Oscillations

Oscillatory motion is motion that repeats itself. An object oscillates if it moves back and forth along a fixed path between two extreme positions. Oscillations are everywhere in the world around you. Examples include the vibration of a guitar string, speaker cone or tuning fork; the swinging of a pendulum, playground swing or grandfather clock; the oscillating air in an organ pipe, the alternating current in an electric circuit, the rotation of a neutron star (pulsar), neutrino oscillations (subatomic particle), the up and down motion of a piston in an engine, the up and down motion of an electron in an antenna, the oscillations of the electron cloud in an atom, the vibration of atoms in a solid (heat), the vibration of molecules in air (sound), the vibration of electric and magnetic fields in space (light).

The Force

The dynamical trademark of all oscillatory motion is that the net force causing the motion is a restoring force. If the oscillator is displaced away from equilibrium in any direction, then the net force acts so as to restore the system back to equilibrium.

Definition: A simple harmonic oscillator is an oscillating system whose restoring force is a linear force — a force $F$ that is proportional to the displacement $x$:

$$F = -kx.$$

The force constant $k$ determines the strength of the force and measures the “springiness” or elasticity of the system. Slinkys, long pendula, and loose drumheads have small $k$ values. Car coils, short pendula, and tight drumheads have large $k$ values.

The Motion

The motion of all mechanical systems is determined by Newton’s law of motion $F = ma$. For a simple harmonic oscillator system, the equation of motion is $-kx = ma$, or

$$x = -(m/k) a.$$

This linear relation between position $x$ and acceleration $a$ is the kinematic trademark of simple harmonic motion. Note that the proportionality constant $(m/k)$ between $x$ and $a$ is the ratio of the inertia $(m)$ of the system to the elasticity $(k)$ of the system. Also note that $\sqrt{m/k}$ has units of time: $[kg/(N/m)]^{1/2} = \text{seconds}.$

The quantity $\sqrt{m/k}$ sets the scale of time for all simple harmonic motion.

How does the position $x(t)$ of a simple harmonic oscillator depend on time $t$? The motion equation $x = -(m/k)a$ determines the motion function $x(t)$. If you know how to take the derivative of a sine function, then you can show that...
The solution of \( x = -(m/k) \frac{d^2x}{dt^2} \) is \( x(t) = A \sin(2\pi t/T) \) where \( T = 2\pi \sqrt{m/k} \).

The amplitude \( A \) determines the “strength” (maximum displacement) of the oscillation. The period \( T \) is the time for one oscillation. Note that the sine function \( x(t) = A \sin(2\pi t/T) \) repeats itself whenever \( t \) increases by the amount \( T \): \( x(t+T) = x(t) \). This is the precise definition of “period” and explains why the ratio of system parameters \( m/k \) is the vital temporal quantity.

**Summary: The Essence of Simple Harmonic Motion**

IF the force acting on mass \( m \) has the linear (simple) form: \( F = -kx \)

THEN the motion of the mass will be sinusoidal (harmonic): \( x = A \sin(2\pi t/T) \)

AND the period \( T \) depends solely on the mass/force ratio: \( T = 2\pi \sqrt{m/k} \)

**Part I. Pendulum**

*Pendulum Basics*

In lecture, you derived the well known formula for the period of a simple pendulum:

\[ T = 2\pi \sqrt{L/g} \]

The derivation consists of applying Newton’s Law \( F = ma \) to a mass \( m \) suspended from a lightweight (massless) string of length \( L \) in a gravitational field of strength \( g \).

The dynamical essence of the derivation is the following. If you pull a pendulum a small horizontal distance \( x \) away from its vertical resting (equilibrium) position, then the force \( F \) restoring the pendulum back to equilibrium is proportional to the displacement \( x \): \( F = -(mg/L) x \).

If you compare the pendulum force function \( F = -(mg/L) x \) with the general force function \( F = -kx \), then you reach an important conclusion:

*The force constant that characterizes the pendulum system of mass \( m \) and length \( L \) is \( k = mg/L \).*

Once you have the force constant, it is easy to get all the motion properties! To get the period of the pendulum, simply substitute the pendulum constant \( k = mg/L \) into the general period formula \( T = 2\pi \sqrt{m/k} \). When you do this, note how the mass \( m \) cancels! You are left with \( T = 2\pi \sqrt{L/g} \). This is how the famous pendulum formula is derived.
A. Making a Grandfather Clock

Galileo made the first quantitative observations of pendulum motion by timing the swing of a chandelier hanging from a cathedral ceiling. He used his pulse to measure the period. His observations and theory led to the construction of the first pendulum clock by Christiaan Huygens in 1650.

