Rotational Motion

Mechanics is the study of motion. All motion in the Universe can be classified into three basic types: translation, rotation, and oscillation. So far, you have studied translational motion – the motion of a particle that moves along a straight or curved path. In rotational motion, a system of particles turns about an axis. In oscillatory motion, an object moves back and forth.

Rotational motion is everywhere. When you push a door, it rotates. When you pedal a bike, the wheel rotates. When you start an engine, many parts rotate. Electrons rotate in an atom. Galaxies rotate in the universe. The earth translates and rotates. Ice skaters translate and rotate. Oxygen molecules translate, rotate, and oscillate.

Part I. Angular Position, Velocity, and Acceleration

In previous labs, you used a motion sensor to measure the translational (linear) variables x, v, a. Here, you will use a rotary motion sensor to measure the rotational (angular) variables \( \theta \), \( \omega \), \( \alpha \).

Definition of Motion Variables

<table>
<thead>
<tr>
<th>Variable</th>
<th>Translational Motion</th>
<th>Rotational Motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Position</td>
<td>x</td>
<td>( \theta )</td>
</tr>
<tr>
<td>Velocity</td>
<td>( v \equiv \frac{dx}{dt} )</td>
<td>( \omega \equiv \frac{d\theta}{dt} )</td>
</tr>
<tr>
<td>Acceleration</td>
<td>( a \equiv \frac{dv}{dt} )</td>
<td>( \alpha \equiv \frac{d\omega}{dt} )</td>
</tr>
</tbody>
</table>

A. Angular Position \( \theta \)

The angle \( \theta \) is the natural quantity to represent the position of a solid object rotating about a fixed axis. This position variable has units of radians and is defined as the ratio of two lengths involving a circle: arc length \( s \) over the radius \( r \). Note that \( \theta = 1 \) radian is defined by \( s = r \).

\[ \theta \equiv \frac{s}{r} \]

The rotary motion sensor measures \( \theta \) as a function of time \( t \). Start Logger Pro and open the file Torque. Activate the sensor by clicking on Collect or hitting the Enter key. Gently rotate the rod with your hand and watch the angular position graph recorded by the sensor. Rotate the rod
clockwise and counter-clockwise, move it fast, move it slow, hold it at rest, let it spin freely and note how each of these different rotational motions are displayed on the $\theta(t)$ graph – the angular "worldline" of the rod.

Click Collect again. Starting from rest, rotate the rod with your hand. Stop the rod when the rod has made one revolution (one complete circle). Report the change in the angular position of the rod recorded by your rotary motion sensor.

$$\theta_f - \theta_i = \phantom{0} - \phantom{0} = \phantom{0} \text{ radians.}$$

What is the value of this angular displacement in degrees? Carefully show your calculation that converts radians into degrees.

$$\theta_f - \theta_i = \phantom{0} \text{ degrees.}$$

B. Angular Velocity $\omega$

Angular velocity is the rate of change of angular position with respect to time: $\omega = \frac{d\theta}{dt}$.

Graphically speaking, $\omega$ is the slope of the $\theta(t)$ curve.

Give the rod a gentle push with your hand and then let go. With no force (torque) acting on the rod, except for negligible friction, the angular speed $\omega$ will remain constant. Both you and the computer will monitor the motion. Click on Collect. While the rotary motion sensor is collecting $\theta(t)$ data, use your stopwatch to measure the time $\Delta t$ it takes for the rod to complete five revolutions. Find $\omega$ two different ways:

1. **Stopwatch Measurement.**

$$\frac{\Delta \theta}{\Delta t} = \phantom{0} \text{ rad} / \phantom{0} \text{ s} = \phantom{0} \text{ rad/s}.$$  

2. **Motion Sensor Data.** Your $\theta(t)$ graph should be straight sloping line (since $\omega$ is constant). Select the good (straight line) data region of your $\theta(t)$ graph. Find the best-fit line through the data. Record the slope of the line.

$$\text{Slope of } \theta(t) \text{ Line} = \phantom{0} \text{ rad/s}.$$  

Percent Difference between “$\Delta \theta/\Delta t$” and “Slope of $\theta(t)$” is $\phantom{0} \%$. 

