Newton’s Third Law, Momentum, Center of Mass

Part I. Newton’s Third Law

Atomic Springs

When you push against a wall, you feel a force in the opposite direction. The harder you push, the harder the wall pushes back on you. The amazing fact is that the force exerted by the wall on you is exactly equal and opposite to your push. How does the wall “know” to push this way? In general how can inanimate objects, such as walls, floors, and tables, push and pull other objects?

The answer lies in the atomic world. Solids are made of atoms held together in a lattice structure by electric forces (atomic bonds). Physicists view solids as a set of balls (atoms) connected by springs (bonds):

![Atomic Spring Diagram]

The length of one “atomic spring” is about one nanometer \((10^{-9} \text{ m})\). The number of atoms in a typical solid such as a book or a table is about \(1,000,000,000,000,000,000,000,000\).

When you push on a solid, the atomic springs compress, i.e., the electron clouds of the atoms overlap. The harder you push, the more the springs compress, and thus the harder the springs push back on you. What you actually “feel” is the electric repulsion between the atoms in the surface of the solid and the tip of your finger. It is similar to the magnetic repulsion you feel when you push one magnet toward another magnet.

It is important to realize that nothing actually “touches” when two objects come into “contact”, i.e., when you push on something. If two atoms actually touched – one nucleus on top of another nucleus – then nuclear fusion would result thereby creating an atom-bomb explosion! Force fields touch, not ponderable matter. Again, think about two magnets repelling at a distance.

This “atomic springiness” endows matter with an elastic property. When you put a book on the table or push on the wall, the surface of the table and the wall bend slightly (greatly exaggerated in the picture below). Think of putting a bowling ball on a mattress (box springs).
The “bent” table acts like a compressed spring that exerts a force back on the book. The “action” of pushing forward creates a “reaction” of pushing backward. This is the origin of the ubiquitous “Normal Force.”

Not all solids are elastic. Materials such as clay, putty, and dough are inelastic – they do not “spring back” after they have been compressed or stretched.

**Law III: “Action and Reaction”**

Qualitatively speaking, a force is a ‘push’ or a ‘pull’. Rigorously speaking, a force is an interaction between two objects. This interaction obeys a deep law of mechanics:

**Newton’s Third Law:** If one object exerts a force on a second object, then the second object exerts an equal and opposite force on the first. In symbols,

\[ F_{12} = -F_{21} \]

\[ F_{12} \equiv \text{Force on object 1 due to object 2.} \quad F_{21} \equiv \text{Force on object 2 due to object 1.} \]

All forces come in pairs. For every “action”, there is an equal and opposite “reaction”. You cannot have a single force in any situation because the force on one object is always due to some other object. Here are some examples of interactions (pairs of forces):

<table>
<thead>
<tr>
<th>Action</th>
<th>Reaction</th>
</tr>
</thead>
<tbody>
<tr>
<td>You push backward on Floor</td>
<td>Floor pushes forward on You (explains how a person moves)</td>
</tr>
<tr>
<td>Tires push backward on Road</td>
<td>Road pushes forward on Tires (explains how a car moves)</td>
</tr>
<tr>
<td>Rocket pushes back on Gas</td>
<td>Gas pushes forward on Rocket (explains how a jet moves)</td>
</tr>
<tr>
<td>Book pushes down on Table</td>
<td>Table pushes up on Book</td>
</tr>
<tr>
<td>Earth pulls down on Moon</td>
<td>Moon pulls up on Earth</td>
</tr>
<tr>
<td>Magnet pulls left on Nail</td>
<td>Nail pulls right on Magnet</td>
</tr>
</tbody>
</table>

Add three more force pairs to this list:

_________________________ ________________________

_________________________ ________________________

_________________________ ________________________
Experimental Tests of Newton’s Third Law

The Perfect Balance

Since $F$ is equal to $ma$, the equal and opposite nature of the force $F$ implies an equal and opposite nature in the motion $ma$. Any pair of mutually-interacting objects (isolated system) must move toward or away from each other in such a way that their $ma$’s are in perfect balance at all times. The symmetry in the motion $m_1a_1 = -m_2a_2$ is a reflection of the symmetry in the force $F_{12} = -F_{21}$.

