

# Waves and Modes

## Part I. Standing Waves

Whenever a *wave* ( sound, heat, light, ... ) is confined to a *finite region of space* (string, pipe, cavity, ... ) , something remarkable happens – the space fills up with a spectrum of vibrating patterns called “standing waves”. Confining a wave “*quantizes*” the frequency.

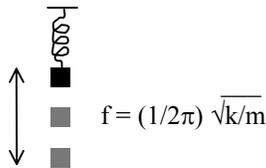
Standing waves explain the production of sound by *musical instruments* and the existence of *stationary states* (energy levels) in atoms and molecules. Standing waves are set up on a guitar string when plucked, on a violin string when bowed, and on a piano string when struck. They are set up in the air inside an organ pipe, a flute, or a saxophone. They are set up on the plastic membrane of a drumhead, the metal disk of a cymbal, and the metal bar of a xylophone. They are set up in the “electron cloud” of an atom.

Standing waves are produced when you ring a bell, drop a coin, blow across an empty soda bottle, sing in a shower stall, or splash water in a bathtub. Standing waves exist in your mouth cavity when you speak and in your ear canal when you hear. Electromagnetic standing waves fill a laser cavity and a microwave oven. Quantum-mechanical standing waves fill the space inside atomic systems and nano devices.

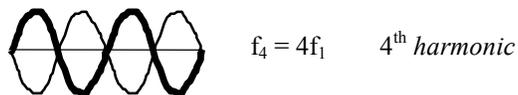
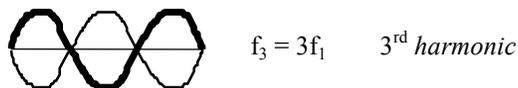
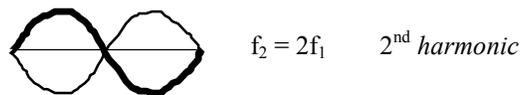
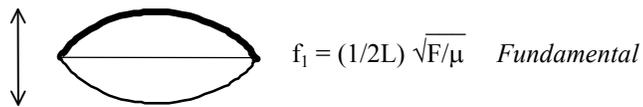
### A. Modes

A mass on a spring has *one* natural frequency at which it freely oscillates up and down. A stretched string with fixed ends can oscillate up and down with a whole *spectrum* of frequencies and patterns of vibration.

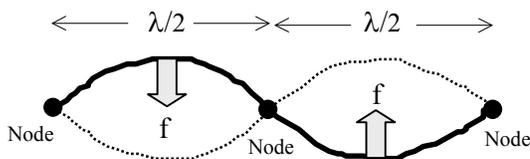
**Mass on Spring**  
(mass  $m$ , spring constant  $k$ )



**String with Fixed Ends**  
(length  $L$ , tension  $F$ , mass density  $\mu$ )



These special “*Modes of Vibration*” of a string are called **STANDING WAVES** or **NORMAL MODES**. The word “*standing wave*” comes from the fact that each normal mode has “*wave*” properties (wavelength  $\lambda$ , frequency  $f$ ), but the wave pattern (sinusoidal shape) does not travel left or right through space – it “*stands*” still. Each segment ( $\lambda/2$  arc) in the wave pattern simply oscillates up and down. During its up-down motion, each segment sweeps out a “*loop*”.



A standing wave is a system of *fixed nodes* (separated by  $\lambda/2$ ) and *vibrating loops* (frequency  $f$ ). In short, a standing wave is a “flip-flopping” sine curve.

All points on the string oscillate at the *same frequency* but with different amplitudes. Points that do not move (zero amplitude of oscillation) are called *nodes*. Points where the amplitude is maximum are called *antinodes*. The mathematical equation of a standing wave is  $y(x,t) = \sin(2\pi x/\lambda) \cos(2\pi ft)$ . The “*shape*” term  $\sin(2\pi x/\lambda)$  describes the sinusoidal shape of the wave pattern of wavelength  $\lambda$ . The “*flip-flop*” term  $\cos(2\pi ft)$  describes the up-down oscillatory motion of each wave segment at frequency  $f$ . Each mode is characterized by a different  $\lambda$  and  $f$ .

## B. Harmonics

The simplest normal mode, where the string vibrates in one loop, is labeled  $n = 1$  and is called the **fundamental mode** or the **first harmonic**. The second mode ( $n = 2$ ), where the string vibrates in two loops, is called the **second harmonic**. The  $n^{\text{th}}$  harmonic consists of  $n$  vibrating loops. The set of all normal modes  $\{n = 1, 2, 3, 4, 5, \dots\}$  is the **harmonic spectrum**. The spectrum of natural frequencies is  $\{f_1, f_2, f_3, f_4, f_5, \dots\}$ . Note that the frequency  $f_n$  of mode  $n$  is simply a whole-number multiple of the fundamental frequency:  $f_n = n f_1$ . The mode with 3 loops vibrates three times as fast as the mode with 1 loop.

Harmonics are the basis of HARMONY in music. The sectional vibrations of a string as *one whole, two halves, three thirds*, and so on, are very special because these vibrations produce musical tones that sound the “most pleasant” when sounded together, i.e. they represent the most *harmonious* combination of sounds. This explains the origin of the word “harmonic”.

