Team: ________________

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Orbital Mechanics Laboratory

Studying the forces of nature – the interactions between matter – is the primary quest of physics. In this "celestial experiment", you will measure the force responsible for whirling the planets along their pristine orbital paths You will come face to face with the celebrated Kepler Problem: *Given an ellipse, find the force*. You will experience first hand Newton's elegant geometric solution – one of the "top ten" calculations in the history of science. In essence, you will prove a fundamental law of nature – the law of gravity – using string, tacks, and a ruler, along with a little help from Galileo, Kepler, and Newton.

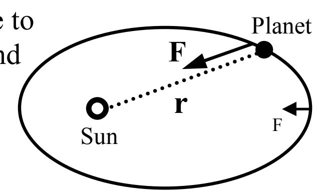
This experiment is a team effort. Each of you will observe a different part (tiny arc) of the whole orbit (large ellipse). By analyzing the shape of your arc, you will compute the force causing the planet to move along the arc. The value of the force depends on the distance between your arc and the sun. By pooling together the f*orce* versus *distance* data from everyone in the class, we can discover the *law of force*.

Part I. Theory

Force and Geometry

In mechanics, there exists a deep connection between force and geometry. A constant gravitational force causes a body to move in a parabolic path – projectile motion. A constant centripetal force causes a body to move in a circle – uniform circular motion.

In 1609, Johannes Kepler reported that the planet Mars moves in an elliptical orbit. We now know that the conic sections – ellipses, parabolas, and hyperbolas – are the geometric shapes associated with celestial motion. What kind of force causes a planet to move in an ellipse? The force cannot be constant. When a planet is close to (far from) the sun, the force is big (small). The line connecting the sun and the planet is the radial coordinate r. The law which specifies how a force function F(r) depends on the radial distance r is called the *force law*. For example, the force law associated with a spiral orbit is $F(r) \sim 1/r^3$. There are two general problems in orbital mechanics:

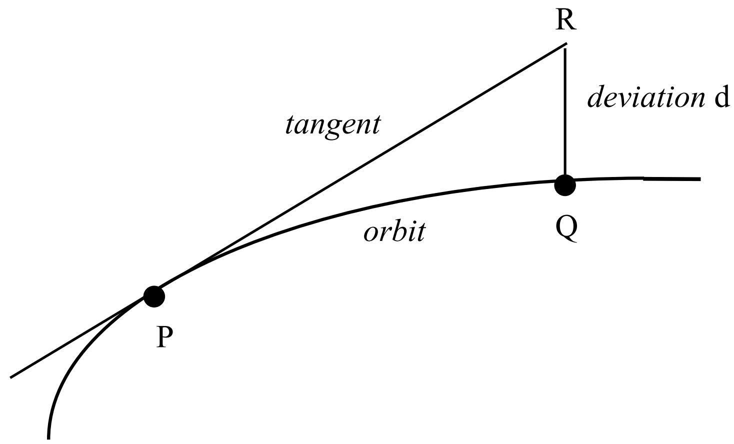


Direct Problem: Given the orbit shape, find the force law.
Inverse Problem: Given the force law, find the orbit shape.

Isaac Newton solved these problems in his *Mathematical Principles of Natural Philosophy*, published in 1687. This book, referred to as *The Principia*, is one of the greatest science books ever written. In this lab, you will solve the direct problem of orbital mechanics using "Newton's Recipe".

Orbital Motion = Inertial Motion + Falling Motion

Consider a planet orbiting the sun. In a certain time interval, the planet moves from point P to point Q along the orbit.



If no force acted on the planet, then the planet would move along the straight line PR with the constant velocity it had at P. Because of the force directed toward the sun, the planet deflects away from the straight line and moves around the sun along the curved path PQ. The deviation d = QR of the curved orbit from the straight tangent provides a measure of the force.

What is the mathematical relation between the force F and the deviation d? Newton's genius was to realize that for small deviations ($d \to 0$), the *variable* force can be treated as a *constant* force in both magnitude and direction. A constant force implies a constant acceleration. A constant acceleration implies projectile motion.
In short,

$$\lim_{Q \to P} \textit{Orbital Motion} = \textit{Projectile Motion} .$$

Thus, tiny orbital arcs are equivalent to parabolic paths. In effect, by using calculus to look at infinitesimal portions of the orbit, Newton replaced the complex ellipses of Kepler with the simple parabolas of Galileo.