Here you will study three different kinds of interaction between a pair of carts: spring force, magnetic force, and contact force. You will observe first hand how the motions of each cart are in “perfect balance”. Verifying this symmetry in the motion is the gold standard test of Newton’s Third Law.

Here is the Force diagram of the interacting carts:

Here is the physics of the Motion:

During the time interval $\Delta t$ during which the carts interact,

$F_{12}$ causes the velocity of $m_1$ to change by an amount $\Delta v_1$.
$F_{21}$ causes the velocity of $m_2$ to change by an amount $\Delta v_2$.

Newton’s Second Law of Motion applied to each cart separately gives

$$F_{12} = m_1\Delta v_1/\Delta t$$
$$F_{21} = m_2\Delta v_2/\Delta t$$

These second-law equations allow you to find the average forces $F_{12}$ and $F_{21}$ during the interaction time $\Delta t$ by observing the change in velocities $\Delta v_1$ and $\Delta v_2$.

Qualitative Experiments on Newton III

Here, you will deduce the values of $F_{12}$ and $F_{21}$ from qualitative estimates of $\Delta v_1$ and $\Delta v_2$. You do not have to quantitatively measure any velocity. Simply observe the motion and note if the velocities look the “same”. If you prefer more accuracy, you can do a quick measurement of distance-over-time using the meter scale and a stopwatch. When observing the motion, here is the “bottom line”:

Look for the “symmetry”, the “perfect balance”, or the “gain = loss” in the motion.
**Experiment 1. Interaction = Spring Force**

**Observe the Motion**

First make sure that the track is level using the leveler. Attach a spring between the carts as shown below. Place cart 1 at the 40-cm mark and cart 2 at the 80-cm mark. The non-magnetic Velcro patches should be facing each other so that the carts stick together when they collide. Release the carts (simultaneously).

Observe the motion. Note the symmetry in the motion, such as where the carts meet. Note that it takes about one second of time from the moment of release to the moment of collision. During this interaction time, the spring force causes the velocity of each cart to change.

Given the following fact about the velocity of cart 1, describe the velocity of cart 2 (fill in the blanks) based on your qualitative observations. Define the positive direction to point to the right.

During the interaction time 1.0 second,

The velocity of cart 1 changes from 0 m/s to 0.30 m/s.

The velocity of cart 2 changes from __________ to __________.

**Find the Action and the Reaction**

Measure the mass of each cart. Record the change in velocity of each cart during the interaction time $\Delta t = 1.0 \, s$.

\[
m_1 = \underline{\hspace{2cm}} \, \text{kg}, \quad m_2 = \underline{\hspace{2cm}} \, \text{kg}.
\]

\[
\Delta v_1 = \underline{\hspace{2cm}} \, \text{m/s}, \quad \Delta v_2 = \underline{\hspace{2cm}} \, \text{m/s}.
\]

Calculate the average values of the spring force acting on each cart during the interaction.

\[
F_{12} = \underline{\hspace{2cm}} \, \text{N}, \quad F_{21} = \underline{\hspace{2cm}} \, \text{N}.
\]
**Experiment 2. Interaction = Magnetic Force**

**Observe the Motion**

Place cart 1 at the left end of the track. Place cart 2 in the middle of the track. Make sure that the cart magnets are facing each other so that the carts repel each other. Give cart 1 an initial velocity of “about 30 cm/s” toward cart 2.

Observe the motion. Note the symmetry. Note that the interaction time is about 0.5 seconds. This is the time interval during which the carts are close enough (< 10 cm) to feel the magnetic force and repel each other. During this time, the magnetic force causes the velocity of each cart to change: cart 1 slows down, cart 2 speeds up. Before and after this time of interaction (when the carts are separated by more than 10 cm) the net force on each cart is approximately zero and thus the carts move at constant velocity. Friction does cause some slowing.