Exercise: Sketch the 6<sup>th</sup> harmonic of the string.

If the frequency of the 5<sup>th</sup> harmonic is 100 Hz, what is the frequency of the 6<sup>th</sup> harmonic?

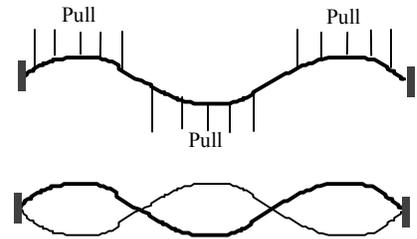
If the length of the string is 3 m, what is the wavelength of the 6<sup>th</sup> harmonic?

### C. Creating a Mode

In general, when you pluck a string, you excite an infinite number of harmonic modes (we will discuss this later). How do you excite *only one* of the modes? There are three different methods:

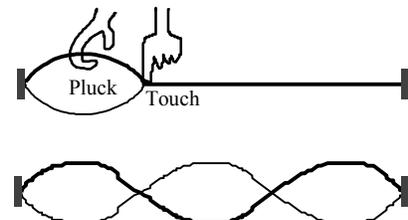
#### 1. The Mathematician Method: “Sine Curve Initial Condition”

If you pull each mass element of a stretched string away from equilibrium (flat string) so that the string forms the shape of a *sine curve*, and then let go, the whole string will vibrate in one normal mode pattern. If you start with any other initial shape – one that is *not* sinusoidal – then the motion of the string will be made up of different modes. Starting out with an exact sine-curve shape is not easy to do – you need some kind of fancy contraption.



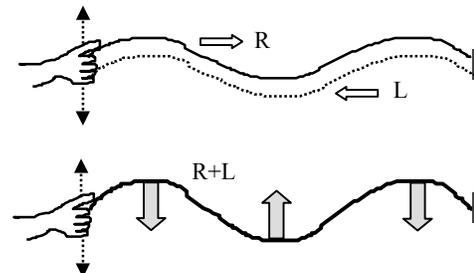
#### 2. The Musician Method: “Touch and Pluck”

Guitar players and violin players do this all the time. They gently touch the string at one point (where you want the node to be) and pluck the string at another point (antinode) to make a “loop”. The oscillation (up and down motion) of the plucked loop will “drive” the rest of the string to form additional equal-size loops which oscillate up and down at the same frequency as the driving loop.



#### 3. The Physicist Method: “Resonance”

If you gently shake (vibrate) the end of a string up and down, a wave will travel to the right (R), hit the fixed end, and reflect back to the left (L). If you shake at just the right “resonance” frequency – one that matches one of the natural frequencies of the string – then the two *traveling waves* (R and L) will combine to produce a *standing wave* of large amplitude:  $R + L = \text{STANDING WAVE}$ .



### D. Resonance

Since RESONANCE is one of the most important concepts in science, we will focus on this method. Resonance phenomena are everywhere: tuning a radio, making music, shattering a crystal glass with your voice, imaging the body with an MRI machine, picking cherries, designing lasers, engineering bridges, skyscrapers, and machine parts, etc. Consider pushing a person in a swing. If the frequency of your hand (periodic driving force) matches the natural frequency of the swing, then the swing will oscillate with large amplitude. It is a matter of timing, not strength. A sequence of “gentle pushes” applied at just the “right time” – in perfect rhythm with the swing – will cause a dramatic increase in the amplitude of the swing. A small stimulus gets *amplified* into a LARGE response.

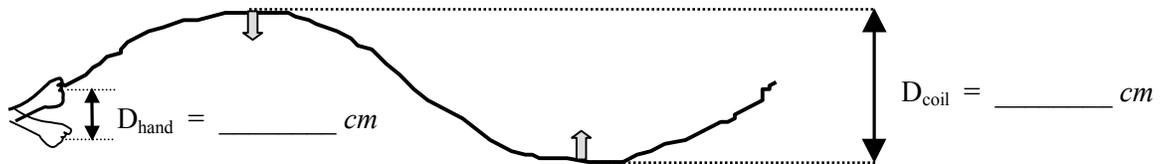
#### Experiment: Shake a Slinky, Make a Mode

Go into the hallway. Stretch the slinky (in the air, not on the floor) so that its end-to-end length is 4 to 5 m. It is okay if the slinky sags a little. Keep one end fixed. **GENTLY** shake (vibrate) the other end *side-to-side* (side-to-side is easier than up-and-down). **DO NOT** shake with a large

force or large amplitude. Remember that resonance is all in the *timing*, not the force! Shake at just the *right frequency* to produce the two-loop ( $n = 2$ ) normal mode. When you have “*tuned into*” this  $n = 2$  state, take special note:

You are now RESONATING with the coil. You and the coil are “one” !