It is now your turn to make a simple “grandfather clock”, also known as a *seconds pendulum*. The pendulum in a grandfather clock swings so that the time interval between the ‘tick’ and the ‘tock’ is exactly one second.

![Diagram of a pendulum with 'Tick' and 'Tock' labels]

Before you build your clock, calculate the length of a seconds pendulum in the space below:

\[ L = \text{________} \text{m} . \]

Based on your theoretical value of \( L \), construct your clock using a piece of string and a 200-gram brass weight for the pendulum bob. Remember to measure \( L \) from the pivot point to the center of mass of the weight. Pull the pendulum 15° – 20° away from the vertical and release. Measure the period with a stopwatch. Remember the accurate method to measure a period: measure the time for five oscillations and then divide by five to get the period.

\[ T = \text{________} \text{s} . \]

How well does your clock keep time, i.e. what is the time interval between the ‘tick’ and the ‘tock’?

\[ \text{Tick-Tock Time} = \text{________} \text{s}. \]
B. Experimental Test of the Relation $T \sim \sqrt{L}$

Construct a new pendulum that is exactly 1/4 the length of your seconds pendulum. Note how the lower pivot clamp allows you to slide the string through the clamp until the 1/4 length is achieved. Use a 200-gram mass as the pendulum bob.

Measure the period of this new pendulum without using a stopwatch. Instead, use your homemade clock that keeps time by ‘tick-tocking’ in seconds. Hang the new pendulum right next to your seconds pendulum, but from the lower pivot so that the two masses are even. Pull both masses a small angle away from the vertical and release at the same time. Carefully observe the parallel motions.

Draw five pictures that show the position of each pendulum at five different times:

\[
\begin{array}{ccccc}
  t = 0 & 0.5 \text{ s} & 1.0 \text{ s} & 1.5 \text{ s} & 2.0 \text{ s} \\
  \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

By comparing the swing of your seconds pendulum to the swing of the 1/4 length pendulum, how many seconds does it take for the 1/4 length pendulum to complete one back and forth motion?

\[ T \text{ (1/4 Length)} = \text{___________________ s} . \]

Based on your measured values of the period $T(L)$ of a pendulum of length $L$ and the period $T(L/4)$ of a pendulum of length $L/4$, can you conclude that period of a pendulum is proportional to the square root of its length: $T \sim \sqrt{L}$, or equivalently $T^2 \sim L$? Explain carefully. \\*Hint: construct ratios.*
C. Experimental Test of “$T$ is independent of $m$”

The theoretical formula $T = 2\pi\sqrt{L/g}$ says two amazing things:

1. The period does not depend on the mass of the pendulum.
2. The period does not depend on the size of the arc through which the pendulum swings.

Recall that this formula is valid for “small” oscillations or small arcs ($x << L$). Let’s see if we can experimentally prove that $T$ is insensitive to $m$.

Use your seconds pendulum. To insure small oscillations, keep the angle of oscillation less than $20^\circ$. Measure the period $T$ for three different masses ($m = 50$ g, 100 g, 200 g). Remember to measure the time for five oscillations and then divide by five to get the period. For each mass, repeat the period measurement four times. Compute the average value of $T$ and the uncertainty. Define the uncertainty as the half-width spread in the $T$ values: $(T_{\text{max}} - T_{\text{min}})/2$.

<table>
<thead>
<tr>
<th>m (grams)</th>
<th>$T$</th>
<th>$T$</th>
<th>$T$</th>
<th>$T$</th>
<th>average $T$ ± uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>±</td>
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<tr>
<td>100</td>
<td></td>
<td></td>
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<td></td>
<td>±</td>
</tr>
<tr>
<td>200</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>±</td>
</tr>
</tbody>
</table>

Provide a graphical display of your measured values of $T$ (average ± uncertainty) on the following range diagram, i.e. draw the three horizontal line segments that represent your three $T$ ranges.

Does your range diagram provide experimental proof that the period of a simple pendulum is independent of the mass of the pendulum? Explain.
Part II. The Mass on Spring Oscillator

Suspend the 50-gram weight hanger from the end of the narrow silver spring. Place the 100-gram slotted brass cylinder on the hanger platform. Place the motion sensor directly under the oscillator. The distance between the sensor disk and the oscillator should be about 25 cm. Open Logger Pro file DisVelAcc. With the mass in the resting (equilibrium) position, zero the motion sensor: click Experiment on the menu bar and select Zero. Pull the mass about 10 cm away from the equilibrium position (x = 0) and release. Record the oscillations with the motion sensor. If needed, change the scales on your graph so that you can clearly see the sinusoidal variations.