Given the following fact about the velocity of cart 1, describe the velocity of cart 2 (fill in the blanks) based on your qualitative observations.

During the interaction time 0.50 seconds,
The velocity of cart 1 changes from 0.30 m/s to 0 m/s.
The velocity of cart 2 changes from ____________ to ____________.

**Find the Action and Reaction**

Measure the mass of each cart. Record the change in velocity of each cart during the interaction time \( \Delta t = 0.50 \text{ s} \).

\[
\begin{align*}
m_1 &= \text{__________ kg} & m_2 &= \text{__________ kg} \\
\Delta v_1 &= \text{__________ m/s} & \Delta v_2 &= \text{__________ m/s}
\end{align*}
\]

Calculate the average values of the magnetic force acting on each cart during the interaction.

\[
\begin{align*}
F_{12} &= \text{__________ N} & F_{21} &= \text{__________ N}
\end{align*}
\]
Experiment 3. Interaction = Contact Force (Electron Cloud Repulsion)

Observe the Motion

Place cart 1 at the left end of the track. Place cart 2 at the center. Make sure that the non-magnetic Velcro patches (hooks – loops) are facing each other so that the carts stick together after the collision. Give cart 1 an initial velocity of “about 30 cm/s” toward cart 2.

Observe the motion. The interaction time is about 0.05 seconds. During this time of “contact”, the contact force causes the velocity of each cart to change: cart 1 slows down, cart 2 speeds up. Remember that nothing actually touches. During the contact – the “smacking” of the carts – the distance between the repelling electron clouds in 1 and 2 is a few nanometers!

Given the following fact about the velocity of cart 1, describe the velocity of cart 2 by filling in the blanks.

During the interaction time 0.05 seconds,
The velocity of cart 1 changes from 0.30 m/s to 0.15 m/s.
The velocity of cart 2 changes from _________ to _________.

Find the Action and the Reaction

Measure the mass of each cart. Record the change in velocity of each cart during the interaction time \( \Delta t = 0.05 \text{ s} \).

\[
\begin{align*}
m_1 &= \underline{ \quad \quad } \text{ kg} , \\
m_2 &= \underline{ \quad \quad } \text{ kg} , \\
\Delta v_1 &= \underline{ \quad \quad } \text{ m/s} , \\
\Delta v_2 &= \underline{ \quad \quad } \text{ m/s} .
\end{align*}
\]

Calculate the average values of the contact force (“repelling electron clouds”) acting on each cart during the interaction.

\[
\begin{align*}
F_{12} &= \underline{ \quad \quad } \text{ N} , \\
F_{21} &= \underline{ \quad \quad } \text{ N} .
\end{align*}
\]
Quantitative Experiments on Newton III

Here you will use two motion sensors interfaced to two computers. Sensor 1 will measure the velocity and acceleration of cart 1. Sensor 2 will measure the velocity and acceleration of cart 2. Place the sensors on the table off each end of the track as far as possible. Elevate each sensor (place on wooden block) so that they can see each cart.

Open Logger Pro file Changing Velocity 1. Change the setting on Sensor 2 so that it sees objects that are moving to the right as moving in the positive direction: On the computer that Sensor 2 is connected to, click on “Setup” (on menu bar), then “Sensors”, then “Details”, then “Reverse Direction.
The graph window displays two graphs: Distance and Velocity. Change this to the following two graphs: Velocity and Acceleration. Change the time scale to go from 0.0 to 3.0 s. Change the velocity scale to go from −0.50 to 0.50 m/s. Change the acceleration scale to go from −2.0 to 2.0 m/s². You are now ready to record the motion.