Given that the force is constant, the relation between force F, deviation d, and time t is

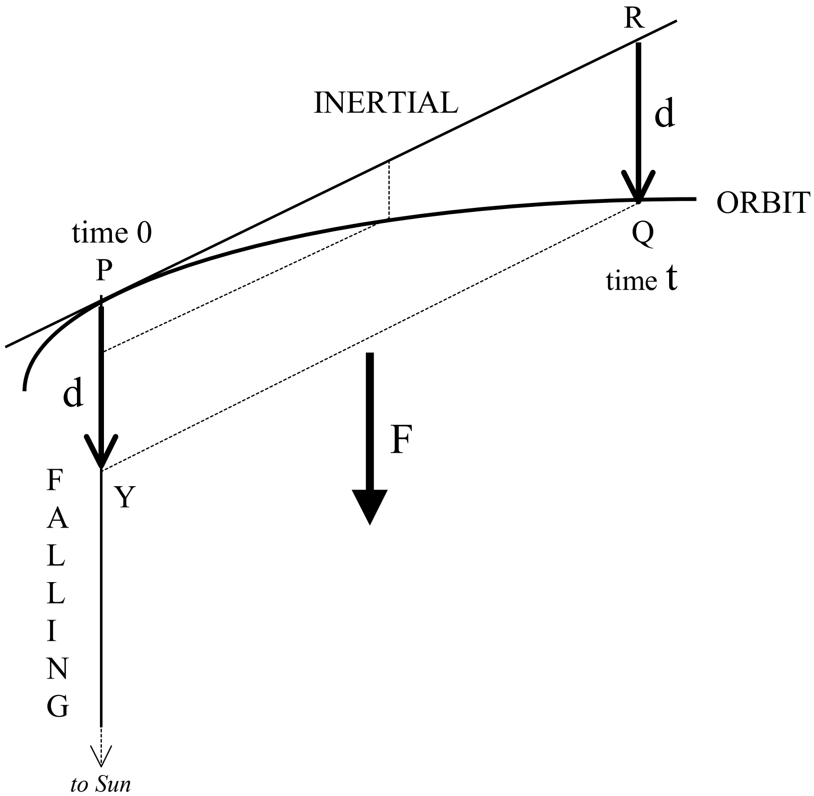
$$F = 2m\, d / t^2 \; . \qquad (1)$$

The mass m of the planet is a constant factor. The important content of the force formula in Eq. (1) is the proportional relation:

$$F \propto d / t^2 \; . \qquad (2)$$

Equation (2) says two things: (1) For a given time interval t during which the planet moves along an infinitesimal orbital arc, the force is proportional to d = the deviation of the arc from the tangent. (2) For a given deviation d, the force is inversely proportional to t^2 = the square of the time it takes for the deviation to occur.

When you look at the basic force formula $F \propto d/t^2$ in Eq. (2), the following image should appear in your *minds eye*:



Just like projectile motion, orbital motion (for short times) can be viewed as a combination of two imaginary motions: inertial motion due to the constant velocity alone (no force) and falling motion due to the constant force alone (no velocity). The planet continually falls beneath the tangent line (inertial path). The direction of the fall (deviation) coincides with the direction of the constant force (toward the sun).

In time t, the planet undergoes the displacement PR due to the velocity alone and the displacement PY (or equivalently RQ) due to the force alone. The actual displacement PQ in time t is the combination (vector sum) of these virtual displacements: **PQ** = **PR** + **RQ**.

This geometric relation is equivalent to the kinematic relation $\mathbf{r}-\mathbf{r}_0 = \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2$.

