Team:

Waves and Music

Part I. Standing Waves

Whenever a *wave* (sound, heat, light, ...) is confined to a *finite region of space* (string, pipe, cavity, ...) , something remarkable happens – the space fills up with a spectrum of vibrating patterns called "standing waves". Confining a wave *"quantizes"* the frequency.

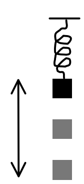
Standing waves explain the production of sound by *musical instruments* and the existence of *stationary states* (energy levels) in atoms and molecules. Standing waves are set up on a guitar string when plucked, on a violin string when bowed, and on a piano string when struck. They are set up in the air inside an organ pipe, a flute, or a saxophone. They are set up on the plastic membrane of a drumhead, the metal disk of a cymbal, and the metal bar of a xylophone. They are set up in the "electron cloud" of an atom.

Standing waves are produced when you ring a bell, drop a coin, blow across an empty soda bottle, sing in a shower stall, or splash water in a bathtub. Standing waves exist in your mouth cavity when you speak and in your ear canal when you hear. Electromagnetic standing waves fill a laser cavity and a microwave oven. Quantum-mechanical standing waves fill the space inside atomic systems and nano devices.

A. Normal Modes. Harmonic Spectrum.

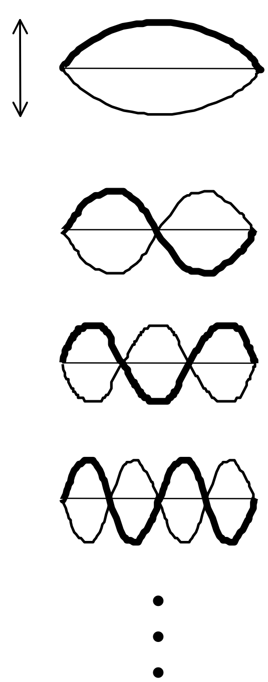
A mass on a spring has *one* natural frequency at which it freely oscillates up and down. A stretched string with fixed ends can oscillate up and down with a whole *spectrum* of frequencies and patterns of vibration.

Mass on Spring
(mass m, spring constant k)



$f = (1/2\pi)\sqrt{k/m}$

String with Fixed Ends
(length L, tension F, mass density μ)



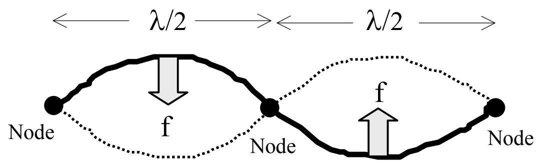
$f_1 = (1/2L)\sqrt{F/\mu}$ *Fundamental*

$f_2 = 2f_1$ 2nd *harmonic*

$f_3 = 3f_1$ 3rd *harmonic*

$f_4 = 4f_1$ 4th *harmonic*

These special *"Modes of Vibration"* of a string are called **STANDING WAVES** or **NORMAL MODES**. The word *"standing wave"* comes from the fact that each normal mode has *"wave"* properties (wavelength λ , frequency f), but the wave pattern (sinusoidal shape) does not travel left or right through space – it *"stands"* still. Each segment ($\lambda/2$ arc) in the wave pattern simply oscillates up and down. During its up-down motion, each segment sweeps out a *"loop"*.



A standing wave is a system of *fixed nodes* (separated by $\lambda/2$) and *vibrating loops* (frequency f). In short, a standing wave is a "flip-flopping" sine curve.

All points on the string oscillate at the *same frequency* but with different amplitudes. Points that do not move (zero amplitude of oscillation) are called *nodes*. Points where the amplitude is maximum are called *antinodes*. The mathematical equation of a standing wave is $y(x,t) = \sin(2\pi x/\lambda)\cos(2\pi ft)$. The "shape" term $\sin(2\pi x/\lambda)$ describes the sinusoidal shape of the wave pattern of wavelength λ. The "flip-flop" term $\cos(2\pi ft)$ describes the up-down oscillatory motion of each wave segment at frequency f. Each mode is characterized by a different λ and f.

The simplest normal mode, where the string vibrates in <u>one loop</u>, is labeled n = 1 and is called the **fundamental mode** or the **first harmonic**. The second mode (n = 2), where the string vibrates in <u>two loops</u>, is called the **second harmonic**. The n^{th} harmonic consists of n vibrating loops. The set of all normal modes {n = 1, 2, 3, 4, 5, ... } is the **harmonic spectrum**. The spectrum of natural frequencies is $\{f_1, f_2, f_3, f_4, f_5, ...\}$. Note that the frequency f_n of mode n is simply a <u>whole-number multiple</u> of the fundamental frequency: $f_n = nf_1$. The mode with 3 loops vibrates three times as fast as the mode with 1 loop.

<u>Exercise</u>: Sketch the 6^{th} *harmonic* of the string.

If the frequency of the 5^{th} *harmonic* is 100 *Hz*, what is the frequency of the 6^{th} *harmonic*?

If the length of the string is 3 *m* , what is the wavelength of the 6^{th} *harmonic*?

B. Why are Normal Modes Important ?

One of the most far-reaching principles in physics is this:

The general motion of any vibrating system can be represented as a sum of normal mode motions.

Normal modes are important for three reasons:

1. The motion of each mode is **SIMPLE** , being described by a Simple Harmonic Oscillator (trig) function: $\cos 2\pi ft$.

2. The set of modes serves as a **BASIS** for all motion: Any Wave/Vibrational Motion = Sum of Normal Mode Motions. Reducing the complex *"generic whole"* into a set of simple *"harmonic parts"* provides deep insight into nature.