Your shaking hand is perfectly in-sync with the coil’s very own natural vibration. The frequency of your hand matches  $f_2$  of the coil. Note how the distance  $D_{\text{hand}}$  that your hand moves back and forth is much smaller than the distance  $D_{\text{coil}}$  that the coil moves back and forth. The driven slinky system has the ability to *amplify* a small input ( $D_{\text{hand}}$ ) into a large output ( $D_{\text{coil}}$ ). This is the trademark of resonance: a weak driving force (hand moves a little) causes a powerful motion (coil moves a lot). Estimate the values of  $D_{\text{hand}}$  and  $D_{\text{coil}}$  by simply observing the motions of your hand and the coil when you are resonating with the coil. Compute the “*Amplification Factor*” for this driven coil system.



*Amplification Factor* :  $D_{\text{coil}} / D_{\text{hand}} = \underline{\hspace{2cm}}$  .

## Part II. Why are Modes Important?

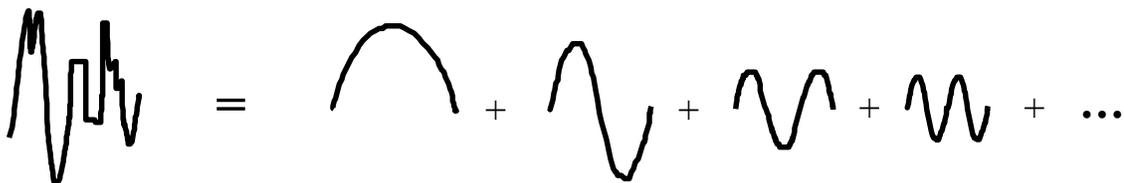
One of the most far-reaching principles in theoretical physics is this:

*The general motion of any vibrating system can be represented as a sum of mode motions.*

Reducing the complex “generic whole” into a set of simple “harmonic parts” provides deep insight into nature. In essence, normal modes are important for two reasons:

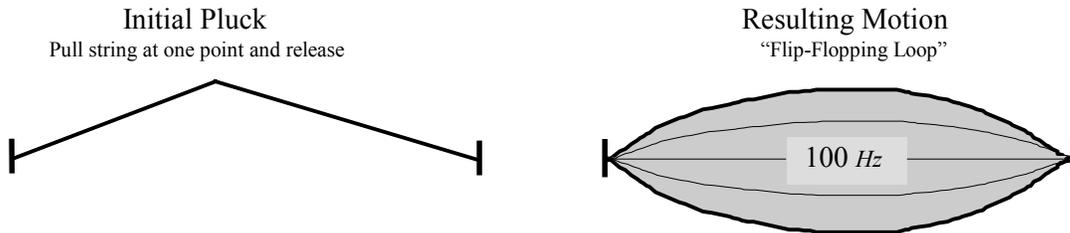
- (1) The motion of each mode is SIMPLE , being described by a *simple harmonic oscillator* (trig) function:  $\cos(2\pi ft)$  .
- (2) The set of modes serves as a BASIS for any kind of wave/vibrational motion:

*Any Motion* = *Sum of Mode Motions*

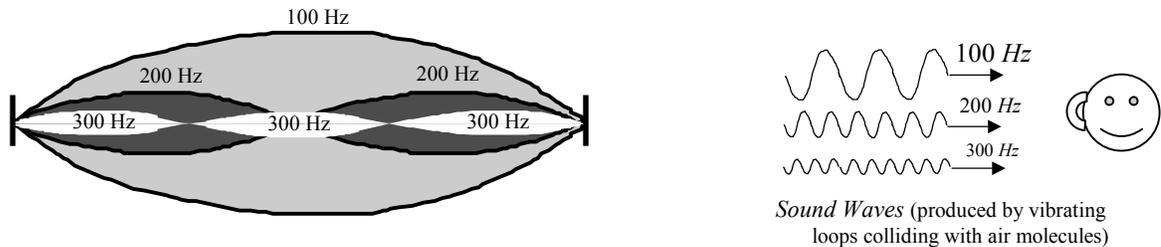


## A. Plucked String = Mixture of Harmonic Parts

Suppose you pluck the “A-string” of a guitar and look closely at the shape of the vibrating string. You will see the fundamental mode (1st harmonic) “flip flopping” at the rate of about 100 vibrations per second. At such a high frequency, the string looks like one big blurred loop:



However, what you see is **NOT THE WHOLE PICTURE !** In reality, the string is vibrating with a whole spectrum of normal mode shapes and frequencies all at the same time: 1-loop @ 100 Hz , 2-loop @ 200 Hz , 3-loop @ 300 Hz , 4-loop @ 400 Hz , etc. The “actual” picture of a plucked string (showing only the first three harmonics) is:



In the language of *Music*, when you pluck a guitar string, the resulting musical tone consists of the **Fundamental Tone** (frequency  $f$ ) together with a whole series of higher pitched and generally fainter **Overtone**s ( $2f$ ,  $3f$ ,  $4f$ , etc.)

It is difficult to see the 2-loop, 3-loop, 4-loop vibrations , ... and to hear the corresponding overtones  $2f$ ,  $3f$ ,  $4f$ , ... because the higher-harmonic *amplitudes* are usually much smaller than the 1<sup>st</sup> harmonic *amplitude*. The 1-loop vibration is the shape you tend to see (and the tone you tend to hear) because it is the “loudest” component in the harmonic mixture – the one that “shakes” the air and your eardrum the most. If you listen carefully in a quiet room (no background noise) to the plucked A-string of a guitar, you can hear the first overtone 200 Hz (second harmonic) which is one “**Octave**” above the fundamental tone 100 Hz. Some musicians can hear several overtones.