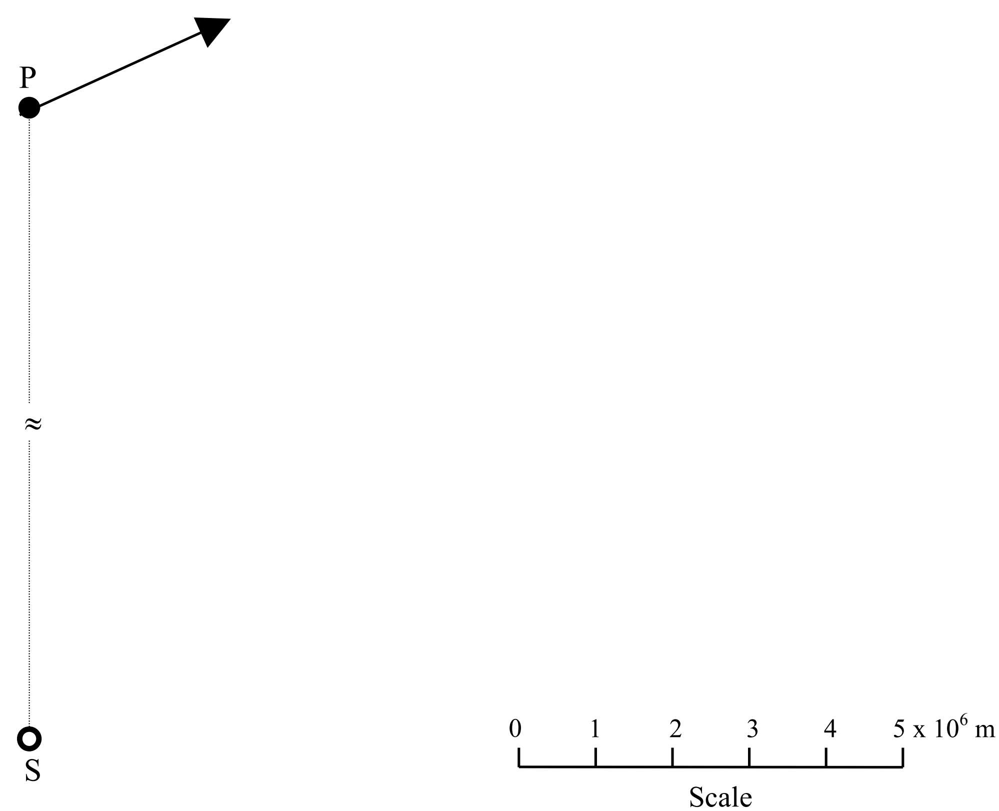
Exercise 1.

Derive Eq. (1). Carefully justify each step in your derivation. *Hint*: Imagine "dropping" the planet from rest at a point P in the orbit. The sun is directly below P. Derive the formula that gives the distance d the planet (mass m) falls in the time t due to the approximately constant force F.

Exercise 2.

A planet of mass 2.0 x 10^{24} kg orbits a star. At point P in the orbit, the planet is moving with a speed 4.0 km/s in the direction shown in the picture. The star (S) is directly below the point P. The force exerted by the star on the planet at P is 9.0 x 10^{23} N. During a time interval of one hour, the planet moves from point P to point Q. During this time, the force does not change appreciably in magnitude and direction. Calculate the location of the future point Q and show it in the picture. *Draw all relevant displacements to scale.*

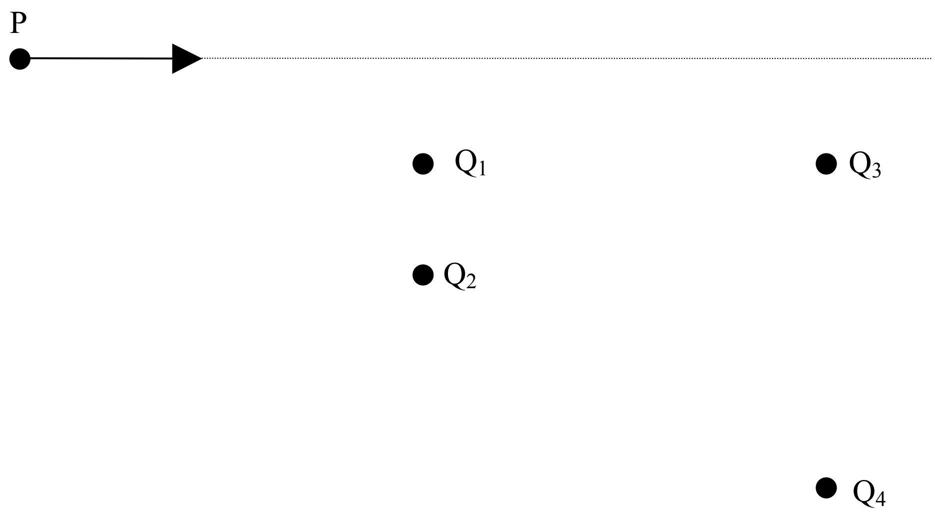
Hint: First find the location of the planet if there were no force and then find the distance it falls below this point due to the force.



Exercise 3.