3. The set of modes that characterize the vibration of a certain object is the acoustic (or optical) **"FINGERPRINT"** of that object. Every Object in Nature has its own "Ring" – its very own spectrum of sound (or light) frequencies that it emits when disturbed.

When you drop a spoon and a pencil on the floor, you hear two different sounds. When you "jolt" a neon atom and a mercury atom, you see red light emitted from neon and blue light from mercury. When a piano, a violin, and a saxophone play the same note at the same loudness, you hear three different sounds. When different instruments play "middle C" for example, the frequency of the fundamental tone ($f_1 = 264$ *Hz*) is the same, BUT the intensities of the Higher Harmonics or **"Overtones"** ($f_2 = 528$ *Hz* , $f_3 = 792$ *Hz* , $f_4 = 1056$ *Hz*, …) that make up the spectrum of the middle C note are different for each instrument.

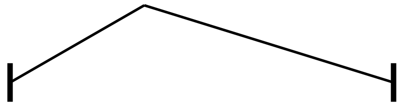
The "PITCH" of a musical note is equal to the frequency of the FUNDAMENTAL tone (n = 1). The "QUALITY OF SOUND" is determined by the number and the strength of the OVERTONES (n = 2, 3, 4, 5, …) that blend with the fundamental tone.

The *"proper tones"* (spectrum of natural frequencies) that identify a given object depend on the size, shape, density, and elasticity of the object. The spectrum is *not* always *harmonic* – *not integer multiples* of a fundamental frequency. The normal modes of vibration of the air inside a flute or an organ pipe are similar to those of a string. However, the normal modes that characterize the standing-wave vibrations of a drumhead or a bell are *inharmonic* – this explains why drums and bells (percussion instruments) sound "less harmonious" than violins and flutes (string and wind instruments). The "proper tones" of an atom – the frequencies of its emitted light – form an inharmonic spectrum.

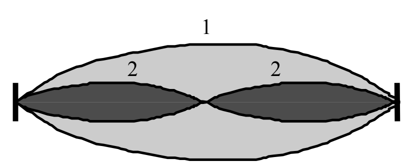
Plucked String = Superposition of Normal Modes

If you simply pluck a string, then you will excite an *infinite number* of modes – all at the *same time*! The motion of a plucked string (triangular wave) consists of a mixture of normal modes (sine waves) characterized by different frequencies. In the language of *Music*, when you pluck a guitar string, the resulting composite tone consists of the *Fundamental Tone* together with a whole series of fainter *Overtones*.

Initial Pluck
(Pull string at one point and release)



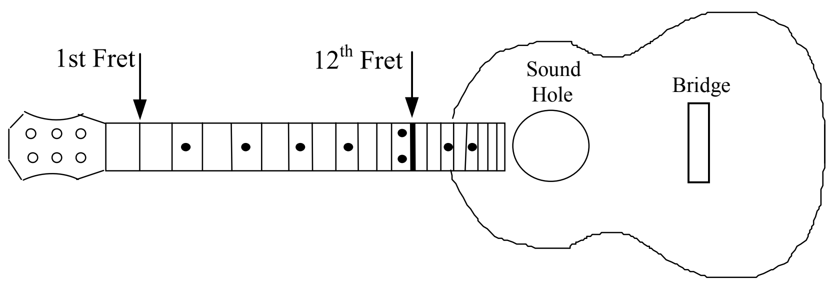
Resulting Motion
(Superposition of all modes, only n=1,2 are shown)



How do you know that the plucked string is really made up of all these harmonics? *"Fourier's Theorem"* provides the mathematical proof. Here you will perform an experimental proof.

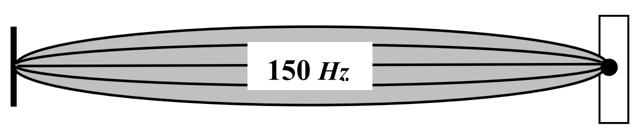
GUITAR: HANDLE WITH CARE

In this lab, you will use a guitar to study the physics of waves and the connection between ART and SCIENCE.   The guitar is not a scientific instrument. It is a fragile musical instrument. It is highly susceptible to dirt, grease, scratches, and cracks.



Experimental Proof of   "Plucked String = Mixture of Harmonic Parts"

All team members should gather as close as possible to the guitar so everybody can see and hear the phenomena. Pluck (using your finger or a guitar pick) the third thickest string on the guitar at any point over the sound hole. Look at the shape of the vibrating string. Listen to the sound it produces. You will see the *fundamental mode* (1st Harmonic) "flip flopping" at the rate of 150 *vibrations per second*! At such a high frequency, the string looks like one big blurred loop:



However, what you see is **NOT THE WHOLE PICTURE !**   In reality, the string is vibrating with a whole spectrum of normal mode shapes and frequencies all at the same time: 1-loop @ 150 *Hz* , 2-loop @ 300 *Hz* , 3-loop @ 450 *Hz* , 4-loop @ 600 *Hz* , etc . It is difficult to see the 2-loop pattern (and the other overtones) because the amplitude of the 2-loop pattern is smaller than the amplitude of the 1-loop pattern. Sound waves with all these harmonic frequencies are produced by these vibrating loops. The waves propagate through the air and vibrate your eardrum, which you perceive as musical tones. The *fundamental* tone (150 *Hz*) or "pitch" is the tone you tend to hear because it is the *loudest* component in the harmonic mixture.