C. Angular Acceleration $\alpha$

Angular acceleration is the rate of change of angular velocity with respect to time: $\alpha \equiv \frac{d\omega}{dt}$. Graphically speaking, $\alpha$ is the slope of the $\omega(t)$ curve.

Note that the spinning axle of the rotary motion sensor coincides with the rotation axis of the system. This axle goes through the center of a pulley located directly above the sensor. This pulley has three “steps”, each with a different radius. Wrap the thread around the middle step (step 2) of this three-step pulley. Feed the thread over the plastic “wagon wheel” clamped to the sensor. This wheel merely changes the direction of the thread (tension force) from horizontal to vertical. When we say “pulley” in this lab, we are referring to the three-step pulley around the rotation axis and not to the plastic wheel.

Attach a 50-gram mass to the end of the thread. Release the mass. Click on Collect. Note that your $\theta(t)$ graph is curved! More precisely, $\theta(t)$ is a parabola. Remember: a curving position-time graph is the kinematic trademark of an accelerating system. Your $\omega(t)$ graph should be a straight sloping line because the acceleration of the system – which is due to the constant force of gravity – is constant. Select the good data region of your $\omega(t)$ graph. Find the best-fit line and report the slope:

$$\alpha = \text{slope of } \omega(t) \text{ line} = \frac{\text{ }}{\text{rad/s}^2}.$$ 

D. Connection between Translational and Rotational Variables

From the definition of angle, $\theta \equiv \frac{s}{r}$, it follows that $s = r\theta$. This means that if the pulley of radius $r$ rotates (turns) through the angle $\theta$, then the hanging mass will translate (fall) over the linear distance $x = r\theta$. As the curved length $s$ of thread unwinds from the pulley, it becomes the straight length $x$: 

$$x = s.$$
Given the position relation \( x = r\theta \), it follows that the velocity relation is \( \frac{dx}{dt} = rd\theta/dt \) and the acceleration relation is \( \frac{d^2x}{dt^2} = rd^2\theta/dt^2 \), or more simply \( v = r\omega \) and \( a = r\alpha \), respectively. Note that the radial length \( r \) is the “bridge” between the “linear world” \((x,v,a)\) and the “angular world” \((\theta,\omega,\alpha)\): \( x = r\theta \), \( v = r\omega \), \( a = r\alpha \).

**Experimental Test of \( x = r\theta \) and \( a = r\alpha \)**

Start the system at rest with the hanging mass at the very most top point. The instant you let go of the rod, start the stopwatch and measure the time \( t \) it takes for the rod to rotate through an angle \( \theta \) of five revolutions. During this time, the mass falls a linear distance \( x \). Measure \( x \) with the meter stick. Report your measured values of \( t \), \( x \), and \( \theta \):

During the time \( t = \underline{\hspace{2cm}} \) s, the

*Linear* displacement of the *translating* mass is \( x = \underline{\hspace{2cm}} \) m.

*Angular* displacement of the *rotating* rod is \( \theta = 5 \text{ rev} = \underline{\hspace{2cm}} \text{ rad} \).

Use the kinematic relations for uniformly accelerated motion, \( x = \frac{1}{2}at^2 \) and \( \theta = \frac{1}{2}\alpha t^2 \), to calculate \( a \) and \( \alpha \) from your measured values of \( t \), \( x \), \( \theta \). Show your calculations:

*Linear* acceleration of the *translating* mass: \( a = \underline{\hspace{2cm}} \frac{m}{s^2} \).

*Angular* acceleration of the *rotating* rod: \( \alpha = \underline{\hspace{2cm}} \frac{\text{rad}}{s^2} \).

Measure the diameter \( 2r \) of the pulley (middle step) with the vernier caliper. Report the value of the radius of the pulley:

\( r = \underline{\hspace{2cm}} \) m.