A body moves from point P to point Q_1 along an arc PQ_1 due to the force F_1. A different force, F_2 , F_3 , and F_4 , causes the same body to move along a different arc, PQ_2 , PQ_3 , and PQ_4 , respectively. At the initial point P, the velocity of the body points to the right and has the same magnitude for each of the four different motions. Each of the four forces is constant and points downward. Given $F_1 = 400$ N , find F_2 , F_3 , and F_4 .

Hint: You do not need to measure any distances. Merely note the *relative* values from the picture, such as $d_2 = 2d_1$, $t_3 = 2t_1$, etc.

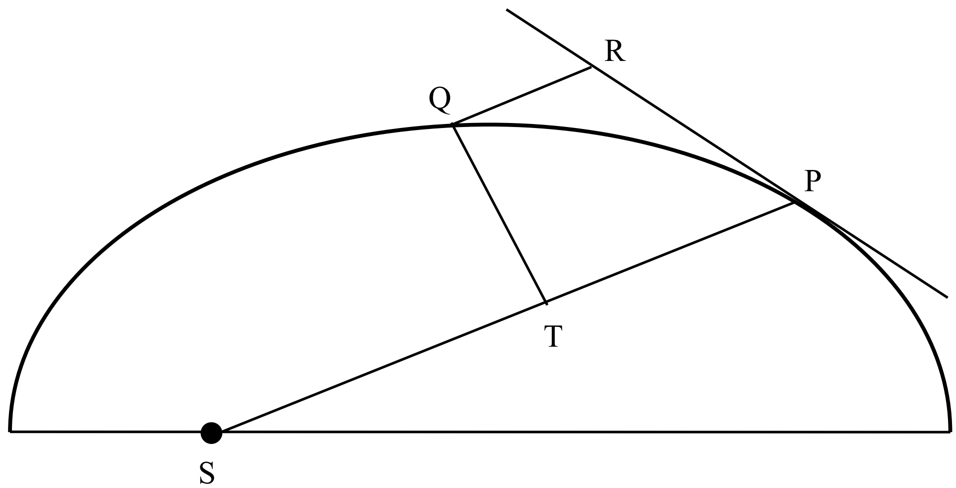


Newton's Force Formula

Newton's version of the force formula $F \propto d/t^2$ in Eq. (2) is

$$F \;\propto\; \frac{QR}{(SP \times QT)^2} \;. \qquad (3)$$

Equation (3) is the most important formula in *The Principia*. The three geometric quantities (QR, SP, QT) that appear in the formula characterize the shape of the orbit around point P. These *shape parameters* are defined in the following *Newton Diagram*:



Radius SP ≡ distance between the sun S and the planet P.
Deviation QR ≡ deviation of the curved orbit PQ from the straight tangent PR.
Height QT ≡ height of the "time triangle" SPQ.

QR is ***parallel*** to SP. QT is ***perpendicular*** to SP.

Newton's formula in Eq. (3) is a *purely geometric* measure of force. The two kinematic quantities in $F \propto d/t^2$, namely d and t, are replaced by the two geometric quantities, QR and SPxQT, in Eq. (3). The geometric measure of time, $t \propto$ SPxQT, stems from Kepler's Law of Areas:

The time it takes a planet to move from P to Q around the sun S is proportional to the area of the sector SPQ swept out by the line connecting the sun and planet.

In the limit $Q \rightarrow P$, the arc PQ is equal to the chord PQ and the area swept out is equal to the area of the triangle SPQ. The triangle SPQ has base SP and height QT. Thus, the time-area relation is *time* $\propto$ ½ (*base* x *height*) or $t \propto (SP \times QT)$.

Note that Newton's force formula in Eq. (3) measures the *relative* value of the force at a particular point P in the orbit. The exact force is proportional to this relative force. The dimension of Newton's force measure $QR/(SPxQT)^2$ is $1/(\text{length})^3$. Indeed, Newton measured force in units of m^{-3} ! He never knew about *"Newton"* units. This is perfectly fine because, like Newton, we are only interested in comparing the force values at different points in the orbit.

So, like Sir Isaac, we will measure force in the purely geometric units of m^{-3} .

In what follows, the symbol F will denote Newton's force measure $QR/(SPxQT)^2$.

In summary, Newton's geometric diagram and force formula allows one to solve the following fundamental problem of orbital mechanics:

> *Given*: The orbital path of a planet and the location of the sun.
> *Find*: The force acting on the planet.

