Part I. Non Constant Force – A Case Study

How does $F(x)$ depend on $x$?

So far in this course, all the forces you have studied have been constant forces – independent of position. Many forces in nature depend on position. The gravitational force on a planet depends on its distance from the sun. The magnetic force on a mag-lev train depends on its distance from the tracks. In this lab, you will encounter one of the most important position-dependent forces in physics: the force due to a spring. The mechanical system that you will study consists of a cart connected to two springs moving on a metal track:

The origin of the $x$-axis is chosen to be the point where the net force on the cart is zero. The cart will remain at rest at $x = 0$. At this so-called “equilibrium point”, the left spring pulls on the cart with a force that exactly balances the pull of the right spring. Thus $F = 0$ at $x = 0$. With your hand, pull the cart away from $x = 0$ to another position, such as $x = 5$ cm or $x = -5$ cm and feel the force. The force always restores the mass back toward $x = 0$:

As you pull the cart further away from $x = 0$, the restoring force back to $x = 0$ becomes stronger due to the increasing expansion and compression of the springs. Thus the magnitude of $F$ is not constant: $F$ depends on $x$.

Since the force $F$ on the cart depends on the position $x$ of the cart, this physical quantity is represented by a mathematical function $F(x)$. Given any value of $x$, the function $F(x)$ specifies
the value of the force at that particular x. Based on your qualitative study so far, all you know
about this function is that \( F(0) = 0 \) and that the magnitude of \( F(x) \) is an increasing function. Your
experimental goal is to find the precise form of this force function. Does \( F(x) \) have the form
\( F = 3x \), or \( F = 5x^2 \), or \( F(x) = \ln(1+x) \)? You will find \( F(x) \) two different ways: (1) Dynamically,
by observing the back-and-forth motion of the cart. (2) Statically, by using a spring scale.

1. Discovering the Law of Force

In physics, a force law is a mathematical rule that specifies how force depends on position. From
the previous Force and Motion Lab, you learned how physicists discover the force laws of
nature: observe the motion, deduce the force. In this experiment, your quest is to

\[
\text{Deduce how the force } F(x) \text{ depends on } x \text{ from a measurement of how the position } x(t) \text{ depends on } t.
\]

For any constant force, the motion is described by a quadratic function of time: \( x(t) = \frac{1}{2} at^2 \).
For gravity (near the earth’s surface), the well known value of the constant acceleration is \( a = 9.8 \)
m/s\(^2\). The geometric trademark of constant-force motion is the parabolic shape of the x-t curve.
For the non-constant force due to springs, what is the shape of \( x(t) \)?

Sinusoidal Motion

Pull the cart from \( x = 0 \) to about \( x = 20 \) cm and release. Use the motion sensor to record the
position \( x(t) \) of the cart as a function of time. [Open Logger Pro file Moving Along] Does your
x-t curve look like a “sine curve”? A sine it is!

There is a deep connection between force and geometry:

- Gravitational Force ⇔ Elliptical Motion
- Spring Force ⇔ Sinusoidal Motion
- Electromagnetic Force ⇔ Helical Motion

As the cart moves back and forth along the x axis, the motion repeats itself. The cart oscillates or
vibrates. The position function \( x(t) \) is a periodic function of time \( t \). Oscillatory motion is
everywhere in Nature: pendulum, tuning fork, guitar string, sound waves, light waves, radio
waves, electron in antenna, atomic clock, molecular vibrations (heat), nuclear magnetic
resonance (NMR), pulsating stars (pulsars), oscillating universe (big bang-big crunch).

The mathematical equation describing the x-t curve recorded by your motion sensor is

\[
x(t) = A \sin \left(2\pi t/T\right).
\]

Qualitatively speaking, the parameters \( A \) and \( T \) specify the “height” and the “width” of the sine
curve:
More precisely, the amplitude $A$ is the maximum displacement of the cart away from the equilibrium point. The period $T$ is the time it takes the cart to complete one oscillation (one back-and-forth motion). Note that the function $x(t) = A\sin(2\pi t/T)$ is bounded, $-A \leq x \leq A$, and periodic, $x(t+T) = x(t)$.