Are your measured values of the *linear* variables \((x, a)\) related to the *angular* variables \((\theta, \alpha)\) via the connection equations \( x = r\theta \) and \( a = r\alpha \)?

% difference between \( x = \underline{\hspace{2cm}} \) m and \( r\theta = \underline{\hspace{2cm}} \) m is \( \underline{\hspace{2cm}} \)%.

% difference between \( a = \underline{\hspace{2cm}} \) m and \( r\alpha = \underline{\hspace{2cm}} \) m is \( \underline{\hspace{2cm}} \)%.
Part II. Torque and Angular Acceleration

Force causes linear acceleration. Torque causes angular acceleration.

A. Basic Principles

Why is the handle on a door located far away from the hinge? Why is it easier to loosen a nut using a long wrench? Why are long wheel-base cars more stable than short wheel-base cars? Why do tightrope walkers use long poles?

The “ability” of a force to rotate an object about an axis depends on two variables:

1. The magnitude of the force $F$.
2. The distance $r$ between the axis of rotation and the point where the force is applied.

Try opening a door by applying the same force $F$ at different distances $r$ from the hinge. You will quickly realize that the resulting motion of the door – the acceleration $\alpha$ – depends on $F$ and $r$. It turns out that the “turning ability” of a force is simply the product of $F$ and $r$. The technical name for this turning ability is torque.

Definition: The torque $\tau$ exerted by a force $F$ that is applied at a point $r$ relative to the origin is the cross product of $r$ and $F$:

$$\tau \equiv r \times F.$$ 

It makes qualitative sense that a large (small) torque $\tau$ will cause a large (small) acceleration $\alpha$, but what is the quantitative relation between $\tau$ and $\alpha$? The deep law of physics is that $\tau$ is directly proportional to $\alpha$:

$$\tau \sim \alpha.$$ 

What could be simpler – a linear relation between cause ($\tau$) and effect ($\alpha$). If you double $\tau$, then $\alpha$ will double. $\tau$ does not go like $\alpha^2$!

The Physics behind the $\tau \sim \alpha$ relation is as follows. Consider the thread that is pulling on the pulley, thereby causing the pulley and rod to rotate. The pulley and rod consist of many mass elements (particles). The picture below shows the pulley divided into its constituent mass elements. Focus on the element of mass $m$ on the rim of the pulley where the external force $F$ (tension in thread) is applied:
This particle moves (as do all particles) according to Newton’s Law of Motion: \( F = ma \). If you multiply \( F = ma \) by \( r \) you get \( rF = mra \), or \( rF = mr^2(a/r) \). Now \( rF \) is equal to the torque \( \tau \) and \( a/r \) is equal to the angular acceleration \( \alpha \). Thus for one mass element, “\( F = m\alpha \)” can be written as “\( \tau = mr^2\alpha \)”.

If you now sum over all the mass elements (1+2+3+...) in the pulley and rod, the sum over Newton’s law \( \tau = mr^2\alpha \) for each element becomes:

\[
\tau_1 + \tau_2 + \tau_3 + ... = m_1 r_1^2 \alpha + m_2 r_2^2 \alpha + m_3 r_3^2 \alpha + ...
\]

or simply

\[
\tau = I \alpha .
\]

In this rotational equation, \( \tau \) is the net external torque on the system, \( \alpha \) is the angular acceleration of the system, and \( I \) is the Rotational Inertia of the system:

\[
I \equiv m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ...
\]

\( \tau = I\alpha \) is the fundamental dynamical equation of rotational motion. Remember that it is essentially Newton’s second law \( F = ma \) multiplied by \( r \). You now see exactly why \( \tau \sim \alpha \)!