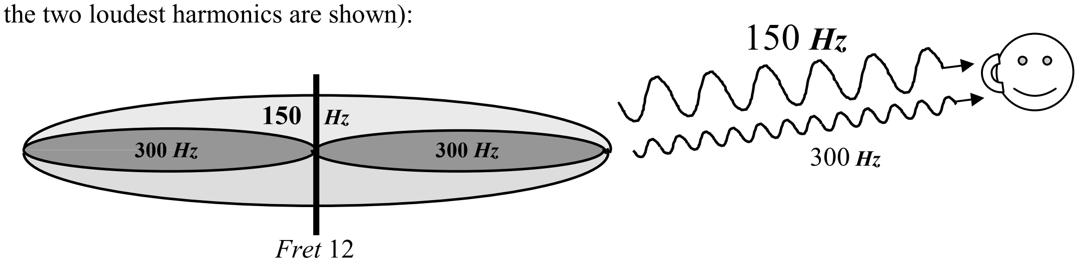
Destroying the Fundamental**.**   Pluck the string again and then *lightly touch* the string at its exact midpoint, which is located at the 12^{th} Fret (see picture above). Just a gently touch will do – *do not* press the string against the fingerboard.

Listen carefully   **. . .**   THE STRING IS STILL SINGING A TONE !

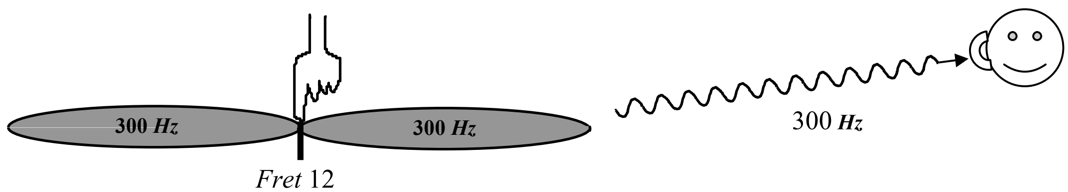
The faint tone you hear is the *second harmonic*. In music, the second harmonic tone (n = 2) is one **"OCTAVE"** above the fundamental tone (n =1). While the guitar is singing the octave, touch the string at any point away from the midpoint. The octave will disappear.

The Physics Explanation of this experiment is as follows:

When you pluck the string near the sound hole, the string vibrates with many frequencies all at once (only the two loudest harmonics are shown):



If you then touch this flip-flopping string at the midpoint (*Fret* 12) , which is an *antinode* of n=1 but a *node* of n=2, you **destroy** (crush) the one-loop mode (150 *Hz*) but the two-loop mode (300 *Hz*) survives. The 300-*Hz vibrating loops* emit a 300-*Hz sound wave* that you hear loud and clear as the octave tone:



If you then touch anywhere away from the midpoint (node), you dampen the two-loop vibration.

<u>Musicians</u> say "SOFT and MELLOW". <u>Physicists</u> say "MISSING HARMONICS".

Use the guitar pick to pluck the guitar string very near its endpoint (bridge) and listen to the sound. Now pluck the string at its midpoint and listen. In both cases, the *Pitch* (150 *Hz*) and *Loudness* are the same, but the *Quality of Sound* is very different. Circle the word that best describes the sound quality:

Pluck near End: *MELLOW* or *TINNY* Pluck at Midpoint: *MELLOW* or *TINNY*

Physics Explanation. The frequencies that are present (and their prominence) depends on where you pluck the string:

Harmonic:	1	2	3	4	5	6
Pluck near Endpoint:	150 Hz	300 Hz	450 Hz	600 Hz	750 Hz	900 Hz
Pluck at Midpoint:	150 Hz	**Missing**	450 Hz	**Missing**	750 Hz	**Missing**

Many strong high-frequency overtones ⇒ "TINNY"
Few and weak overtones, strong fundamental ⇒ "MELLOW"

<u>Midpoint Theorem in Music</u>: If you pluck a string at its *Midpoint* , then all the *Even Harmonics* (n = 2, 4, 6, 8, …) will be *Missing* from the spectrum.

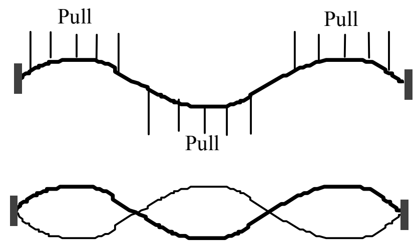
<u>Theoretical Proof</u>: If you want to produce the even harmonics, then you must not cause the midpoint to move (such as by plucking that point) because the midpoint is a motionless point (NODE) for the even harmonics.

<u>Experimental Proof</u>: Pluck the string at its midpoint (12^{th} Fret) and then lightly touch the midpoint. What is the result? Explain this result by drawing the plucked string showing n = 1,3 on the same picture (n=2 is missing!)

C. Creating a Normal Mode

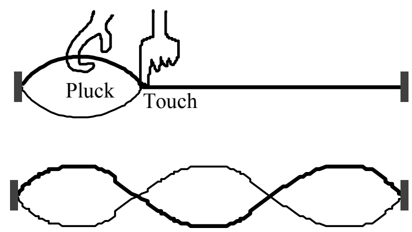
Plucking a string excites an infinite number of normal modes. How do you excite *only one* of the modes? There are three different methods:

1. <u>The Mathematician Method</u>: "Sine Curve Initial Condition"



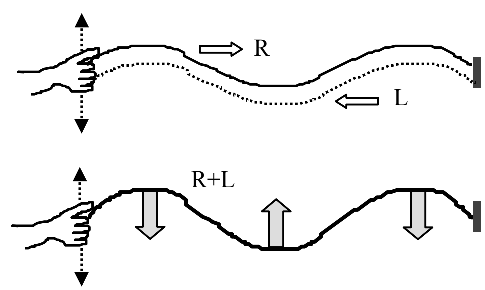
If you pull each mass element of a stretched string away from equilibrium (flat string) so that the string forms the shape of a *sine curve*, and then let go, the whole string will vibrate in one normal mode pattern. If you start with any other initial shape – one that is *not* sinusoidal – then the motion of the string will be made up of different modes. Starting out with an exact sine-curve shape is not easy to do – you need some kind of fancy contraption.