Quality of Sound. The **pitch** of a musical note is equal to the frequency of the *fundamental tone*. The **quality of sound** is determined by the number and the strength of the *overtone*s that blend with the fundamental tone. When a guitar, piano, violin, and saxophone play the same note (middle C for example) at the same loudness, you hear four different sounds. Why? Although the frequency of the fundamental tone ( $f_1 = 262$  Hz) is the same, the *intensities* of the overtones ( $f_2 = 524$  Hz ,  $f_3 = 786$  Hz ,  $f_4 = 1048$  Hz , ...) are different for each instrument.

Proving the Fact that a “Plucked String = Mixture of Harmonic Modes”

Here is a thought experiment to prove this amazing fact of MUSIC/PHYSICS. A plucked guitar string is vibrating in the lowest two modes: 100 Hz *fundamental* and 200 Hz *overtone (octave)*. Suppose you lightly touch this vibrating string at its exact midpoint. What happens? What do you hear? Answer these questions by drawing a picture of the string modes before and after the touch. *Hint*: The midpoint of a string is an antinode for the first harmonic and a node for the second harmonic.



What happens to the fundamental loop when you touch the string?

What tone do you hear after the touch?

**B. Fourier’s Theorem. Harmonic Analysis.**

How do you “really know” that a plucked string is made up of a bunch of harmonic parts:



*Fourier’s Theorem* says that any periodic motion can be represented as a sum of simple harmonic motions. “*Fourier analysis*” or “*harmonic analysis*” refers to the decomposition of a vibration (periodic function) into its harmonic components (sine functions). Fourier analysis permeates all of science and engineering. This far-reaching mathematical tool is used in applications ranging from building a music (Moog) synthesizer, to designing circuits, to analyzing waveforms, to processing signals in electronics, to processing light in optical systems, and to calculating a quantum jump in atomic systems.

Computer Experiment: Adding Harmonics

You will explore how just the right mixture of smooth harmonic modes (sine waves) can produce the pointed shape of a plucked string (triangular wave). Start the program *Logger Pro* and open the file “*Fourier*”. You will see three graphs showing the first three *base shapes* (harmonics) of a guitar string of length 100 cm. Note that the first harmonic  $y_1$  is “loud”, while the second and third harmonics ( $y_2$  and  $y_3$ ) are “fainter”. Use the *Examine Icon* [x=?] to find the *amplitudes* ( $A_1, A_2, A_3$ ) and the *wavelengths* ( $\lambda_1, \lambda_2, \lambda_3$ ) of each harmonic.

$$A_1 = \underline{\hspace{2cm}} . \quad A_2 = \underline{\hspace{2cm}} . \quad A_3 = \underline{\hspace{2cm}} .$$

$$\lambda_1 = \underline{\hspace{2cm}} . \quad \lambda_2 = \underline{\hspace{2cm}} . \quad \lambda_3 = \underline{\hspace{2cm}} .$$

Use the *Examine Icon* to find the y-values of the harmonic functions for each of the x-values listed in the following table. Use your calculator to sum the y-values.

*Fourier Analysis Table*

Point on String x (cm)	1 <sup>st</sup> Harmonic y <sub>1</sub> (cm)	2 <sup>nd</sup> Harmonic y <sub>2</sub> (cm)	3 <sup>rd</sup> Harmonic y <sub>3</sub> (cm)	Sum of Harmonics y <sub>1</sub> + y <sub>2</sub> + y <sub>3</sub> (cm)
0				
5				
10				
15				
20				
25				
50				
70				
90				
100				

Use Graphical Analysis to plot  $y_1 + y_2 + y_3$  (vertical axis) versus x (horizontal axis). Change the y-axis range to go from 0 to 50. PRINT this *Sum-Over-Harmonics* graph.

Where is the Guitar String Plucked?

Based on the shape of your *Sum-Over-Harmonics* graph, what point on the string ( $0 < x < 100$ ) is pulled and released? x = \_\_\_\_\_ .

**Fourier Essence.** Note how each harmonic part ( $y_1, y_2, y_3$ ) consists of simple wavy curves. In sharp contrast, note how the sum of the harmonics ( $y_1 + y_2 + y_3$ ) consists of straight lines meeting at a sharp edge. This is a remarkable feature of adding sine curves:

*By compounding smooth and rounded curves, one can generate straight lines and acute angles !*

As you add more and more harmonics, the edges become sharper and the lines become straighter. Open the file “*Adding Sines*” to see the sine curves  $y_1, y_2, y_3$  plotted on the same graph. PRINT this graph. Carefully look at  $y_1, y_2, y_3$  to understand the “*Fourier Essence*”. Note how in a certain region all the sine curves bend in the same direction. When this happens, the *curvatures mutually reinforce each other* producing a huge curvature, that is an acute angle. In other places, the curves bend in opposite directions – the *curvatures mutually weaken each other* – so that a straight line results.