Furthermore, this derivation of \( \tau = I\alpha \) tells you the precise value of the proportionality constant between \( \tau \) and \( \alpha \): \( I = \sum mr^2 \), i.e. the sum of \( mr^2 \) over all the mass elements in the system.

\[
\tau = I\alpha \quad is \ the \ rotational \ analogue \ of \quad F = ma.
\]

B. Apply the Torque \( \tau \), Measure the Acceleration \( \alpha \), Prove the Law \( \tau = I\alpha \)

You will apply a torque \( \tau \) to the pulley by pulling on the thread. This torque will cause the pulley – and everything attached to the pulley (axle, rod, brass weights) – to undergo rotational motion with angular acceleration \( \alpha \). You will measure \( \tau \) with the spring scale (and vernier caliper). You will measure \( \alpha \) with the rotary motion sensor. You will discover for yourself the deep relation between \( \tau \) and \( \alpha \).
Apply $\tau$. Measure $\alpha$.

Make sure that the spring scale reads ‘zero’ when it is in the horizontal position and nothing is attached to the small hook. Remove the hanging mass from the thread and attach the small hook of the spring scale to the thread. Pull the scale so that the scale reads a constant force $F$ during your entire pull. Make sure to keep the thread and scale horizontal while you are pulling. Lower the long vertical rod on which the rotary apparatus is mounted to a level that makes it easier to read the spring scale and pull in the horizontal direction. Maintaining a constant force may be a bit tricky. It is okay if the force fluctuates slightly – sometimes you pull too hard, sometimes too soft, but the average will be just right! Note that the scale reading gives the tension in the thread.

The torque due to the tension force $F$ acting at the distance $r$ from the rotation axis is $rF$.

While you are pulling, record the motion using the rotary motion sensor. Find the angular acceleration $\alpha$ of the system by finding the slope of the best-fit line through the velocity $\omega(t)$ data. Be sure to select the good data region of your $\omega(t)$ graph – the region where $\omega(t)$ is a straight line, i.e. where the slope is constant.

Report your results in the $\tau$ and $\alpha$ Table. Do this measurement for three values of the force ($F = 0.1N, 0.3N, 0.5N$) on the middle-step pulley and three values of the force ($F = 0.1N, 0.3N, 0.5N$) on the largest-step pulley. Measure the radius $r$ of each step using the vernier caliper.

$\tau$ and $\alpha$ Table

<table>
<thead>
<tr>
<th>$r$ (m)</th>
<th>$F$ (N)</th>
<th>$\tau = rF$ (Nm)</th>
<th>$\alpha$ (rad/s$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td></td>
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<td></td>
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<tr>
<td>0.5</td>
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</tr>
<tr>
<td>0.1</td>
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<tr>
<td>0.3</td>
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<td></td>
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<tr>
<td>0.5</td>
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</tbody>
</table>

Analysis of $\tau$-$\alpha$ Data

1. Use Graphical Analysis to plot $\tau$ (y axis) versus $\alpha$ (x axis). Include the point (0, 0) on your graph. Delete the connecting lines between the data points. Simply by looking at the overall trend in the seven data points ($\alpha, \tau$) that appear on your graph, can you claim to have experimentally “proved” the fundamental theoretical relation of rotational dynamics:

"$\tau$ is directly proportional to $\alpha$".

Explain.
2. Note that there are two points on your graph that are essentially “on top of each other”. These two points correspond to the same angular acceleration, but to completely different applied forces. Write the values of $\alpha$ and $F$ at these two points:

\[
F_1 = \underline{} \text{ N} \quad \alpha_1 = \underline{} \text{ rad/s}^2
\]

\[
F_2 = \underline{} \text{ N} \quad \alpha_2 = \underline{} \text{ rad/s}^2
\]

Carefully explain how these different forces ($F_1 \neq F_2$) can cause the same motion ($\alpha_1 \approx \alpha_2$).