2. <u>The Musician Method</u>: "Touch and Pluck"



Guitar players and violin players do this all the time. They gently touch the string at one point (where you want the node to be) and pluck the string at another point (antinode) to make a "loop". The oscillation (up and down motion) of the plucked loop will *"drive"* the rest of the string to form two other equal-size loops which oscillate up and down at the same frequency as the driving loop.

3. <u>The Physicist Method</u>: "Resonance"



If you gently shake (vibrate) the end of a string up and down, a wave will travel to the right, hit the fixed end, and reflect back to the left. If you shake at just the right "resonance" frequency – one that matches one of the natural frequencies of the string – then the two *traveling waves* (Right and Left) will combine to produce *a standing wave* of large amplitude: R + L = STANDING WAVE.

Since RESONANCE is an important concept in science, we will focus on this method. Resonance phenomena are everywhere: tuning a radio, shattering a crystal glass with your voice, imaging the body with an MRI machine, building bridges, picking cherries, designing lasers, etc. Consider pushing a person in a swing. If the frequency of your hand (periodic driving force) matches the natural frequency of the swing, then the swing will oscillate with large amplitude. It is a matter of <u>timing</u> , <u>not strength</u>. A sequence of *"gentle pushes"* applied at just the *"right time"* – in perfect rhythm with the swing – will cause a dramatic increase in the amplitude of the swing. A small stimulus gets *amplified* into a LARGE response.

<u>Experiment:</u> *Shake a Slinky Coil , Make a Standing Wave*

Stretch the coil so that its end-to-end length is about 3 *m*. Keep one end fixed. **<u>Gently</u>** shake (vibrate) the other end *side-to-side*. <u>DO NOT</u> shake with a large force or a large amplitude. Remember that resonance is all in the *timing*, not the force! Your oscillating hand should not move more than a couple of inches. Shake at just the *right frequency* to produce the two-loop (n = 2) normal mode. When you have *"tuned into"* this n = 2 state, take special note:

You are now RESONATING with the coil. You and the coil are "one" !

Your shaking hand is perfectly in-sync with the coil's very own natural vibration. The frequency of your hand matches f_2 of the coil. Note how the *amplitude* of your shaking *hand* is about 5 *cm* , whereas the *amplitude* of the vibrating *coil* is about 50 *cm* !! This is the trademark of resonance: a feeble effort (hand moves a little) translates into a powerful output (coil moves a lot).

D. Discovering a Harmonic Spectrum: Measure f_n , Calculate f_n

Use the slinky as before – keep one end fixed, make the length 3 *m*, shake the other end. Shake at the *"right frequencies"* to create the first four harmonics n = 1, 2, 3, 4. Unlike all other quantities that you have measured in the past physics labs, there is no error in measuring the quantity n. For example, n could be 2 or 3, but not 2½. Thus, the uncertainty in n is exactly equal to zero: $n \pm 0$!!!

For each harmonic, measure the period of your oscillating hand. Note that the period of your hand is equal to the period of the "flip-flopping" loops on the slinky. To find the period, use a *stopwatch* to measure the time it takes your hand to complete ten (or twenty) side-to-side motions. Divide by ten (or twenty) to get the period. Summarize your results in the following table. Sketch the shape of each mode you observed. Record your measured values of the period and the frequency of each mode.

Mode Number	Mode Shape	Period (*s*)	Frequency (*Hz*)
1			
2			
3			
4			

The Theory

So far, you have *measured* the spectrum of natural frequencies of the slinky system. You will now *calculate* this spectrum based on the theory of waves. There are three basic parts to the theory of wave motion:

Kinematics: $v = \lambda f$ (since *velocity* = *distance* over *time* = λ/T).

Dynamics: $v = (F/\mu)^{1/2}$ (from solving $F = ma$ for a string with *tension* F and *mass density* μ).

Fitting Condition: $n(\lambda/2) = L$ (see picture on page 1 for proof that mode n consists of n *half-wavelengths* "stuffed" inside the *length* L).

Note: Even though a standing wave does not travel, the concept of velocity still makes sense because when you shake a string, you generate a traveling wave that propagates along the string. The velocity v of this traveling wave (whose back-and-forth motion forms the standing wave) defines the "v" that appears in the standing wave equation $f_n = nv/2L$.

Derive the Frequency Relation

According to theory, the velocity v of any periodic wave satisfies the relation $v = \lambda f$. In the fundamental mode, one loop – which corresponds to one-half of a wavelength – just *"fits"* inside the length L of the string. Given this *"fitting condition"* $\frac{1}{2}\lambda = L$, the velocity relation becomes $v = 2Lf$. Thus the frequency of the fundamental (n =1) mode is

$$f_1 = v/2L \, .$$

Given the general fitting condition $n(\lambda/2) = L$, the frequency of all the higher modes (overtones) of the string are *integer multiples* of the fundamental frequency: $f_n = nf_1$. So to calculate the *entire* theoretical spectrum (an infinite number of frequencies $f_n = nv/2L$) , all you need to know are the values of two kinematic quantities: L and v.

Measure L. Record the length of the stretched slinky coil on which your team observed the first four harmonics.

L = ____________ *m* .

Find v. The velocity of a wave on a string (or coil) does not depend on the shape of the wave. All small disturbances propagate along the string at the same speed. So the best way to measure the value of v is to make a wave which is easiest to observe. The simplest kind of wave is a *single pulse*.