- (1) Mark these two special regions on your printed graph of the sine curves  $y_1, y_2, y_3$ .
- (2) Label one region “*Reinforcing Curvatures*” and the other “*Canceling Curvatures*”.
- (3) Sketch  $y_1 + y_2 + y_3$  on this graph.

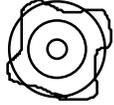
### C. Proper Tones: Every Object in Nature has its Very Own “Ring”

When you drop a spoon and a pencil on the floor, you hear two different sounds. When you pluck a violin and a cello (or blow into a flute and an oboe, or hit a drum and a bell) you hear two different tones. When you jolt a *Neon* atom and a *Mercury* atom (in a high voltage lamp), you see two different colors of light (red and blue).

*In general, every object in Nature has its very own set of “natural frequencies” at which it vibrates when disturbed (plucked, bowed, hit, blown, jolted, etc).*

The special vibration frequencies that identify a given object – the “acoustic fingerprint” of the object – are called the Proper Frequencies (or Proper Tones or Eigenfrequencies) of the object. This sound spectrum is *not* always *harmonic* – *not integer multiples* of the fundamental frequency. The proper “tones” of an atom are the frequencies of light emitted by the vibrating electron cloud of the atom. Each atom (H, C, N, O, ...) has a different light spectrum (optical fingerprint). Atomic spectra are always inharmonic.

The following table lists the proper frequencies of the lowest four modes of vibrations of several objects.

Pluck String	Blow Air in Pipe	Hit Drumhead	Hit Tuning Fork	Jolt H Atom
				
100 Hz	100 Hz	100 Hz	100 Hz	$4.6 \times 10^{14}$ Hz Red
200 Hz	200 Hz	160 Hz	630 Hz	$6.2 \times 10^{14}$ Hz Green
300 Hz	300 Hz	215 Hz	1720 Hz	$6.9 \times 10^{14}$ Hz Blue
400 Hz	400 Hz	230 Hz	3370 Hz	$7.3 \times 10^{14}$ Hz Violet
⋮	⋮	⋮	⋮	⋮

Why do percussion instruments (drums, forks, bells, ...) sound “*less harmonious*” than string and wind instruments (guitars, violins, flutes, clarinets, ...) ?

**Frequency of Object  $\sim 1/\text{Size of Object}$ .** The proper tones of an object depend on the size, shape, mass, and elasticity of the object. For a string,  $f \sim 1/L$ . In a musical instrument, the size (length) of the mode-filled region is changed by pressing frets, opening valves, covering holes, and moving tubes. In general, the fundamental vibration frequency of an object is inversely proportional to the size of the object.

*Large* objects (tuba, trombone, cello, large bell, bowling ball, ...) produce *low*-pitched sounds. *Small* objects (flute, piccolo, violin, small bell, coin, ...) produce *high*-pitched sounds. Producing higher frequencies requires smaller objects. Vibrating *crystals* ( $10^{-2}$  m) emit *radio waves* ( $10^6$  Hz). Vibrating *atoms* ( $10^{-8}$  m) emit *heat waves* ( $10^{12}$  Hz). Vibrating *electrons* ( $10^{-10}$  m) emit *light waves* ( $10^{14}$  Hz).

**Biophysics Experiment.** Say the vowels *a, e, i, o, u*. Carefully note how you change the size and shape of your mouth cavity (using your jaw, tongue, and lips) to produce each vowel sound. The *highest* frequency vowel “*eee*” corresponds to the *smallest* cavity size.

For which vowel do you make the largest mouth cavity? \_\_\_\_\_ .

## D. Three Applications

### 1. Guitar Tone

The length of a guitar string is  $65\text{ cm}$ . The E-string (the thickest string) on a guitar has a mass of  $5.4\text{ grams}$  and is under a tension of  $95\text{ Newtons}$ . Because of this force and mass, the velocity of a wave on the E-string is  $107\text{ m/s}$ . Calculate the frequency of the fundamental tone that is produced when you pluck the E-string of a guitar. Show your calculation.

*Wave Facts:* (1) velocity = wavelength  $\times$  frequency. (2) In the fundamental mode, the length of the string forms one loop =  $\frac{1}{2}$  wavelength.

$$f_1 = \text{_____ } Hz .$$

### 2. Flute Length

A flute (open pipe) is designed to play middle C ( $262\text{ Hz}$ ) as the fundamental tone when all the holes in the flute are covered. Calculate the length of a flute from the mouthpiece (one open end) to the far end (other open end). Show your calculation. Is your answer reasonable?

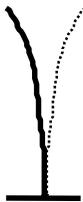
*Wave Facts:* The velocity of a wave in the pipe (*sound in air* at  $20^\circ\text{C}$  and  $1\text{ atm}$  pressure) is  $343\text{ m/s}$ . For any periodic wave, velocity = wavelength  $\times$  frequency. For the fundamental mode, one-half of a wave-length just fits inside the pipe length.

$$L = \text{_____ } cm .$$

### 3. Rod Modes

A tuning fork consists of two vibrating rods. Vibrating rods are everywhere in nature: metal antennas, wind chimes, wood beams, flag poles, glass rods, xylophone bars, tall buildings, long bridges, water towers. The following sketch shows the first two modes of a tuning fork rod that is fixed at one end (*node*) and free at the other end (*antinode*). Sketch a picture of the third mode.