A. x(t) and a(t) are “In Sync”. Finding T from x(t)/a(t).

The kinematic trademark of simple harmonic motion is that the position x(t) and the acceleration a(t) are proportional to each other, and opposite in direction, for all time t:

\[
a(t) = -\omega^2 x(t).
\]

The proportionality constant between a(t) and x(t) is the square of the angular frequency \(\omega = \frac{2\pi}{T}\) of the system. When x is zero, a is zero. When x is maximum (minimum), a is minimum (maximum). If the magnitude of x doubles during a certain time interval, then the magnitude of a will double during that same time interval. Note that the velocity is completely out of sync (out of phase) with x and a. When x is zero, v is max, etc.

Examine your graphs to be a first-hand witness to this “in-sync dance” between x and a. Do your x(t) and a(t) curves satisfy the motion equation \(a(t) = -\omega^2 x(t)\)? Use the Examine Icon [x=?] to find the values of x(t) and a(t) at the six times \(t = a, b, c, d, e, f\) shown below. Any six points that are approximately equally-spaced within one period of the sine curve are perfectly fine.

Record your measured values of x and a along with the ratio \(x/a\) in the table:
According to the theory, \( x(t) \) and \( a(t) \) both vary in time, but the ratio \( x(t)/a(t) \) is independent of \( t \). Your experimental data should support this theoretical fact. Based on your data, what is the constant value of \( x(t)/a(t) \) that characterizes your mass-spring system?

\[
x(t)/a(t) = \frac{\text{(measured value)}}{\pm \text{(uncertainty)}} \text{ (s}^2\text{)}.
\]

Calculate the period \( T \) of your oscillator from your measured value of \( x(t)/a(t) \). Remember, \( a(t) = -\omega^2 x(t) \) and \( \omega = 2\pi/T \). Show your calculation.

\[
T = \frac{\text{(measured value)}}{\pm \text{(uncertainty)}} \text{ s}.
\]

**B. Finding \( T \) from m/k**

The mass of your mass-spring system is \( m = 0.150 \text{ kg} \). The force constant \( k \) is equal to the spring constant of the spring in your system. Measure the value of \( k \). You have measured spring constants before. Hang a mass from the spring and measure the stretch. Repeat for several masses. Plot \( F \) versus \( x \) and find the best-fit slope. Hand in the graph showing the best-fit line and the “slope box”. Report your \( k \) value. Find the \( m\text{-over-}k \) value of your oscillator.

\[
k = \frac{\text{(measured value)}}{\pm \text{(uncertainty)}} \text{ N/m}.
\]

\[
m/k = \frac{\text{(measured value)}}{\pm \text{(uncertainty)}} \text{ s}^2.
\]

Calculate the period of your oscillator from your measured value of \( m/k \). Show your calculation.

\[
T = \frac{\text{(measured value)}}{\pm \text{(uncertainty)}} \text{ s}.
\]

**C. Summary**

You have measured the period of a mass-spring oscillator using two different methods. As a third method, measure the period directly with a stopwatch. Summarize your results by listing all three of your \( T \)-values below.

**Motion** Method. Measure \( x(t) \) and \( a(t) \), compute \( x(t)/a(t) \) : \( T = \frac{\text{(measured value)}}{\pm \text{(uncertainty)}} \text{ s} \).

**Force** Method. Measure \( m \) and \( k \), compute \( m/k \) : \( T = \frac{\text{(measured value)}}{\pm \text{(uncertainty)}} \text{ s} \).

**Timing** Method. Measure time directly with clock : \( T = \frac{\text{(measured value)}}{\pm \text{(uncertainty)}} \text{ s} \).
Part III. Damped Oscillator – A Class Research Project

So far, we have neglected the effect of friction on the motion. In physics, an oscillator subject to friction is called a “damped oscillator”. Friction causes the motion to decay. The amplitude of the oscillations decreases as time goes on. The height of the sine curve gets smaller and smaller. In this experiment, you will study the effect of air friction on the motion of your mass-spring oscillator.

Any object moving through the air experiences a ‘drag force’ due to the myriad of impacts between the object and the air molecules. As a result of the air-object collisions, the object slows down and the air molecules speed up (the air gets hotter). Mechanical energy of the object is transformed into thermal energy of the air.

