Entanglement Evolution in a Heisenberg Spin Dimer

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The transitory and steady-state entanglement of a two-qubit XY Heisenberg magnetic spin system immersed in an external magnetic field are analyzed in terms of the entanglement measure of concurrence. The dependence of the system entanglement on coupling strength, decay rate, and applied magnetic field is numerically simulated for two distinct models of system evolution involving decoherence to the environment. An applied magnetic field was found to consistently reduce the system entropy. The external magnetic field increases entanglement for small strength, and if too strong, it reduces the entanglement. The maximum entanglement is formed from a balance of external magnetic field and the decay rate. The transitory level of entanglement in the system is found to oscillate with a frequency that is proportional to the magnetic field strength before reaching steady-state entanglement. It was found that collective decay produces more entanglement than individual decay.

Index Terms—**Entanglement, Heisenberg, magnetics, spin-dimer.**

I. Introduction

Quantum entanglement is an important phenomenon in the study of quantum systems and in applications of quantum information. Entanglement is a quantum mechanical phenomenon in which the individual quantum states of a set of qubits cannot be measured independently. Entanglement is a correlation between the measurement of the two qubits as measured on the same basis. Quantum entanglement has been demonstrated to exist between qubits even at large separations, for instance, distances such as between a satellite and the surface of Earth [1].

Heisenberg spin chains are important systems which manifest entanglement phenomena. Physical realizations of spin chains could potentially provide applications to quantum computing [2], [3]. Experimentally many molecules have been shown to behave as spin chains [4]–[7], which are well characterized by the Heisenberg spin chain model.

This article investigates the behavior of a Heisenberg spin dimer immersed in an external magnetic field. Entanglement in such systems has been studied for various models [8]–[10]. The additional insight presented in this article is to quantify the role of the external magnetic field and decay in terms of entanglement creation.

II. Model

The model of a Heisenberg XY spin dimer composed of two qubits with nearest-neighbor interactions immersed in an external magnetic field is given by

$$H_{xy} = \sum_{n=1}^{2} J_x S_n^x S_{n+1}^x + J_y S_n^y S_{n+1}^y + B S_n^z \tag{1}$$

where $S_n^{(x,y)} = (1/2)\sigma_n^{(x,y)}$ with $\sigma_n^{(x,y)}$ denoting the three Pauli spin matrices and n denoting the nth position on the spin chain. The J's (J_x, J_y) denotes the magnitude of the exchange interaction between two qubits in each dimension with positive values denoting anti-ferromagnetic interaction and negative values denoting ferromagnetic interaction [10]. The external magnetic field is directed along the z-axis which is described by the term (BS_n^z) in the Hamiltonian, where B is the strength of the magnetic field.

If the following parameters are defined as $J = (J_x + J_y)/2$ and $r = (J_x - J_y)/(J_x + J_y)$, and the raising and lowering operators for the spin system $(S^{\pm})$ are defined according to: $S_n^{\pm} = S_n^x \pm i S_n^y$, then (1) can be rewritten as [10]:

$$\begin{aligned} H = {} & B(S_1^z + S_2^z) + J(S_1^+ S_2^- + S_1^- S_2^+) \\ & + Jr(S_1^+ S_2^+ + S_1^- S_2^-). \end{aligned} \tag{2}$$

The system state is described in terms of a density matrix ρ which evolves according to the master equation either by individual decay [see (3)] or collective decay [see (4)]. Individual decay means the qubit transitions to the ground state uncorrelated to each other. The collective decay means the transition of both qubits coupled to an optical cavity mode [11], [12]

$$\frac{d\rho}{dt} = -i[H, \rho] + \gamma \mathcal{D}[S_1^-]\rho + \gamma \mathcal{D}[S_2^-]\rho \tag{3}$$

$$\frac{d\rho}{dt} = -i[H, \rho] + \gamma \mathcal{D}[S_1^- + S_2^-]\rho. \tag{4}$$

In (3) and (4), the Lindblad superoperator describes the irreversible decay process of the system to the environment and is defined as

$$\mathcal{D}[\mathcal{A}]\mathcal{B} = ABA^\dagger - \frac{1}{2}\{A^\dagger AB\}. \tag{5}$$

The parameter γ defines the decay rate of the two qubits.