Mode 1 ( $100\text{ Hz}$ )



Mode 2 ( $630\text{ Hz}$ )



Mode 3 ( $1720\text{ Hz}$ )



Mode 2 is called the “*Clang Mode*”. Because of the high inharmonic overtone frequencies of a tuning fork ( $f_2 = 6.3f_1$  for a fork, compared to  $f_2 = 2f_1$  for a string), a tuning fork makes a bright clinking (sharp metallic) sound when struck.

## Part III. Discovering a Harmonic Spectrum

### A. Experiment: Measuring the Spectrum

Use the slinky as before – keep one end fixed, make the length 4 to 5 meters, shake the other end. Record the length:  $L = \underline{\hspace{2cm}} m$ . Shake at the “right frequencies” to create the first three harmonics  $n = 1, 2, 3$ . Note: the uncertainty in measuring the integer  $n$  is exactly equal to zero:  $n \pm 0$ !

For each harmonic, measure the period of your oscillating hand. Note that the period of your hand is equal to the period of the “flip-flopping” loops on the slinky. To find the period, use a *stopwatch* to measure the time it takes your hand to complete ten side-to-side motions. Divide by ten to get the period. Summarize your results in the following table. Sketch the shape of each mode you observed. Record your measured values of the period and the frequency of each mode.

Mode Number	Mode Shape	Period (s)	Frequency (Hz)
1			
2			
3			

Do your measured frequencies satisfy the harmonic law  $f_n = nf_1$ ? Explain.

### B. Theory: Calculating the Spectrum

So far, you have *measured* the spectrum of natural frequencies of the slinky system. You will now *calculate* this spectrum based on the theory of waves. There are three basic ingredients that make up the theory of wave motion:

Kinematics:  $v = \lambda f$  (since *velocity* = *distance over time* =  $\lambda/T$ ).

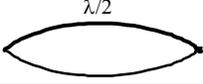
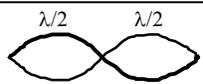
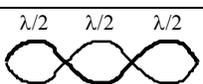
Dynamics:  $v = (F/\mu)^{1/2}$  (from solving  $F = ma$  for a string with *tension*  $F$  and *mass density*  $\mu$ ).

Harmonics:  $n(\lambda/2) = L$  (Mode  $n$  consists of “ $n$  half-wavelengths just fitting inside length  $L$ ”).  
For a proof of this “fitting condition”, see the mode pictures on page 1.

*Note*: Even though a standing wave does not travel, the concept of velocity still makes sense because when you shake a string, you generate a traveling wave that propagates along the string. The velocity  $v$  of this traveling wave (whose back-and-forth motion forms the standing wave) defines the “ $v$ ” that appears in the standing wave equation  $f_n = nv/2L$ .

**Derive Frequency Formula.** If you combine the *kinematic* relation  $v = \lambda f$  with the *harmonic* relation  $n(\lambda/2) = L$ , then you get the *frequency spectrum* of any string-like object:  $f_n = \underline{nv/2L}$ .

Do not forget the physics behind this important spectral formula: the frequency of any periodic wave is  $f = v/\lambda$  and the wavelength of a standing periodic wave ( $n$   $\frac{1}{2}$ -wave loops) is  $\lambda = 2L/n$ . The following table summarizes the mode physics of a vibrating string (showing only  $n=1,2,3$ ).

Mode	Wavelength ( $n$ $\frac{1}{2}$ -waves = string length)	Frequency $f = v/\lambda$
1 <sup>st</sup> Harmonic 	$1 \lambda/2 = L$	$f_1 = 1 v/2L$
2 <sup>nd</sup> Harmonic 	$2 \lambda/2 = L$	$f_2 = 2 v/2L$
3 <sup>rd</sup> Harmonic 	$3 \lambda/2 = L$	$f_3 = 3 v/2L$

*Bottom Line:* The overtones are integer multiples of the fundamental:  $f_n = n f_1$ , where  $f_1 = v/2L$ . So to calculate the entire theoretical spectrum (an *infinite* number of frequencies  $f_n = n v/2L$ ), all you need to know are the values of two kinematic quantities: L and v !

**Record L.** Length of stretched slinky on which you observed  $n=1,2,3$ :  $L = \underline{\hspace{2cm}} m$ .

**Find v.** The velocity of a wave on a string (or coil) does not depend on the shape of the wave. All small disturbances propagate along the string at the same speed. So the best way to measure the value of  $v$  is to make a wave which is easiest to observe. The simplest kind of wave is a *single pulse*.

Stretch the coil as before so that it has the same length  $L$ . Keep *both* ends fixed. At a point near one end, pull the slinky sidewise (perpendicular to the length) and release. This will generate a *transverse traveling wave*. Observe this wave disturbance (pulse) as it travels down the slinky, reflects off the other end, and travels back to its point of origin. Use a *stopwatch* to measure the time it takes for the pulse to make four round trips, i.e. 4 down-and-back motions along the entire length of the coil. Divide by 4 to get the round-trip time  $t_r$ . Calculate the velocity  $v$  of the wave on your coil using the fact that the wave pulse travels a distance  $2L$  (down  $L$  + back  $L$ ) during the round trip time.