3. The physics equation $\tau = I\alpha$, written as “$\tau = I\alpha + 0$”, has the same mathematical form as the generic equation of a line “$y = mx + b$”. If your $\tau$-$\alpha$ graph does not look linear, then consult with your instructor. Find the best-fit line through your seven data points ($\alpha$, $\tau$). PRINT your graph showing the best-fit line along with the values of the slope and the intercept. Report the equation of your line here:

\[
\tau = \underline{} \alpha + \underline{}
\]

What is the rotational inertia of your system (rod + brass weights + pulley + axle)?

\[
I = \underline{} \text{ kg m}^2
\]
Part III. Rotational Inertia

Rotational inertia I has two meanings: \( I = \tau/\alpha \) and \( I = \sum \text{mr}^2 \). In Part II, you focused on the dynamic meaning of I in terms of force and motion \((\tau/\alpha)\). Here you will investigate the geometric meaning of I in terms of mass and distance \((\text{mr}^2)\).

The “Mass Distribution” Meaning of I: \( I = \sum \text{mr}^2 \)

Translational Inertia is \( M = m_1 + m_2 + m_3 + \ldots \)

Rotational inertia is \( I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots \)

In the I-equation, \( r_i \) is the perpendicular distance of the mass element \( m_i \) from the axis of rotation. Note that the rotational inertia \( I \) of a rotating body depends on how the mass is distributed about the axis of rotation. An object whose mass is located far away from the axis of rotation has a larger \( I \) – and thus is harder to turn (accelerate) – than an object with the same mass that is distributed close to the axis of rotation.

Together, the “amount of mass” \( M = \sum m \) and the “distribution of mass” \( I = \sum \text{mr}^2 \) tell you everything about the linear and angular response of a system to external forces and torques:

- \( M \) measures an object’s resistance to changing its linear velocity: \( F\Delta t = M\Delta v \).
- \( I \) measures an object’s resistance to changing its angular velocity: \( \tau\Delta t = I\Delta \omega \).

How do you calculate \( \sum \text{mr}^2 \)?

For point masses, you simply add the \( \text{mr}^2 \) for each point. For solid objects (continuous distribution of mass), the sum over mass elements \( \sum \text{mr}^2 \) can be performed using calculus (integration), i.e. \( \sum \text{mr}^2 \rightarrow \int r^2 \text{dm} \). Objects that are far from the rotation axis (relative to their size) can be treated as point masses. Consider the brass weight on the end of the rod in your apparatus. The mass of the weight is \( M \) and its center is located a distance \( D \) from the rotation axis. Since the distance \( D \) is much greater than the width of the weight, each atom in the weight and each \( m \) in \( \sum \text{mr}^2 \) is approximately the same distance \( (r = D) \) away from the axis. This makes the sum easy to calculate: \( \sum \text{mr}^2 = \sum mD^2 = D^2 \sum m = D^2 M \).
To calculate the rotational inertia of your rotating system, the following rotating objects and formulas will help. These objects represent good models of the various parts of your system.

- **Weight far from axis**
  \[ \sum mr^2 = MD^2 \]

- **Solid Cylinder (Disk or Axle) about axis**
  \[ \sum mr^2 = \int dm \cdot r^2 = \frac{1}{2} MR^2 \]

- **Thin Rod about perpendicular axis through center**
  \[ \sum mr^2 = \int dm \cdot r^2 = \frac{1}{12} ML^2 \]

**The Experiment: Measure “mr² of the Parts” to find “I of the Whole”**

Your goal is to find the value of \( I = \sum mr^2 \) that characterizes the rotational inertia of your system. Everything (each mass element) that is rotating in the system contributes to the rotational inertia. As the following schematic shows, your system consists of five rotating parts:

Since \( I \) is the sum over all the moving “mr²” in your system, the total value of \( I \) is equal to the sum over the \( I \) values of the separate moving parts:

\[
(\sum mr^2)_{\text{System}} = (\sum mr^2)_{\text{Weights}} + (\sum mr^2)_{\text{Rod}} + (\sum mr^2)_{\text{Pulley}} + (\sum mr^2)_{\text{Axle}}.
\]

\[
I = I_{\text{Weights}} + I_{\text{Rod}} + I_{\text{Pulley}} + I_{\text{Axle}}.
\]
Find the value of each $I_{\text{part}}$ by filling in the *Rotational Inertia Table* that appears on the next page. You do not have to separate the parts from your rotational apparatus. The dissected parts of an identical apparatus are available for you to study on the table in the back of the room where the mass scale is located.