Experiment 1.  Magnetic Interaction (Equal Mass Carts)

Give cart 1 an initial velocity toward cart 2 (initially at rest):

![Diagram of tracks and carts](image)

Velocity Curves

Record the motion of each cart. Your velocity curves (v as a function of time) should have the following overall appearance:

*Velocity of Cart 1*

*Velocity of Cart 2*
Each velocity curve is divided into three distinct regions:

1. **Constant Velocity** before the interaction (initial).
2. **Changing Velocity** during the interaction (time interval $\Delta t$).
3. **Constant Velocity** after the interaction (final).

Note how the velocity curves display the three motion parameters ($\Delta t$, $\Delta v_1$, $\Delta v_2$) that determine the force of interaction. If your $v$ graphs do not look like the above $v$ graphs, then you may have to adjust the location and/or tilt of your motion sensors. You can also try different initial positions of cart 2 along the track. Consult with your instructor if your velocity curves do not clearly show the values of $\Delta t$, $\Delta v_1$, $\Delta v_2$.

PRINT your graphs. Write the numerical values of the initial velocity, the final velocity, and the change in velocity of each cart directly on the corresponding parts of your velocity curves.

**Momentum Balance**

From your measured values of the velocity change, calculate the momentum change of each cart during the interaction.

*Momentum Lost by Cart 1:*

$$- m_1 \Delta v_1 = - (\text{kg})(\text{m/s}) = \rule{15cm}{0.5pt} \text{kgm/s}.$$  

*Momentum Gained by Cart 2:*

$$m_2 \Delta v_2 = (\text{kg})(\text{m/s}) = \rule{15cm}{0.5pt} \text{kgm/s}.$$  

Does the momentum in your system seem to be “conserved”? Explain.

**Force Curves**

During the interaction, the magnetic force on each cart varies with time. This time-dependent force function starts at zero (when the carts are far apart), then rapidly rises (as the carts get closer), then reaches a maximum (when the carts almost touch), and then falls back to zero (as the carts move farther apart). The force on cart 2 as a *function of time* has the following overall shape:
Note that the force curve is characterized by two parameters (height and width): the maximum force \( F_{\text{max}} \) and the interaction time \( \Delta t \). The force \( F_{\text{max}} \) occurs when the carts are closest to each other with their magnets repelling most strongly. As always, \( \Delta t \) is the time interval during which one cart feels the force due to the other cart. The average value of the time-dependent force (over the time \( \Delta t \)) is roughly \( \frac{1}{2} F_{\text{max}} \).

What is the exact shape of this interaction curve? Force curves and power curves are very important in science and engineering. Since \( F \) is equal to \( ma \), you can deduce the \( F \) curve from your measured \( a \) curve: simply multiply \( a \) by \( m \). Since \( m \) is a constant, the \( ma \) curve has the same shape as the \( a \) curve. Only the vertical scale changes. For example, suppose that the mass of the cart is 0.50 kg. If the \( a \) curve has height 2.4 m/s\(^2\) and width 0.73 s, then the \( F \) curve has height 1.2 N and width 0.73 s.

Look at the \( a_1 \) curve and \( a_2 \) curve recorded by your motion sensors. The \( F_{12} \) curve is equal to the \( a_1 \) curve multiplied by \( m_1 \). The \( F_{21} \) curve is equal to the \( a_2 \) curve multiplied by \( m_2 \). Show the calculation of the heights of your \( F \) curves (\( F_{\text{max}} = ma_{\text{max}} \)) directly on your printed \( a \) curves. Sketch your force curves below. Write the numerical values of the height (maximum force) and width (interaction time) on each curve.

**Conclusion:**
Experiment 2. Magnetic Interaction (Unequal Masses)

Make cart 2 more massive than cart 1 by placing the black metal bar weight on top of cart 2. Give cart 1 an initial velocity toward cart 2 (initially at rest):

\[ \text{v} \]

\[ \text{1 magnet} \quad \text{magnet} \quad \text{2} \]

Record the motion of each cart. Make sure that your graphs clearly display the before, the during, and the after parts of the motion. You may have to adjust the motion sensors and/or the initial location of cart 2 on the track.