Examine the $x$-$t$ curve recorded by your motion sensor. Read off the values of $A$ and $T$ that characterize the sinusoidal motion of your cart:

$A = \underline{\hspace{2cm}}$ m .  \hspace{1cm} $T = \underline{\hspace{2cm}}$ s .

**The Period of Motion $T$ Determines the Law of Force $F(x)$**

*Amazing Fact:* If you know the time $T$, then you can figure out the force $F(x)$. Here are the key steps that allow you to deduce the force function $F(x)$ from the motion function $x(t)$:

\[
F = m \ a \\
= m \ \frac{dv}{dt} \\
= m \ \frac{d^2x}{dt^2} \\
= m \ \frac{d^2}{dt^2} A\sin(2\pi t/T) \\
= m \ [-(2\pi/T)^2 \ A\sin(2\pi t/T)] \\
= m \ [-(2\pi/T)^2 \ x]
\]

Can you justify each step (equal sign)? This derivation illustrates the power of Newton’s *physics* and *calculus* in analyzing nature.

The last line in the derivation reveals precisely how $F$ depends on $x$. In summary, by inserting the motion function $x = A\sin(2\pi t/T)$ into Newton’s law, $F = m \frac{d^2x}{dt^2}$, we have deduced the following force function:

\[
F(x) = -(4\pi^2 m/T^2) \ x .
\]
Physicists write this force law in the form

\[ F = -kx, \]

where the force constant \( k \) is

\[ k = 4\pi^2m/T^2. \]

The magnitude of the force \( F \) on the cart increases in direct proportion to the distance \( x \) that the cart is displaced from the equilibrium point (\( x=0 \)). If you double \( x \), then you double \( F \). The constant \( k \) is the proportionality constant between \( F \) and \( x \). The minus sign in \( F = -kx \) indicates that the force is always opposite to the displacement. A cart on the right (left) side of \( x=0 \) experiences a force that is directed to the left (right).

Physicists say that the spring force is a “linear force” because the force function \( F = -kx \) is a linear function of \( x \). In contrast, the force of gravity and the force of electricity are “inverse-squared forces” (\( F = k/x^2 \)). Whereas the spring force gets stronger as the distance increases, the forces of gravity and electricity get weaker as \( x \) increases.

If you know the value of the force constant \( k \), then you know everything about the spring force in your system. Different spring systems have different values of \( k \). Stiff springs (car suspensions) have large values of \( k \). Weak springs (toy slinkys) have small values. What is the value of \( k \) that characterizes the chemical bond – the electromagnetic spring – between the vibrating oxygen atoms in an \( O_2 \) molecule?

Note that the relation “\( k = 4\pi^2m/T^2 \)” says that “\( k \) goes like \( 1/T^2 \):

\[ \text{strength of force } \sim \text{ inverse square of period}. \]

Why should the force be related to the inverse of time? This relation makes intuitive sense. A stiff spring (large \( k \)) vibrates faster (small \( T \)) than a weak spring. If the cart moves back-and-forth rapidly (small \( T \)), then the velocity changes rapidly (large \( a \)), and thus the force (\( ma \)) must be strong (large \( k \)). Symbolically speaking, \( F = ma \) and \( a = \frac{d^2x}{dt^2} \), and thus roughly speaking, \( F \) goes like \( 1/(dt)^2 \).

**Measuring the Force Constant with a Stopwatch**

**Bottom Line:** To find the law of force \( F(x) = -(4\pi^2m/T^2)x \), you need to measure the mass \( m \) of the cart and the period \( T \) of the motion.

Measure the mass of the cart. \[ m = \quad \text{kg}. \]

Measure the period with a stopwatch. More specifically, measure the time it takes for the cart to complete five oscillations (five back-and-forth motions). Divide this five-cycle time by five to obtain the period. Why five? Ten would be good too. Because of the uncertainty in starting and stopping the watch and counting the oscillations, it is more accurate to measure a long time rather than a short time. For example, if the uncertainty in measuring the time with the
stopwatch is 0.5 seconds, then the measurement 5 ± 0.5 seconds is a 10% error, whereas the measurement 50 ± 0.5 seconds is only a 1% error. Report the value of the period of your oscillating cart.

\[ T = \underline{\text{_______________}} \text{ s} . \]

Is this stopwatch value of T consistent with (within 5% of) the graphical value of T that you obtained from the “width” of your x-t curve? If not, see your instructor.

