

$$a \text{ (measured)} = \underline{\text{______________}} \pm \underline{\text{______________}} \text{ m/s}^2.$$  

Compare your value of $a$(measured) with the design goal of $a = g/30$. What is the percent difference? Write down the *range* of experimental numbers (low to high) that characterize your measured acceleration. Does the theoretical number $g/30$ fall within this range?