The System

Each four-member team at a table will attach a cardboard surface of area A to their oscillator. Unscrew the bottom disk from the weight hanger rod. Put the rod through the hole in the board. Screw the disk back onto the rod. Place the 100-gram slotted brass cylinder on top of the board. You now have a damped harmonic oscillator:

As your oscillator moves up and down, imagine the cardboard surface pushing through the “sea” of air. The numerous collisions (about \(10^{22}\) per second!) between the surface and the air molecules (tiny “balls” of nitrogen and oxygen) are the mechanical source of the air friction.

Each team will get a different surface area A to study the effect of air friction on the decay of the motion. The goal of the class is to find how the amount of decay depends on the magnitude of the area. This is the kind of research conducted by scientists and engineers who study the aerodynamic design of cars, planes and rockets, and the hydrodynamic design of ships and submarines.

Half Life

How are you going to quantify the “amount of decay”? You will measure the half-life of the oscillator. Scientists use the concept of half-life to quantify the “lifetime” of anything that decays, such as a pendulum with friction, a circuit with resistance, an atom in an excited state, an unstable (radioactive) nucleus, and a bacteria colony.
The half-life is the time it takes for the amplitude of the motion to decay to one half of some initial value. The following position-time curve of a damped oscillator illustrates the concept of a half-life:

Note that the amplitude (height) decreases from $H$ to $H/2$ in 4 cycles (periods). The half-life is therefore $4T$, where $T$ is the period. It is easier to measure the half-life simply by the dimensionless number:

$$N_{1/2} = \text{Number of cycles it takes for the amplitude to decrease by a factor of } 1/2 .$$

For the above example, you would report the half-life as $N_{1/2} = 4$. Note that a small value of $N_{1/2}$ corresponds to a large amount of decay and a short lifetime. If there is no damping whatsoever, then $N_{1/2} = \infty$. An undamped (friction free) oscillator “lives” forever.

**The Quest**

Now that there are two experimental variables ($A$, $N_{1/2}$) that can be measured, the interesting question is: How are $A$ and $N_{1/2}$ related? One thing is certain. More surface area (big $A$) means larger damping (small $N_{1/2}$). The half-life $N_{1/2}$ must decrease as the area $A$ increases. Examples of decreasing functions: $N_{1/2} = 5.4/A$, $N_{1/2} = 3.9/A^2$, $N_{1/2} = 7.3/A^{1.5}$, $N_{1/2} = e^{-2A}$.

**Class Research Goal:** Find the function $N_{1/2}(A)$ that describes how $N_{1/2}$ depend on $A$.

**The Experiment**

Place the motion sensor directly under your oscillator. Make sure that the distance between the sensor and the board is greater than 25 cm. Open the Logger Pro file *Moving Along*. Change the graph scales so that $t$ goes from 0.0 s to 30.0 s and $x$ goes from −0.15 m to 0.15 m. With the oscillator at rest, zero the motion sensor: Click on *Experiment* and select *Zero*. Pull the mass away from the equilibrium position ($x = 0$) about 10 cm and release. Record the motion $x(t)$. PRINT the graph.
Find the half-life \( N_{1/2} \) of your oscillator. Start with the first maximum on your x-t curve and find the height \( H \) (amplitude) of this maximum. Examine your decaying sine curve to find the maximum point in the future that has the height \( H/2 \). Clearly mark these two maximum points on your printed graph. Label the heights of the two horizontal lines that go through these points. Count the number of cycles (periods) between the \( H \) point and the \( H/2 \) point. Record the values of the surface area and the half-life of your damped harmonic oscillator.

\[
A = \text{______________ cm}^2. \quad N_{1/2} = \text{__________}. 
\]

Write your results in the class data table on the chalkboard. Copy the class data table here:

<table>
<thead>
<tr>
<th>A (cm²)</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_{1/2} )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

After all teams have entered their data points \((A, N_{1/2})\) into the table, plot \( N_{1/2} \) (y axis) versus \( A \) (x axis) using Graphical Analysis. Delete the connecting lines between the points on your graph. Find the best-fit curve through the data points. Try the power law fit: \( y = ax^b \). Remember that “\( y \) as a function of \( x \)” in the curve-fit program is “\( N_{1/2} \) as a function of \( A \)” for your system. If the power-law analysis gives the fit parameters to be \( a = 2 \) and \( b = -3 \), then the \( N_{1/2} \) function would be \( N_{1/2} = 2/A^3 \). PRINT your graph showing the best-fit curve. Report the \( N_{1/2} \) function for your damped oscillator.

\[
N_{1/2} (A) = \text{__________________________}. 
\]

Note: There is no theoretical formula to compare your experimental result to. Your Physics 150 team is the first in the world to discover the \( N_{1/2} \) formula!