PRINT your graphs. Write the numerical values of the initial velocity, the final velocity, and the change in velocity of each cart directly on the corresponding parts of your velocity curves.

Momentum Balance

From your measured values of the velocity change, calculate the momentum change of each mass during the interaction.

Momentum Lost by Cart 1:

\[-m_1 \Delta v_1 = - (\text{kg})(\text{m/s}) = \text{___________ kgm/s}.

Momentum Gained by Cart 2:

\[m_2 \Delta v_2 = (\text{kg})(\text{m/s}) = \text{___________ kgm/s}.

Is the velocity lost by 1 equal to the velocity gained by 2? Does the momentum of your system seem to be “conserved”? Explain.
**Force Curves**

Find the forces acting on $m_1$ and $m_2$ as *functions of time*. The $F_{12}$ curve is equal to the $a_1$ curve multiplied by $m_1$. The $F_{21}$ curve is equal to the $a_2$ curve multiplied by $m_2$. Show the calculation of the maximum forces $m_1a_{1\text{max}}$ and $m_2a_{2\text{max}}$ directly on your printed $a_1(t)$ and $a_2(t)$ graphs. Be careful with the mass-times-acceleration multiplication: $m_1$ is *not* equal to $m_2$.

Sketch your measured force curves below. Write the numerical values of the height (*maximum force*) and width (*interaction time*) on each curve.

---

**Conclusion:**
Part II. Conservation of Momentum

What does “action = reaction” have to do with “momentum = constant”? Everything! From your experimental investigations into Newton’s Third Law, you discovered that the “amount of mv” lost by one object is equal to the “amount of mv” gained by the other object. This means that the “total amount of mv” in the system cannot be created or destroyed – it is conserved!

The Physics behind Momentum Conservation is simple. Just combine Newton’s Laws II and III:

Write the Force law \( F_{12} = -F_{21} \) as the Motion law \( m_1\Delta v_1/\Delta t = -m_2\Delta v_2/\Delta t \).

Cancel the mutual time \( \Delta t \) of interaction, and there you have it, one of the most celebrated principles in science:

\[
\Delta(m_1v_1) = -\Delta(m_2v_2)
\]

Momentum Gained by Object 1 = Momentum Lost by Object 2

This “gain = loss” relation means that the Total Momentum \( m_1v_1 + m_2v_2 \) of the isolated system cannot change:

\[
\Delta(m_1v_1 + m_2v_2) = 0 \quad \text{implies} \quad m_1v_1 + m_2v_2 = \text{constant}.
\]

Our derivation of “\( m_1v_1 + m_2v_2 = \text{constant} \)” is valid for any kind of isolated system – a system for which the net external force is zero. The system could be two billiard balls, two colliding galaxies, or the system of \( 10^{28} \) air molecules in our lab room. Physicists write the general Principle of Momentum Conservation simply as \( P = \text{constant} \).

A Set of Collision Experiments

You will study different collisions between your two carts. During each collision (interaction), the carts exert strong internal forces on each other. Remember that these internal forces cannot change the total momentum of the system (since \( F_{12} + F_{12} = 0 \)). If you neglect the external force of friction (since it is much weaker than the internal force), then you can conclude that the total momentum before the collision must equal the total momentum after the collision:

\[
[ m_1v_1 + m_2v_2 ]_{\text{initial}} = [ m_1v_1 + m_2v_2 ]_{\text{final}}.
\]

\[
[ p_1 + p_2 ]_{\text{initial}} = [ p_1 + p_2 ]_{\text{final}}
\]

This simple equation tells you a whole lot about the before and after motion in the collision process – without having to know anything about the complicated internal collision forces (springs, magnets, electrons, velcro, etc.) that are causing the motion!
**Experiment 1. Fly Apart Reaction: Magnetic Explosion**

Here is the initial state of the system:

![Diagram of two carts touching]

Push the equal-mass carts together with your hands until the magnets on each cart “touch”. In this initial state, the magnetic repulsion is ready to “explode” the system. Release both carts at the same time and observe them “fly apart”. Based solely on your observation (no quantitative measurements), sketch three momentum curves (momentum as a function of time): $p_1$ of cart 1, $p_2$ of cart 2, and $p_1+p_2$ of the system. Label your three curves.