Newton's recipe is very general. It is not confined to celestial motion. It works for any kind of motion due to a centripetal force – a force directed toward a fixed point (force center). Newton's recipe only requires two ingredients: the shape of the path and the center of the force.

Remember, Newton's fundamental force formula, $F \propto QR/(SPxQT)^2$, is the geometric version of $F \propto d/t^2$ and stems from four basic principles:

1. Parabolic Approximation: F is constant for small t and d.
2. Kinematic Relation: $d = ½at^2$ (*Galileo*).
3. Dynamic Law: F = ma (*Newton*).
4. Area Law: $t \propto$ area (*Kepler*).

<u>Exercise 4</u>.

Given the orbital arc PQ and the location of the Sun S shown below, calculate the numerical value of the force $F = QR/(SPxQT)^2$. Measure all distances in *meters*. For example, the distance SP is about 0.12 *m* .

Part II. The Experiment – Constructing the Orbit

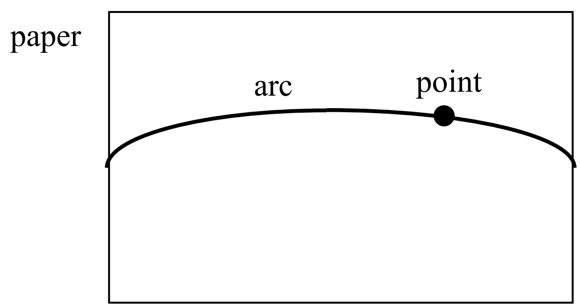
Each team (two people) is responsible for constructing their own orbital arc – a small piece of the whole orbit. The instructor will assign each team one particular region of the elliptical orbit (numbered 1 – 9) where you will draw your arc. The instructor will ask you (and the other people at your table) to come to the "orbit board" sometime during the beginning of the lab.

Steps for Drawing Your Orbital Arc

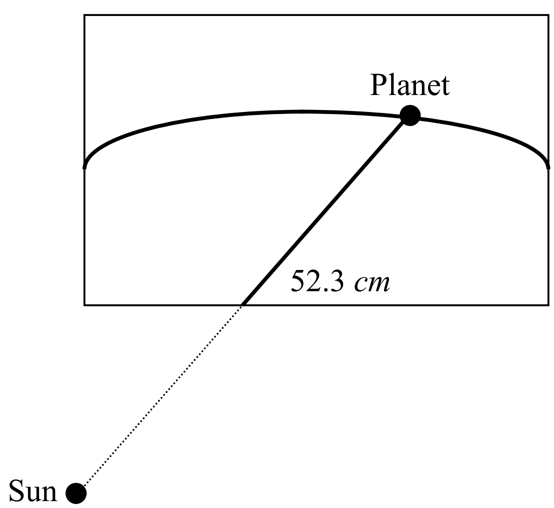
1. Tape a piece of paper to the orbit board in your assigned region.
   The arc should appear in the *top half* of the paper. This leaves space under the arc to draw lines.

2. Draw an arc on the paper <u>using the pen held taut against the string</u>.
   Do not merely trace over the curve under your paper. Let the pen and string do all the work!

3. Mark an orbit point on your arc.
   Choose your point to be 2-3 inches away from the right side of your paper. This will allow you space on the left side to locate future points and drop deviations.

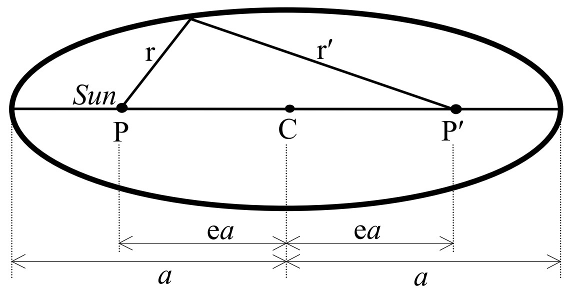


4. Draw the radial line on your paper. Measure the radial distance r between the Sun and your point (planet) and write the value (such as r = 52.3 cm) on your paper.



The Ellipse: A Geometry Lesson

Definitions. An *ellipse* is a curve along which the sum of the distances to two fixed points is constant. The fixed points, P and P′, are called the *foci* of the ellipse. The equation of the ellipse is r + r′ = 2*a* (constant).
For planetary ellipses, *the Sun is at one focus*.