Based on your measured values of m and T, what is the force constant that characterizes your spring system? Show your calculation.

\[ k = \underline{\text{_______________}} \text{ N/m}. \]

Given this value of k, compute the magnitude of the force (F = −kx without the minus sign) at those values of x listed in the following table.

**Force deduced from the Motion**

<table>
<thead>
<tr>
<th>x (m)</th>
<th>0</th>
<th>0.02</th>
<th>0.04</th>
<th>0.06</th>
<th>0.08</th>
<th>0.10</th>
<th>0.12</th>
<th>0.14</th>
<th>0.16</th>
</tr>
</thead>
<tbody>
<tr>
<td>F = kx (N)</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. **Measuring F(x) with a Spring Scale**

Now that you have deduced the force function F(x) from the motion data x(t), you can check your result by directly measuring F(x) using a spring scale. This may seem like an easy method to find the force, but remember, someone had to calibrate the force meter to convert a length measurement (stretch of spring) into a force quantity (Newtons). Finding force from \( F = ma \) is the gold standard for measuring force.

First make sure the spring scale is “zeroed”. When the scale is held in a horizontal position and nothing is connected to the hook, the force should read “zero”.

Start with the cart at rest at \( x = 0 \). The two springs should still be connected to the ends of the cart and the track. Attach the spring scale to the right end of the cart. While keeping the scale as horizontal as possible (parallel to the track), pull on the scale case and displace the cart from \( x = 0 \) to \( x = 0.02 \) m. Hold the cart at 0.02 m and record the scale reading \( F_s \) in the table below. Next pull the cart to 0.04 m, hold, and record the reading, etc. Note that since the cart is at rest at each location, the force exerted on the cart by the scale is equal and opposite to the force exerted
on the cart by the two attached springs. The subscript “s” on \( F_s \) denotes that this force is measured with the spring scale.

\[ \text{Force measured with the Spring Scale} \]

<table>
<thead>
<tr>
<th>( x ) (m)</th>
<th>0</th>
<th>0.02</th>
<th>0.04</th>
<th>0.06</th>
<th>0.08</th>
<th>0.10</th>
<th>0.12</th>
<th>0.14</th>
<th>0.16</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_s ) (N)</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Use the program \( \text{Graphical Analysis} \) to plot \( F_s \) versus \( x \) (\( F_s \) on the y axis). Delete the line segments (connect-the-dots) that the program automatically inserts between neighboring data points. Your data points should fall on a line. Analyze the graph to find the best-fit line through the points. The equation of the line is \( F_s = k_s x \). Report the value of the slope:

\[ k_s = \text{_______________ N/m} \]

On the same graph, plot the force function \( F = kx \) that you discovered from your analysis of the oscillatory motion (see the Table \( \text{Force deduced from the Motion} \)). Note that since the minus sign in the force law \( F = -kx \) has been omitted in both graphs, these graphs display the magnitude of the force and not the direction. On your printed graph, label the \( F = kx \) line with the name “\( \text{Force deduced from the Motion} \)”. Label the \( F_s = k_s x \) line with the name “\( \text{Force measured with the Spring Scale} \)”. PRINT your graphs.

\[ \text{Compare Your Measured Force Laws} \]

The force function \( F(x) = -kx \) deduced by analyzing the oscillatory motion is

\[ F(x) = - \text{_______________} x \]

The force function \( F_s(x) = -k_s x \) obtained by directly reading the spring scale is

\[ F(x) = - \text{_______________} x \]

The percent difference between the force constants \( k \) and \( k_s \) is \( \text{__________} \) %.
**Part II. Exploring Friction**

The force of kinetic friction acts on any object sliding across a surface. This friction force $f$ obeys the empirical law:

$$f = \mu N,$$

where $N$ is the normal force and $\mu$ is the coefficient of kinetic friction. The coefficient $\mu$ depends on the nature of the two surfaces in contact. For steel on steel, $\mu = 0.5$. For Teflon on steel, $\mu = 0.04$. For rope on wood, $\mu = 0.3$. For shoes on ice, $\mu = 0.05$. For rubber on dry concrete, $\mu = 0.8$. For rubber on wet concrete, $\mu = 0.3$.