Stretch the coil as before so that it has the same length L. Keep *both* ends fixed. At a point near one end, pull the slinky sidewise (perpendicular to the length) and release. This will generate a *transverse traveling wave*. Observe this wave disturbance (pulse) as it travels down the slinky, reflects off the other end, and travels back to its point of origin. Use a *stopwatch* to measure the time it takes for the pulse to make 4 round-trips, i.e. 4 back-and-forth motions along the entire length of the coil. Divide by 4 to get the round trip time. Calculate the velocity of the wave on your coil.

Round-Trip Time = ____________ *s* . Wave Velocity v = ____________ *m/s* .

Calculate f. From your values of L and v, calculate the frequency of the fundamental mode using the *theoretical* relation derived above. Show your calculation:

f_1 = ____________ *Hz* .

Remember that the theoretical values of the overtones (f_2 , f_3 , f_4) are calculated from f_1 using $f_n = nf_1$.

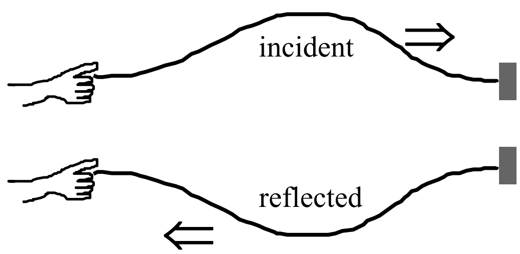
Compare Experimental and Theoretical Frequency Spectrum. Summarize your measured and calculated values of the natural frequencies $\{f_1 , f_2 , f_3 , f_4 , ... \}$ that characterize the slinky system:

n	f_n (*experiment*)	f_n (*theory*)	% Difference
1			
2			
3			
4			

E. What is Happening at Resonance? The "Timing Condition".

As you shake one end of a string (or coil), a *traveling wave* is sent along the string and is then reflected at the other end. The incident and reflected waves overlap (superpose) and add (interfere) to produce a resultant wave. In general, the result is a "jumbled" wave of small amplitude. However, if you *shake just right*, then the result is a highly-ordered standing wave of large amplitude.

What does *"shake just right"* mean? Remember that resonance is all in the *timing*. Suppose your hand generates a crest. When this crest reaches the fixed end, it is reflected as a trough. The returning trough reflects at your hand as a crest. If your hand starts to generate a second crest just as the trough arrives at your hand, the reflected crest and the second crest will be perfectly in step and reinforce each other.



When you create a standing wave by shaking a string , remember this:

Although it looks like the wave is *"standing still"*, what you see is the superposition (resultant sum) of two traveling waves that are *"moving fast"* in opposite directions!

The same physics explains how you can make "ocean waves" in a bathtub or a swimming pool! You must *"splash just right"*: Gently hit the water with your hand. Wait for the ripple to reach the wall and reflect back to you before you splash again. With this periodic timing of your splashes, your little out-going and in-going ripples will soon build up (via constructive interference) to form a huge standing wave!

Timing Condition: Two *traveling* waves (incident ⇒ , reflected ⇐) will combine to produce a *standing* wave if the following resonance (in-sync) condition is satisfied: *The round-trip time* t_r *of the traveling wave must equal an integer multiple of the period* T *of your hand*.

Note: The *Timing Condition* $t_r = nT$ is equivalent to the *Fitting Condition* $L = n\lambda/2$ discussed previously. Can you prove the equivalence?

Experimental Test of the Resonance (Timing) Condition

You have already measured the four special *periods* T of your *hand* that are necessary to create the standing wave modes n = 1, 2, 3, 4 on the slinky system. You have also measured the *round-trip time* t_r of a *traveling wave* on the slinky. How well do your experimental values of these times satisfy the theoretical *Timing Condition*? Compute the ratio of your measured times:

n = 1: t_r / T = ( ) / ( ) =
n = 2: t_r / T = ( ) / ( ) =
n = 3: t_r / T = ( ) / ( ) =
n = 4: t_r / T = ( ) / ( ) =

Compare your measured values of t_r/T to the theoretical values:

Part II. Research Project: Precision Measurement of "Proper Tones" and Wave Speed

The system you will study consists of a string of finite length under a fixed tension. One end of the string is connected to an electric oscillator – a high-tech "shaker". Instead of *you* shaking the string by hand, the electric oscillator moves the string up and down at a precise frequency that you can control. The tension in the string is due to a 150-*gram* mass hanging on the other end. Your *research goal* is to answer two basic questions:

1. What are the *natural frequencies* of this string system?
2. What is the *speed* of a wave on this system?

Frequency Spectrum

Set up standing waves $n = 1 , 2 , 3 , 4 , 5$ by driving the string with the electric oscillator.

Procedure for Resonating with the String: Start with the frequency of the oscillator equal to 0.0 Hz and then slowly increase the frequency. If the oscillator makes a "rattle" sound, then decrease the amplitude. Record your measured values of f_1 , f_2 , f_3 , f_4 , f_5 in the table:

n	1	2	3	4	5
f_n (*Hz*)					

Do the natural frequencies (*proper tones*) of the string system form a *harmonic spectrum*? *Explain*.

Wave Speed

The wave speed on the string is too fast to measure with a stopwatch. "Flick" the string and try to observe the traveling pulse. Setting up standing waves is the gold-standard technique to measure the speed of any kind of wave – from sound in air (300 *m/s*) to light in a vacuum (300,000,000 *m/s*). When you set up a standing wave, you are in effect "freezing" (taking a snapshot of) the speeding wave so you can easily observe the *shape* of the wave and measure the *wavelength*.