Round-Trip Time  $t_r = \underline{\hspace{2cm}} s$ . Wave Velocity  $v = \underline{\hspace{2cm}} m/s$ .

**Calculate f.** From your measured values of  $L$  and  $v$ , calculate the harmonic frequencies  $f_1, f_2, f_3$  using the *theoretical* relation derived above. Show your calculation:

### C. Compare Experimental and Theoretical Spectrum.

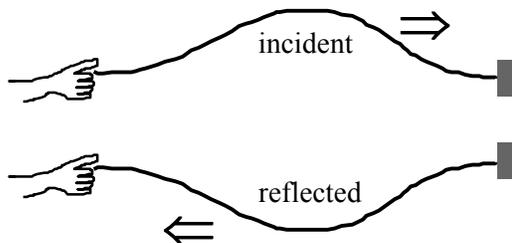
Summarize your measured (experiment) and calculated (theory) values of the natural frequencies  $\{f_1, f_2, f_3, \dots\}$  that characterize the slinky system.

$n$	$f_n$ (experiment)	$f_n$ (theory)	% Difference
1			
2			
3			

### D. What is Happening at Resonance? The “Timing Condition”.

As you shake one end of a string (or slinky), a *traveling wave* is sent along the string and is then reflected at the other end. The incident and reflected waves overlap (superpose) and add (interfere) to produce a resultant wave. In general, the result is a “jumbled” wave of small amplitude. However, if you *shake just right*, then the result is a highly-ordered standing wave of large amplitude.

What does “*shake just right*” mean? Remember that resonance is all in the *timing*. Suppose your hand generates a crest. When this crest reaches the fixed end, it is reflected as a trough. The returning trough reflects at your hand as a crest. If your hand starts to generate a second crest just as the trough arrives at your hand, the reflected crest and the second crest will be perfectly in step and reinforce each other.



When you create a standing wave by shaking a string, remember this:

Although it looks like the wave is “*standing still*”, what you see is the superposition (resultant sum) of two traveling waves that are “*moving fast*” in opposite directions!

The same physics explains how you can make “ocean waves” in a bathtub or a swimming pool! You must “*splash just right*”: Gently hit the water with your hand. Wait for the ripple to reach the wall and reflect back to you before you splash again. With this periodic timing of your splashes, your little out-going and in-going ripples will soon build up (via constructive interference) to form a huge standing wave!

Timing Condition: Two *traveling* waves (incident  $\Rightarrow$ , reflected  $\Leftarrow$ ) will combine to produce a *standing* wave if the following resonance (in-sync) condition is satisfied: *The round-trip time  $t_r$  of the traveling wave must equal an integer multiple of the period  $T$  of your hand.*

Note: The *Timing Condition*  $t_r = nT$  is equivalent to the *Fitting Condition*  $L = n\lambda/2$  discussed previously. Can you prove the equivalence?

#### Experimental Test of the Resonance (Timing) Condition

You have already measured the three special *periods*  $T$  of your *hand* that are necessary to create the standing wave modes  $n = 1, 2, 3$  on the slinky system. You have also measured the *round-trip time*  $t_r$  of a *traveling wave* on the slinky. Compute the ratio of your measured times:

$$\begin{aligned} n = 1: \quad t_r / T &= ( \quad ) / ( \quad ) = \\ n = 2: \quad t_r / T &= ( \quad ) / ( \quad ) = \\ n = 3: \quad t_r / T &= ( \quad ) / ( \quad ) = \end{aligned}$$

How well do your measured times satisfy the *Timing Condition*? Compare your experimental values of  $t_r/T$  with the theoretical values ( $t_r/T = n$ ):

## Part IV. Precision Measurement of “Proper Tones” and Wave Speed

The system you will study consists of a string of finite length under a fixed tension. One end of the string is connected to an electric oscillator – a high-tech “shaker”. Instead of *you* shaking the string by hand, the electric oscillator moves the string up and down at a frequency that you can control with *high precision*. The tension in the string is due to a precisely known mass hanging on the other end. Your “*research goal*” is to answer two basic questions:

1. What are the *natural frequencies* of this string system?
2. What is the *speed* of a wave on this system?

### Frequency Spectrum

Hang 150 *grams* from the end of the string.

Set up standing waves  $n = 1, 2, 3, 4, 5$  by driving the string with the electric oscillator using the following procedure.

*Procedure for Resonating with the String:* Start with the frequency of the oscillator equal to 1.0 *Hz*. Set the amplitude of the oscillator equal to one-half its maximum value. Look at the end of the string attached to the oscillator. It is going up and down once every second – just like your hand shaking the slinky! Now slowly increase the frequency until the first ( $n=1$ ) resonance is achieved where the string forms one big “flip-flopping” loop. If the oscillator makes a rattle sound, then decrease the amplitude. Continue to increase the frequency and successively “*tune into*” each of the overtones  $n = 2, 3, 4, 5$ . Record your measured values of  $f_1, f_2, f_3, f_4, f_5$ :

n	1	2	3	4	5
$f_n$ (Hz)					

Do these natural frequencies (*proper tones*) of the string system form a *harmonic spectrum*? Explain.