![Graph showing momentum over time for multiple carts]

**Experiment 2. Velocity Exchange Reaction**

Sketch the three momentum curves for the following collision between equal-mass carts. Here is the initial state:

![Diagram of carts with magnets]

![Graph showing momentum over time for cart collision]

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Experiment 3. Big Mass Colliding with Small Mass

Here is the initial motion:

\[ \text{Masses: } m_1 = \underline{\phantom{000}} \text{ kg.} \quad m_2 = \underline{\phantom{000}} \text{ kg.} \]

Run the experiment. Make the initial \( v \) somewhere between 30 and 50 cm/s. Use two motion sensors (located off each end of the track) to record the velocities. Write your initial and final velocities on the pictures shown below. Fill in the Momentum Table.

\[
\begin{array}{c|c}
\text{Initial Velocities} & \text{Final Velocities} \\
\hline
\text{1} & \text{2} & \text{1} & \text{2} \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
\text{Initial Velocities} & \text{Final Velocities} \\
\hline
\underline{\phantom{000}} \text{ m/s} & 0 \text{ m/s} & \underline{\phantom{000}} \text{ m/s} & \underline{\phantom{000}} \text{ m/s} \\
\end{array}
\]

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Momentum Table} & \text{Initial} & \text{Final} \\
\hline
p_1 & \underline{\phantom{000}} & \underline{\phantom{000}} \\
p_2 & \underline{\phantom{000}} & \underline{\phantom{000}} \\
p_1 + p_2 & \underline{\phantom{000}} & \underline{\phantom{000}} \\
\hline
\end{array}
\]

How well (percent difference) do your results provide an experimental proof of the Law of Momentum Conservation?
**Experiment 4. Totally Inelastic (Sticking) Collision**

Here is the initial state of the system of two “equal-mass” carts:

```
1  2
velcro hooks  velcro loops  Rest
```

Masses: \( m_1 = \underline{\phantom{000}} \text{ kg} \), \( m_2 = \underline{\phantom{000}} \text{ kg} \).

Run the experiment. Make the initial \( v \) somewhere between 30 and 50 cm/s. Use one motion sensor (located off the left end of the track) to record the velocity. Write your initial and final velocities on the pictures below. Fill in the **Momentum Table**.

<table>
<thead>
<tr>
<th>Initial Velocity</th>
<th>Final Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 )</td>
<td>( 2 )</td>
</tr>
<tr>
<td>( \underline{\phantom{000}} \text{ m/s} )</td>
<td>( 0 \text{ m/s} )</td>
</tr>
<tr>
<td>( \underline{\phantom{000}} \text{ m/s} )</td>
<td>( \underline{\phantom{000}} \text{ m/s} )</td>
</tr>
</tbody>
</table>

**Momentum Table**

<table>
<thead>
<tr>
<th></th>
<th>Initial</th>
<th>Final</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_1 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( p_2 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( p_1 + p_2 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

How well (percent difference) do your results provide an experimental proof of the **Law of Momentum Conservation**?
Part III. The Center of Mass: A Case Study

What is the Center of Mass?

Place cart 1 (loaded with the bar weight) on the track so that the center of cart 1 is at \( x_1 = 20 \ cm \). The x axis is defined by the meter scale fixed to the track. Place cart 2 (no added weight) so that its center is at \( x_2 = 80 \ cm \). The masses of the carts are \( m_1 = 1.0 \ kg \) and \( m_2 = 0.5 \ kg \).