To get an accurate value of the wave speed v, find v using *two* different methods:

A. Space-Time Method: Measure λ and f for each mode n. Compute $v = \lambda f$.

Note: You have already measured f. To find λ for each n , measure the length L of your string and use the *Fitting Condition* "mode n consists of n half-wavelengths stuffed inside L": $n(\lambda/2) = L$.

n	λ (*m*)	f (*Hz*)	$v = \lambda f$ (*m/s*)
1			
2			
3			
4			
5			

average v = ___________ *m/s*.

B. Force-Mass Method: Measure F and μ of string. Compute $v = (F/\mu)^{1/2}$.

The *tension* F in the string is equal to the weight of the hanging mass. To find the *mass density* μ of the string (μ ≡ *mass* per unit *length*), use the sample string on the back table. Do not remove the string attached to the oscillator and weight.

Note: The sample string came from the same spool of string as the actual string and therefore has the same value of μ. The sample string may have a larger *mass* and a larger *length* than the actual string , but the ratio *mass* / *length* is the same.

F = _________ *N* . μ ≡ mass / length = (*kg*) / (*m*) = __________ *kg*/*m*.

Wave Speed $v = (F/\mu)^{1/2}$ = ________________ *m*/*s* .

Compare your λf result with your $(F/\mu)^{1/2}$ result for v:

Part III. The Physics of Music

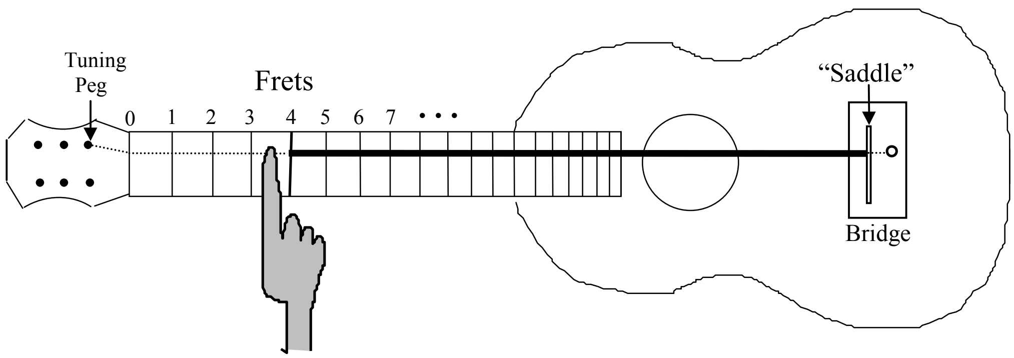
A. Whole Numbers and Harmony

Pythagoras discovered that two identical strings of different lengths when sounded together produce an effect that is "pleasant" to the ear if the lengths of the two strings are in the *ratio of two small whole numbers*. The most "harmonious" or "consonant" sound is produced when the ratio is 1:2 , i.e. one string is twice as long as the other string. The next most harmonious combination of tones occurs for the ratios 2:3 , 3:4 , 4:5. The smaller the integers, the better the harmony. Of course, what is "pleasant to the ear" is a matter of perception. However, there is a scientific explanation of harmony which correlates the *"amount of discord"* with the *"rate of beats"* between the overtones.

Pythagoras' discovery was the first example where basic numbers of mathematics (1, 2, 3, 4, ...) were used to describe a phenomenon of nature. It connected Art and Science. Whoever thought that by measuring *simple lengths*, one could produce *beautiful sounds*! In his *"music of the spheres"*, Pythagoras even hypothesized that the size of the orbits of the planets (celestial spheres) formed ratios of small whole numbers. In *Quantum Mechanics* – one of the scientific revolutions of the 20th Century – whole numbers (1, 2, 3, 4, ...) called "quantum numbers" are used to label the *discrete set of states of an atom* in the same way that whole numbers are used to label the *discrete set of harmonics of a string*.

Pythagorean Experiment on a Guitar

The length of a guitar string is changed by pressing the string against the fingerboard (neck) of the guitar at one of the *"fret"* locations.



Pressing the String: When the string is pressed against the fingerboard between frets 3 and 4 (see picture), the string makes contact with fret 4 and thus fret 4 becomes one of the fixed endpoints of the vibrating string. The other fixed endpoint is located at the "saddle" (see picture) – the thin white strip on which the string rests. So when we say " PRESS THE STRING AT FRET 4 ", we mean press the string against the fingerboard BETWEEN FRETS 3 and 4 – this will automatically "clamp" the string at fret 4.

In this experiment, you will use the thickest (top) string of the guitar, called the "E-string". Strictly speaking, this is the "low" E-string. Practice pressing this E-string at different frets and plucking the shortened string to produce different *musical tones*.

Team up with another group so you can analyze the two tones produced by two identical guitar strings – but of different lengths. If the room is noisy, then your team can go into the hallway to listen to the tones.

Tuning the Strings. Make sure that the *open* (unfretted) E-strings on the two guitars produce the same tone when plucked. The two guitar players should sit next to each other as close as possible and face the other six team members. This will allow all members to see the strings and hear the sounds. Pluck the two open E-strings and listen to the combination of tones. Pluck several times to get a good listen. If the two tones have a different pitch (frequency), then change the tension in one string (by *slowly* turning the tuning peg) until the two tones sound identical. If the two frequencies are slightly different, then you should hear "BEATS" – a pulsating sound that fades in and out. The disappearance of beats is synonymous with the equality of tone.

Caution: Only a *very slight change* in tension should be required. DO NOT increase (or decrease) the tension too much – this may break the string or bridge. Consult your instructor or another team if you have any questions about tuning the E-strings.