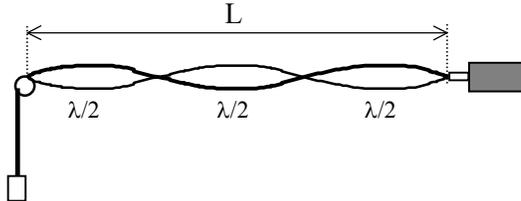
### Wave Speed

The wave speed on the string is too fast to measure with a stopwatch. “Flick” the string and try to observe the traveling pulse. Good Luck. Setting up standing waves is the gold-standard technique to measure the speed of any kind of wave – from sound in air (300 *m/s*) to light in a vacuum (300,000,000 *m/s*). When you set up a standing wave, you are in effect “freezing” (taking a snapshot of) the speeding wave so you can easily observe the *shape* of the wave and measure the *wavelength*.

To get an accurate value of the wave speed  $v$ , find  $v$  using *two* different methods:

**A. Space - Time Method:** Measure  $\lambda$  and  $f$  for each mode  $n$ . Compute  $v = \lambda f$ .

You have already measured  $f$  for each  $n$ . To find  $\lambda$  for each  $n$ , measure the length  $L$  of your string (between its two “fixed” endpoints) and then compute  $\lambda$  from the Fitting Condition: *mode  $n$  consists of  $n$  half-waves stuffed inside  $L$* . The picture below shows the  $n = 3$  fit.



$L = \underline{\hspace{2cm}} \text{ m} .$

$n$	$\lambda$ (m)	$f$ (Hz)	$v = \lambda f$ (m/s)
1			
2			
3			
4			
5			

average  $v = \underline{\hspace{2cm}} \text{ m/s} .$

**B. Force - Mass Method:** Measure  $F$  and  $\mu$  of string. Compute  $v = (F/\mu)^{1/2}$ .

The *tension*  $F$  in the string is equal to the weight of the hanging mass. To find the *mass density*  $\mu$  of the string ( $\mu \equiv \text{mass per unit length}$ ), use the sample string on the back table. Do not remove the string attached to your oscillator and weight.

*Note:* The sample string came from the same spool of string as the actual string and therefore has the same value of  $\mu$ . The sample string may have a larger *mass* and a larger *length* than the actual string, but the ratio *mass / length* is the same.

Tension:  $F = \underline{\hspace{2cm}} \text{ N} .$

Mass density:  $\mu \equiv \text{mass} / \text{length} = ( \quad \text{ kg} ) / ( \quad \text{ m} ) = \underline{\hspace{2cm}} \text{ kg/m} .$

Wave Speed:  $v = (F/\mu)^{1/2} = \underline{\hspace{2cm}} \text{ m/s} .$

Compare your two results for  $v$  [*space-time*  $\lambda f$  and *force-mass*  $(F/\mu)^{1/2}$  ]:

## Part V. Design Project: Create a 3-Loop Flip-Flop @ 50 Hz

The system consists of your stretched string, electric oscillator, and hanging weight. The application requires that you set up a *3-loop* standing wave pattern on the string that vibrates at the rate of *50 cycles per second*. The system parameter that you can change is the *tension* in the string.

### The Theory

First work out the theory, then do the experiment. Recall the three essential elements of wave theory: (1) *Kinematics*:  $v = \lambda f$ . (2) *Dynamics*:  $v = \sqrt{F/\mu}$ . (3) *Harmonics*:  $n\lambda/2 = L$ .

The *string* parameters are length  $L$ , tension  $F$ , mass density  $\mu$ , number of loops  $n$ . The *wave* properties are velocity  $v$ , frequency  $f$ , wavelength  $\lambda$ .

1. Carefully *derive* the formula that gives  $F$  as a function of the four parameters:  $n, L, f, \mu$ .

$F =$

2. Write the numerical values of the parameters:

$$n = \underline{\quad} . \quad L = \underline{\hspace{2cm}} \text{ m}. \quad f = \underline{\hspace{2cm}} \text{ Hz}. \quad \mu = \underline{\hspace{2cm}} \text{ kg/m}.$$

3. Substitute these values into your  $F$  formula to find the tension value:  $F = \underline{\hspace{2cm}} \text{ N}$ .

4. Compute the value of the hanging mass:  $m = \underline{\hspace{2cm}} \text{ kg}$ .

### The Experiment

1. Set the tension in the string by hanging the amount of mass as predicted by your theory.
2. Turn on the oscillator.
3. Vary the frequency of the oscillator so that the 3-loop mode is established on the string "*loud*" and *clear*.
4. Record the value of this 3-loop frequency:  $f_3 = \underline{\hspace{2cm}} \text{ Hz}$ .
5. Compare this experimental value of  $f_3$  with the *Design Specs* value of 50 Hz.

$$\% \text{ difference between } f_3 \text{ and } 50 \text{ Hz is } \underline{\hspace{2cm}} \%$$