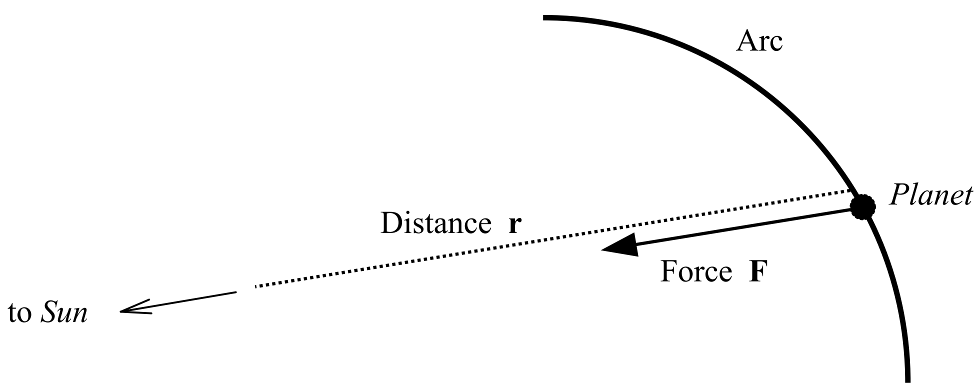


Eccentricity. The long dimension of the ellipse is called the *major axis* and has length 2*a*. The short dimension is called the minor axis. The distance between the center C and the focus P is e*a*, where e is called the *eccentricity* of the ellipse. A circle is a special case of an ellipse in which the foci coincide at the center (P→C←P′). Note that e = 0 for a circle. In general, the value of e varies between 0 and 1 and measures the degree of departure from circularity. For the ellipse pictured above, e = 0.6. The eccentricity of Earth's orbit is e = 0.0167. For Halley's Comet, e = 0.967.

Construction. An ellipse is readily constructed. Two pins (P and P′) are tacked to a board. The ends of a string (length 2*a*) are attached to the two pins. A pencil traces a curve as it is held taut against the string. This construction insures that all points on the curve obey the relation r+r′=2*a*.

Part III. The Analysis – Measuring the Force at Your Orbit Point

The mission of the class – imagine you work for NASA – is to analyze the shape of the orbit and find the law of force. Each team in the class observes and analyzes a different part (arc) of the whole orbit (ellipse).



Quest**:** Find the value of the force **F** acting on the planet at your team's particular value of the sun-planet distance **r**.

Solution: Follow **Newton's "Recipe"** on next page.
Report your results on the **ORBITAL DATA WORKSHEET**.

Newton's Recipe

1. *The inertial path*.
   Draw the tangent line[1] at the point P on the orbit where the force is to be calculated.

2. *The future point*.
   Locate any future point Q on the orbit that is close[2] to the initial point P.

3. *The deviation line*.
   Draw the line segment from Q to R (point on tangent) such that QR is *parallel* to SP.

4. *The time line*.
   Draw the segment from Q to T (point on SP) such that QT is *perpendicular* to SP.

5. *The shape parameters*.
   Measure the values of QR and QT (SP was measured while drawing your arc).

6. *The force measure*.
   Calculate the force measure $QR / (SP \times QT)^2$.

7. *The force sequence*.
   Repeat steps 2 to 6 for several future points[3] to obtain a sequence of forces.

8. *The calculus limit*.
   Find the exact force at P by taking the limit $Q \to P$ of the force sequence[4].

[1] IT IS VITAL TO DRAW AN ACCURATE TANGENT LINE ! All other line segments hinge on your tangent line. See your instructor for hints on drawing "perfect" tangents.

[2] What does "close" mean? Remember that Newton's formula is exact only for calculus quantities – infinitesimal deviations and times. As a general rule, the deviation QR should be less than 10% of the radius SP. To get started, CHOOSE YOUR FIRST Q SO THAT QR IS BETWEEN 3CM AND 5CM.

[3] How should you choose your other future points? CHOOSE AT LEAST FOUR OTHER FUTURE POINTS Q SO THAT THEIR DEVIATIONS QR FROM THE TANGENT ARE ABOUT 4CM, 3CM, 2CM, 1CM. Measuring distances less than 1cm with a ruler involve larger relative errors. You may want to briefly venture into the ultra-infinitesimal (sub-centimeter) world – where calculus rocks – to see how consistent your results are!