**Measuring the Coefficient of Friction**

Your goal is to measure the value of $\mu$ for felt on aluminum. Attach the spring scale to the wooden block covered in felt. Practice pulling the block across the aluminum track at a constant speed. Judge the speed (qualitatively) by covering a constant amount of distance every second of time. Try a speed of about 5 cm per second. Then try a speed of about 10 cm per second. To a very good approximation, the force of friction is independent of speed.

Why pull at constant speed? If the speed of the block is constant, then its acceleration is zero, which means that the net force on the block is zero, which means that the scale reading $F$ (spring force) equals the magnitude of the friction force $f$. In the vertical direction, $N$ equals $mg$. See the force diagram below. This force analysis assumes that the track is level so that there is no component of $mg$ along the track direction. Use the leveler again to make sure your track is level.

---

![Force Diagram](image)

---

Record the value of the friction force $f$ for five different values of the normal force $N$. There is an assortment of brass weights that can be used to load the block in order to change the value of $N = mg$. Note that $m =$ mass of wooden block + mass of brass weights.

<table>
<thead>
<tr>
<th>$m$ (kg)</th>
<th>$N$ (Newtons)</th>
<th>$f$ (Newtons)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Use the program *Graphical Analysis* to graph \( f \) versus \( N \) (\( f \) on y axis, \( N \) on x axis). Find the best-fit line through your five data points. If you compare the mathematical equation of a line \( y = mx + b \) with the physical equation of friction, written in the form \( f = \mu N + 0 \), then it follows that the coefficient of friction \( \mu \) is equal to the slope (\( m \)) and the y-intercept (\( b \)) should be zero. Indeed, \( \mu = f/N \) is the rise over the run of the line. PRINT your graph showing the best-fit line and the value of the slope. Report your measured value of the coefficient of friction:

\[
\mu = \underline{\hspace{2cm}} \quad \text{(felt on aluminum)}.
\]

**Part III. Experimental Design: Lowering a Weight at g/30**

Your goal is to lower a “heavy” weight (100 gram brass cylinder) over a pulley so that the weight falls to the ground with an acceleration that is \( 1/30 \) that of free fall. Here is the schematic of the mechanical system:

![Diagram of mechanical system](image)

**The Theory**

Draw a free body diagram for the hanging mass \( m_H \). Set up Newton’s equation of motion \( F_{\text{net}} = ma \) for this body. Plug the numerical values of \( m_H \) and \( a \) (as required by the *design specs*) into Newton’s equation and solve for the value of the tension \( T \) in the string.

\[
T = \underline{\hspace{2cm}} \quad \text{N}.
\]
Treat the Block-plus-Load system as one body of mass \( m \equiv m_B + m_L \). Draw a free body diagram for the mass \( m \). Set up Newton’s equation \( F_{net} = ma \) for \( m \). Plug the numerical values of \( T \), \( \mu \), and \( a \) into Newton’s equation and solve for the value of \( m \). Then compute the mass of the load using the relation \( m = m_B + m_L \).

\[
m = \underline{\phantom{000}} \text{kg}.
\]

\[
m_L = \underline{\phantom{000}} \text{kg}.
\]

Have your instructor check your theoretical work before you move on to the experimental stage of the design project.

**The Experiment**

Add the amount of load \( m_L \) to the block as predicted by your theory. Distribute this load mass (brass weights) evenly over the top of the block. Release the system from rest and measure the acceleration.

Report your measured value of \( a \). Include the uncertainty in \( a \). Describe how you made the measurement and estimated the uncertainty. Provide all data and/or graphs that you used to find \( a \).

\[
a (\text{measured}) = \underline{\phantom{000}} \pm \underline{\phantom{000}} \text{m/s}^2.
\]

Compare your value of \( a(\text{measured}) \) with the design goal of \( a = g/30 \). What is the percent difference? Write down the range of experimental numbers that characterize your measured acceleration. Does the theoretical number \( g/30 \) fall within this range?