**Part IV. Coupled Oscillators and Normal Modes**

If you couple together two or more oscillators, then the resulting motion gets very interesting. Many systems in nature (mechanical, electrical, atomic) are coupled oscillators. The concept of a “normal mode” simplifies the analysis of the motion of coupled oscillators. It allows you to understand complicated non-sinusoidal vibrational motion in terms of a sum or “superposition” of simple sinusoidal (harmonic) vibrations. The whole motion is broken down into a set of more manageable parts. These simple parts are the “normal modes”.

What exactly is a normal mode?

*A normal mode is a pattern of motion in a system of particles in which every particle vibrates sinusoidally with the same frequency.*
Normal Modes A and B

Examine the coupled pendula system on display in the lab. Note that the spring serves as the coupling mechanism. This system may look simple, but it exhibits a rich variety of motions.

There are two normal modes: mode A and mode B. To “excite” mode A, pull both masses away from the vertical (equilibrium) in the same direction and release at the same time. To “excite” mode B, pull both masses away from the vertical (equilibrium) in opposite directions and release at the same time. Note that in each normal mode, both masses oscillate with the same frequency. Measure the period $T$ and compute the frequency $f$ of each mode:

- **Mode A**
  - $T_A = \underline{\phantom{000}}$ s.
  - $f_A = \underline{\phantom{0000}}$ Hz.

- **Mode B**
  - $T_B = \underline{\phantom{000}}$ s.
  - $f_B = \underline{\phantom{0000}}$ Hz.

**Energy Exchange Motion = Mode A + Mode B**

Pull mass 1 away from the vertical (do not touch mass 2) and release.

Observe this “energy exchange motion”. Note that the total amount of energy in the system remains constant, but it is constantly being passed between the two pendula. Pendulum 2 speeds up at the expense of pendulum 1 slowing down, and vice versa. If there were no friction, then this exchange would continue forever! The energy “sloshes” back and forth between the two parts of the system. The spring provides the transfer mechanism. This simple system models the complex energy transfers that occur in nature.

Measure the period of the energy exchange, i.e. the time it takes for 2, which starts with zero energy, to gain all the energy from 1, and then give it all back to 1. In other words, find the time it takes for pendulum 2 to complete the cycle rest $\rightarrow$ moving $\rightarrow$ rest. Since the energy exchange motion is a combination of the A and B modes, we will label this motion “AB”. Report your measured values of the period $T_{AB}$ and frequency $f_{AB}$ of this energy exchange (AB) motion:

- $T_{AB} = \underline{\phantom{000}}$ s.
- $f_{AB} = \underline{\phantom{0000}}$ Hz.
Relation Between $f_A$, $f_B$, $f_{AB}$

Why is the energy exchange motion a superposition of mode A and mode B? The following picture illustrates how the A and B displacements add to give the energy-exchange displacement:

![Diagram of modes A and B combining to form A+B]

According to the theory (normal mode analysis), the frequency of the energy-exchange motion (AB) is related in a simple way to the frequency of the constituent modes (A and B):

$$f_{AB} = f_B - f_A.$$

Do your experimental values of $f_{AB}$, $f_B$, and $f_A$ obey this theoretical relation? What is the percent difference between $f_B - f_A$ and $f_{AB}$?

Music, Atoms, & Photons

There exists many systems and processes in nature where three frequencies are related via the “normal mode rule” $f_{AB} = f_B - f_A$. Most notably are “beats” in music and “photons” in quantum mechanics.

When two musical notes (sound waves) of slightly different frequency $f_A$ and $f_B$ are played together (superimposed), the result is a sound wave that pulsates at the frequency $f_{AB} = f_B - f_A$. When an electron in an atom makes a quantum jump from a high-energy state B, where it “orbits” at frequency $f_B$, to a low-energy state A where it “orbits” at frequency $f_A$, the atom emits a light wave of frequency $f_{AB} = f_B - f_A$. During the quantum jump, the electron is in a superposition of quantum states (modes): state A + state B. In music, the superposition of sound waves results in “beats”. In quantum mechanics, the superposition of quantum waves results in a “photon”. One of the great triumphs of quantum mechanics was to discover the precise connection between the orbital frequencies ($f_A$, $f_B$) of the electron in an atom and the optical frequency ($f_{AB}$) of the emitted light. The coupled pendulum system that you studied in this lab explains how two tuning forks make sound (beats) and how two atomic states make light (photons)!