*Where is the “I Scale”?*

When an object is placed on the mass scale, or “M-scale”, the scale reads the translational inertia ($M = \sum m$) of the object. Unfortunately, there does not exist an “I-scale” that directly reads the rotational inertia ($I = \sum mr^2$)! You will have to find $I_{\text{part}}$ by using the M-scale to measure the amount of mass and a ruler or caliper to measure the shape of the mass, i.e. the relevant distances from the rotation axis.

Write your measured values of mass and distance directly on the pictures in the *Rotational Inertia Table*.

**Conclusion**

Write your measured values of $I$ that you obtained using two different methods (Parts II and III):

- *Torque/Acceleration Method* ($I = \tau/\alpha$): $I = \underline{\quad} \text{kg m}^2$.

- *Mass \cdot Distance}^2 Method* ($I = \sum mr^2$): $I = \underline{\quad} \text{kg m}^2$.

Compare your two values of $I$. What is the percent difference? Which method of measuring $I$ do you think is more accurate? Explain.

Before proceeding to the next part of the lab, ask your instructor to check your values of $I$. 


## Rotational Inertia Table

<table>
<thead>
<tr>
<th>Rotating Part</th>
<th>Mass-Distance Values</th>
<th>Calculation of $I_{\text{part}}$</th>
<th>Value of $I_{\text{part}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brass Weights</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rod</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pulley</td>
<td></td>
<td></td>
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<tr>
<td>Axle</td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\sum I_{\text{part}} = \_\_\_\_\_\_ kg m^2
\]
Part IV. Design Problem – Fine Tuning I so that \( t \) is Ten

Your goal is to use your *rotating machine* to lower a 50-gram mass, starting from rest, through a distance of 1.5 *meters* in a time of 10.0 \( \pm \) 0.5 *seconds*. Use the second-step pulley. The only machine parameter that you can adjust is the placement of the brass weights along the rod.

Here is a schematic of the rotating machine:

![Schematic of rotating machine](image)

The Theory

The following set of questions guide you through the theoretical analysis to find the value of \( D \equiv \text{distance between the rotation axis and the center of mass of each brass weight} \). Show all your work in answering each question.

1. Calculate the acceleration \( a \) of the hanging mass.

   \[
a = \underline{\quad} \text{m/s}^2.
   \]

2. Calculate the tension \( T \) in the thread. First draw a free body diagram for the hanging mass and then set up \( F = ma \) for the translational motion of the mass. *Note: \( T \neq mg \).*

   \[
   T = \underline{\quad} \text{N}.
   \]
3. Calculate the torque $\tau$ on the pulley due to the tension $T$.

$$\tau = \underline{\text{___________}} \ Nm.$$ 

4. Calculate the angular acceleration $\alpha$ of the rotating system.

$$\alpha = \underline{\text{___________}} \ \text{rad/s}^2.$$ 

5. Calculate the rotational inertia $I$ of the system.

$$I = \underline{\text{___________}} \ \text{kg m}^2.$$ 

6. Calculate the contribution to $I$ from the two brass weights.

*Hint:* $I = I_{\text{Weights}} + I_{\text{Rod}} + I_{\text{Pulley}} + I_{\text{Axle}}$.

$$I_{\text{Weights}} = \underline{\text{___________}} \ \text{kg m}^2.$$ 

7. Calculate $D$.

$$D = \underline{\text{___________}} \ \text{m}.$$ 

**The Experiment**

Place the brass weights on the rod as predicted by your theoretical value of $D$. Start the 50-gram mass at a height of 1.5 meter above the floor. Release the mass from rest. Use a stopwatch to time how long it takes for the mass to touch the floor. Repeat three times. List your three values of time. Report the average value.

$$t = \underline{\text{___________}} \ \text{seconds}.$$ 

Does your measured value of $t$ fall within the 5% window (10.0 ± 0.5 seconds) allowed by the *Design Specs*? If not, consult with another team or your instructor.