SEARCHING for HARMONY. Now that your two strings vibrate in unison, you are ready to play "two-note chords". Again, the two guitar players should sit close to each other and face the "audience". On one guitar, keep the E-string open. On the other guitar, press the E-string at Fret 2 , i.e. between 1 and 2. Pluck the open string and the fretted string at the same time (assign one person to simultaneously pluck both strings over the sound hole). Listen to the combination of two tones. Pluck several times to get a good listen. Does your team perceive the sound as "pleasant" or "dissonant"? Also try Frets 3 , 6 , 7 , 11 , 12.

Among the six shortened strings (Frets 2, 3, 6, 7, 11, 12) that you sounded with the open string, list the two frets that resulted in the "most pleasant" sounds:

Fret _______ . Fret _______ .

Measure the two different lengths of the string associated with these two "best" frets. Also measure the length of the open string. In the picture below, the open string is drawn. Draw the two "best" fretted strings. The length of your fretted line segments should be drawn approximately to scale – relative to the open line segment. Write your measured lengths (in *cm*) directly next to the strings. Compute the length ratio: Open/Fretted.

<u>Ratio of Lengths</u>

"Most Harmonious"

Open ________________ *cm*

Fretted .... *cm*

"Next Harmonious"

Open ________________ *cm*

Fretted .... *cm*

How do your two "harmonious length ratios" compare to the "top two" Pythagorean ratios 2/1 and 3/2 ?

B. The Physics of Pitch

How does the *Musical Pitch* of a *Sound* depend on the *Physical Properties* of the *String* producing the sound?

Pitch is equal to the *frequency of the fundamental mode* of vibration. Frequency f is equal to v/λ. Wavelength λ (of the fundamental mode) is equal to 2L. Thus the pitch formula is $f = v/2L$, where the wave speed is $v = \sqrt{F/\mu}$. Let's summarize this important relation:

<u>The "Pitch Formula" for Stringed Instruments</u>**.** The *pitch* (fundamental frequency) f produced by a vibrating string of *length* L , *tension* F , and *mass density* μ is determined by the formula

$$f = \frac{1}{2L}\sqrt{\frac{F}{\mu}} \; .$$

This formula contains three *"Laws of Vibrating Strings"*:

1. <u>Law of Tension</u>: $f \sim \sqrt{F}$. Pitch increases as tension increases. F=m*a* says big F causes big *a* and big *a* means big f. You encountered this relation when you tuned the E-strings by changing the tension.
2. <u>Law of Mass</u>: $f \sim \sqrt{1/\mu}$. Pitch decreases as mass density increases. *a* =F/m says big m means small *a*. Look at the six strings on the guitar and observe how the thick strings (large μ) produce low bass pitches (small f), while the thin strings (small μ) produce high treble pitches (large f).
3. <u>Law of Length</u>: $f \sim 1/L$. Pitch is *inversely proportional* to Length. This explains why the pitch of a vibrating guitar string increases as you shorten the string by pressing the string against the fretboard. It explains why a violin produces *high* frequencies (200 *Hz* –1000 *Hz*) and a double bass produces *low* frequencies (40 *Hz* –100 *Hz*).

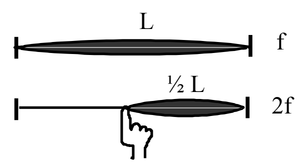
This *Law of Lengths* for string-like objects is a special case of a *General Principle* valid for *all* objects:

$$\text{FREQUENCY of Object} \sim \frac{1}{\text{SIZE of Object}}$$

Large objects (tuba, trombone, cello, large bell, bowling ball, ...) make *low*-pitched sounds.
Small objects (flute, piccolo, violin, small bell, coin, ...) make *high*-pitched sounds. Producing higher frequencies requires smaller objects. Vibrating *crystals* (10^{-2} *m*) emit *radio* waves (10^{6} *Hz*). Vibrating *atoms* (10^{-8} *m*) emit *heat* waves (10^{12} *Hz*). Vibrating *electrons* (10^{-10} *m*) emit *light* waves (10^{14} *Hz*).

Physicists say *"You Doubled the Frequency"*. **Musicians** say *"You Went Up One Octave"*.

When you press a guitar string at Fret 12 , you *decrease* the length of the string by a factor of ONE HALF and therefore (since f ~ 1/L) *increase* the vibration rate of the string by a factor of TWO . This explains the two dots that are embedded in the fingerboard at Fret 12.



Experiment: The 12th Fret = The Special Place to make Octaves. Sensing f $\rightarrow$ 2f .

Guitar Fact: the open E-string on a guitar vibrates at 82.4 *Hz* in the fundamental mode. Pluck the open E-string on your guitar. Then pluck the E-string while pressing the string at Fret 12. Listen to the difference between these two tones. The vibration of your ear drum went from 82.4 *Hz* to 164.8 *Hz*. The *"musical interval"* you hear is exactly one *octave*!

Try this on some of the other strings so that you can experience different octaves. How the human auditory system senses and interprets a doubling in the frequency is a research problem in biophysics, biomedical engineering, musical acoustics, and cognitive science.

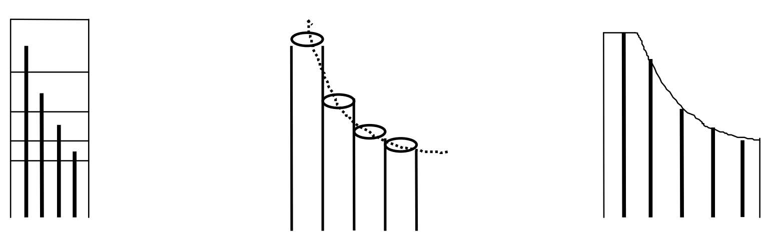
Exercise: What is the pitch of the musical tone that is produced when you pluck the E-string while pressing the string at Fret 7 ? Show your calculation. *Hint*: Construct ratios.