The center of mass \( x_{cm} \) of this system of two carts is

\[
x_{cm} = \frac{1.0 \ kg \cdot 20 \ cm + 0.5 \ kg \cdot 80 \ cm}{1.0 \ kg + 0.5 \ kg} = 40 \ cm.
\]

\[
\text{center of mass}
\]

\[
|\hspace{2cm} 20 \ cm \hspace{2cm} 40 \ cm \hspace{2cm} 80 \ cm |
\]

Discovering a Conservation Law

Now attach a spring between the carts. Make sure the Velcro patches are facing each other so that the carts stick together when they collide. Pull the carts apart so that their centers are at 20 cm and 80 cm as before. This is the initial state.

Release the carts at the same time. Observe the motion. The carts accelerate toward each other, collide, and come to rest (final state). Record the final resting positions \( (x_1 \) and \( x_2) \) of the center of cart 1 and the center of cart 2 in the table below. Calculate the center of mass \( x_{cm} \) of the system of two carts in this final state. Show your calculation:

<table>
<thead>
<tr>
<th></th>
<th>( m_1 )</th>
<th>( m_2 )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_{cm} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>initial</td>
<td>1.0 kg</td>
<td>0.5 kg</td>
<td>20 cm</td>
<td>80 cm</td>
<td>40 cm</td>
</tr>
<tr>
<td>final</td>
<td>1.0 kg</td>
<td>0.5 kg</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Does your experimental data prove that the center of mass is conserved? i.e. Does \( x_{cm} \) (initial) equal \( x_{cm} \) (final) within experimental error (10%)?
The Center of Mass is the Place to Be

Why is \( x_{\text{cm}} \) conserved? The center of mass of any system moves according to Newton’s equation of motion for a system of particles: \( F_{\text{ext}} = M \ddot{x}_{\text{cm}} \). In your experiment, the system consists of two carts. The external force is \( F_{\text{ext}} = \text{friction} \approx 0 \). The internal force is \( F_{\text{int}} = \text{spring} \). Given \( F_{\text{ext}} = 0 \), the motion of the center of mass is trivial: \( \ddot{x}_{\text{cm}} = 0 \). What could be simpler. During the entire time that the carts move (accelerate) toward each other in some complicated way, the center of mass \( x_{\text{cm}} \) does not move at all! Amidst the “chaos” of accelerating parts, namely \( x_1(t) \) and \( x_2(t) \), there is one point of “calmness”: the center of mass \( x_{\text{cm}} \).

The conservation relation “\( x_{\text{cm}} = \text{constant} \)” allows you to know the position of each cart at any time during the motion without having to know anything about the time-dependent internal (spring) force that is causing the motion!

The initial position (20 cm) of cart 1 (1.0 kg) and the initial position (80 cm) of cart 2 (0.5 kg) are marked on the position diagram shown below. The position of cart 1 is marked at three later times of the motion. Mark the position of cart 2 at these same three future times.

\[\begin{align*}
\text{t}_0 & \quad 1.0 \text{ kg} & \quad 0.5 \text{ kg} \\
20 \text{ cm} & \quad 30 \text{ cm} & \quad 80 \text{ cm} \\
\text{t}_1 & \quad 25 \text{ cm} \\
\text{t}_2 & \quad 30 \text{ cm} \\
\text{t}_3 & \quad 35 \text{ cm}
\end{align*}\]

Einstein, Center of Mass, \( E = mc^2 \)

The two Conservation Laws that you discovered in this laboratory, “\( m_1v_1 + m_2v_2 = \text{constant} \)” and “\( m_1x_1 + m_2x_2 = \text{constant} \)” are two of the most important consequences of Newton’s Third Law “\( F_{12} + F_{21} = 0 \)”.

Albert Einstein realized the deep significance of the constancy of \( x_{\text{cm}} \). In his 1906 paper, “The Principle of Conservation of Motion of the Center of Mass and the Inertia of Energy”, Einstein used the relation \( x_{\text{cm}} \text{ (initial)} = x_{\text{cm}} \text{ (final)} \) to derive the most famous equation in science: \( E = mc^2 \). Einstein used this center of mass theorem to analyze the motion of a train car on a frictionless track when a light bulb inside the car was turned on.