[4] How do you find the calculus limit? The five values of your force corresponding to the five deviations (approximately 5cm, 4cm, 3cm, 2cm, 1cm) should be roughly constant or slowly approaching a well-defined limiting value. You may have to extrapolate a bit. Try graphing F vs QR and see where the curve hits the F axis. If your values vary widely or appear random, then you are not in the calculus regime, or more likely, you did not correctly draw lines and/or measure distances. If you are uncertain about the limiting value of the force at your point P, consult your instructor.

Part IV. Discovering the Force Law – Proving the Law of Gravity

Professional physicists are in the business of finding force laws. What is the law that describes the *electromagnetic force* between electrons or the *strong force* between quarks? It is now time for *you* to discover the law describing the *gravitational force* – the force that causes celestial orbs to travel through space along elliptical paths.

So far, each team has found the value of the force F at *one* particular point r (sun-planet distance) in the orbit. Based on all the team's measured values of F at a *few* points, can you find a continuous function F(r) that gives the values of F at *all* points r in the orbit? The mathematical rule that specifies how F depends on r is the Law of Force.

The Class Quest: Find the Force Function F(r).

In mathematics, there exist all kinds of functions. Some are simple such as r^2, sin r, and exp r. Some are complicated such as $r^{3/5}\sin(r\ln r)/\arctan(r^7+r^{10})$. Only a few describe nature. In searching for the fundamental laws of nature, physicists are guided by the Rule of Simplicity: *the mathematical description of physical phenomena should be simple*. A few simple laws describe a myriad of complex phenomena. The "beauty of nature" lies in its inherent simplicity!

Finding F(r)

1. *Create a Data Table*. You will find a data table on the blackboard with two columns: r and F. Each team must enter their pair of numbers (r , F) into the table. After all the data is in, copy the table onto your *Orbital Data Worksheet*.

2. *Make a Graph*. Use *Graphical Analysis* to plot F on the y-axis and r on the x-axis. Delete the connecting line segments between the points.

3. *Fit with a Power Law*. Fit your (r , F) data points with a "power law" function of the form $F = Ar^B$. Why choose a power law? Because *Newton's Universal Law of Gravity* – the theoretical law that you are trying to prove in this Orb Lab experiment – is a power law! PRINT your graph showing the best-fit curve. Report your force law on the *Orbital Data Worksheet*.

4. *Compare with Newton*. According to Sir Isaac, the gravitational force between any two masses, m and M, separated by a distance r is $F = GMm/r^2$, where the gravitational constant is $G = 6.67\times10^{-11}\ Nm^2/kg^2$. Note how F depends on r. Physicists say *"The force of gravity is an inverse-square law of force"* : $F \sim 1/r^2$. If you double r, then F decreases by ¼. Note that the law of gravity has the form $F = \text{constant}\ r^{-2}$, i.e. a *power law* where the exponent is exactly equal to −2 . Compare your power law with Newton's exact law on the *Orbital Data Worksheet*.

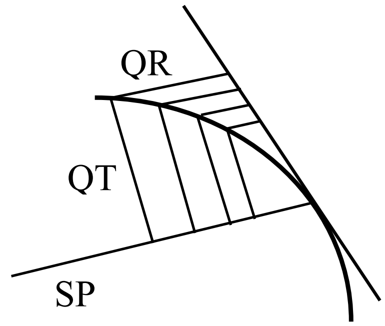
Hand in your **"Force Law Portfolio"** – your elliptical arc, F-versus-r graph, and orbital data worksheet.

ORBITAL DATA WORKSHEET

Sun-Planet Distance (SP)

r = *m*

Shape Parameters



QR ∥ SP
QT ⊥ SP

Force on Planet at One Point in Orbit

Deviation QR (*m*)	**Time Measure** QT (*m*)	**Force Measure** $F = QR/(SP \times QT)^2$ (m^{-3})

$\lim_{QR \to 0} F =$ ______

Bottom Line:

When the planet is at r = __________ , the force is F = ________________ .

Class Data: Variation of F with r

r (*m*)	F (m^{-3})

r (*m*)	F (m^{-3})

r (*m*)	F (m^{-3})

The Force Law

Your Measured Force Law:

$$F(r) \; = $$

Best-fit force function describing ellipse experiment (class data)

Newton's Law of Gravity:

$$F(r) \; = \; \frac{\text{constant}}{r^{2}}$$

Theoretical force function describing all gravitational phenomena

Compare theory (Newton's Law) with experiment (Your Law). What is the percent difference between Newton's power-law exponent and your power-law exponent?