C. The Guitar Maker's Formula

Why do the frets on a guitar get closer together as you move along the neck toward the bridge?

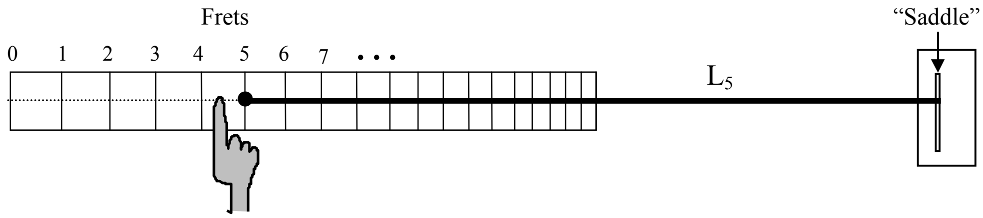
If you want to make a flute, how do you know where to drill the holes? How long should you make the pipes of an organ, the wires in a piano, or the bars of a xylophone?

Guitar Frets *Organ Pipes* *Piano Strings*



Measuring the "Fret Function"

Let L_0 denote the length of the open (unfretted) string. Let L_N denote the length of the guitar string when the string is pressed at fret number N. Remember that L_N is the length between fret N and the saddle.



Measure L_0 , L_1 , L_2 , L_3 , ... , L_{12} . Record your results:

N	0	1	2	3	4	5	6	7	8	9	10	11	12
L_N (*cm*)													

Use Graphical Analysis to graph your *Fret Function*: L_N (y-axis) as a function of N (x-axis).

"Guitar Maker's Formula": According to theory, the location of the frets on a guitar is specified by the following universal formula:

$$L_N = L_0 \left(\sqrt[12]{2} \right)^{-N} .$$

Note that the ratio of *any* two successive lengths, such as L_0/L_1 , L_1/L_2 , or L_8/L_9 , is equal to $\sqrt[12]{2} = 2^{1/12} = 1.059463...$. This means that pressing the string on each successive fret of a guitar shortens the length of the string by a factor of $\sqrt[12]{2}$ (about 6%) and therefore raises the frequency of the note by the same factor. Note that L_N is an *exponential* function of N and can be written in the form $L_N = L_0 \, (0.9438743)^N$.

The 12th Root of 2 : $\sqrt[12]{2} = 1.059463...$ is a **Fundamental Constant** in the World of Music.

The other fundamental constant is the *standard of frequency*: "Concert A" ≡ 440 *Hz*.
Where does the special number $\sqrt[12]{2}$ come from? It comes from the *Equal-Tempered Scale of Music*, which is the basis for modern Western music. In the *Equal-Tempered Scale*, the octave is divided into 12 *equal* steps, i.e. every pair of adjacent tones has the *same frequency ratio*. Since an octave has a frequency ratio equal to **2** , each of the **12** equal steps has a frequency ratio equal to $\mathbf{\sqrt[12]{2}}$. This explains the "magic" number $\sqrt[12]{2}$ in the Guitar Maker's Formula.

Given that the open E-string on a guitar has frequency 82.4 *Hz* , the next note (Fret 1) in the 12-tone equal-tempered scale has frequency $2^{1/12} \times 82.4$ *Hz* = 87.3 *Hz*. The next note (Fret 2) has frequency $(2^{1/12})^2 \times 82.4$ *Hz* = 92.5 *Hz*.

Exercise: List the frequency f of each note played on the E-string of a guitar:

Fret	0	1	2	3	4	5	6	7	8	9	10	11	12
Note	E	F	F#	G	G#	A	A#	B	C	C#	D	D#	E′
f (*Hz*)	82.4	87.3	92.5										

Testing the Guitar Maker's Formula: *" The* $^{12}\sqrt{2}$ *Rule "*

Let's see if we can "extract" the theoretical constant of music $^{12}\sqrt{2}$ from your experimental data. Analyze your L_N versus N graph as follows. Delete the connecting lines between the data points. Use the curve-fit program to fit your data points with the function $y = AB^{-x}$, called the *"fret function"*. If the fret locations that you measured on your guitar meet the *Guitar Maker's Specifications* , namely $L_N = L_0\,(^{12}\sqrt{2})^{-N}$, then the values of your curve-fit constants (A,B) should be $A = L_0$ and $B = {}^{12}\sqrt{2}$.

PRINT your L_N versus N graph showing the best-fit curve and the values of A and B. Compare your experimental constants (A , B) with the theoretical *Guitar Maker's* constants (L_0 , $^{12}\sqrt{2}$) :

% difference between A = _________ and L_0 = _________ is _____ % .

% difference between B = _________ and $^{12}\sqrt{2}$ = _________ is _____ % .

Testing the Standing Wave Formula: *" The* f = v/2L *Rule "*

Look at your table of L values and table of f values. Note how L *decreases* and f *increases* as you press successive frets to play successive notes. If your values of L and f satisfy the theoretical relation f = v/2L , then the product of f and L should *stay the same* as you press successive frets! Use the values of L and f in your tables to calculate the value of the product fL for frets 2 , 5 , 8 , 12 .

Fret 2:	f L	=	__________	×	__________	=	___________ .
Fret 5:	f L	=	__________	×	__________	=	___________ .
Fret 8:	f L	=	__________	×	__________	=	___________ .
Fret 12:	f L	=	__________	×	__________	=	___________ .

Based on your values of fL , what is the *velocity of a wave* on the E-string of a guitar?

v = ____________ *m/s* .

v = ____________ *mph* .

Given this value of v, calculate the tension in the E-string of a guitar. DO NOT remove the E-string from your guitar. Use the sample E-string on the back table. Show all your work:

F = ____________ *N* .

F = ____________ lbs .