The above master equation can be solved numerically to obtain the time evolution of the density matrix and the steady-state solution. The initial state of the system is an unentangled ground state which is defined as $|\Psi\rangle = |gg\rangle$. Several measures of the degree of entanglement in a quantum system exist. Experimentally, quantum entanglement is confirmed by violating the Bell inequality for a classical system [13]. In this article, entanglement is measured using concurrence [14].

Manuscript received January 6, 2019; revised March 14, 2019; accepted September 14, 2019. Date of current version November 18, 2019. Corresponding author: J. Wang (e-mail: jinwang@umich.edu).

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Digital Object Identifier 10.1109/TMAG.2019.2943126

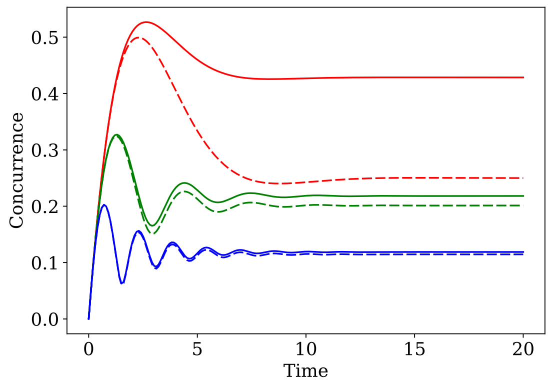

Fig. 1. Concurrence versus time for collective and individual decay. The system initialized in an unentangled state with $J = 0.5$, $r = 0.5$, and $\gamma = 0.5$ for three different magnetic field values (B). The values of B are 0, 1, and 2 from the top to the bottom of the figure. The solid lines indicate results from a collective decay model and the dashed lines indicate results from an individual decay model.

The concurrence is defined by (6) where the λ's represent the eigenvalues of the matrix $\widetilde{\rho}$ which is constructed from the system density matrix ρ. This is the solution to the master equation

$$C = \max(\sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}, 0) \tag{6}$$

$$\widetilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y). \tag{7}$$

One more measure that gives more insight into the system behavior is called entropy. The definition of Von Neumann entropy is given by

$$S(\rho) = Tr(\rho \log(\rho)) = \sum_i \lambda_i \log(\lambda_i). \tag{8}$$

Given the density matrix ρ describing the time evolution of a two-qubit system, a reduced density matrix for either qubit can be formed by applying a partial trace to the density matrix over the basis of the other qubit.

III. Entanglement

The time evolution of the concurrence (entanglement) is presented in Fig. 1 for the two-qubit system in both the individual and collective decay models. The collective decay yields a higher overall entanglement for the same decay rate for individual decay as well as a similar effect of the external magnetic field. This could be due to the collective decay means the qubits decay into the same mode of the cavity which causes photon interactions between the two qubits to be more efficient. The photon interaction causes an entanglement between the qubits. Of course, decay will also cause the entanglement to be lost since not all the emitted photons will be absorbed by the partner qubit. This implies that beyond a certain amount of decoherence, the entanglement will decrease.

In Fig. 2, the dependence of the steady-state concurrence on the magnetic field and on the collective and individual decay rate γ is demonstrated. There are a few observations that can be found from analyzing Fig. 2. The first observation is that when the external magnetic field is zero, the entanglement is nonzero for most cases. Although not shown, in the case of no decay, there is no steady-state entanglement. This implies that both decay and the interaction Hamiltonian are needed to create steady-state entanglement when the B field is zero. This also suggests photons mediate the entanglement between the qubits. The second observation is the role of the magnetic field in the creation of entanglement is multifaceted. When the magnetic field is absent, the qubit spins are randomly oriented, so the photon mediated interaction is not efficient, but is critical to create entanglement. For a field greater than zero increasing to a critical threshold value, the entanglement increases. This is because the qubit spins will tend to align to the external magnetic field, and the photons emitted by decay will more efficiently interact with each other than when the spins are randomly oriented in the absence of field. For a magnetic field at threshold strength, it yields the most entanglement due to the alignment effect and the interaction between the qubits is still significant compared with the external field. However, as the magnetic field strength increases beyond the threshold value, the entanglement gradually decreases since the external field overshadows any interaction between the individual qubits.

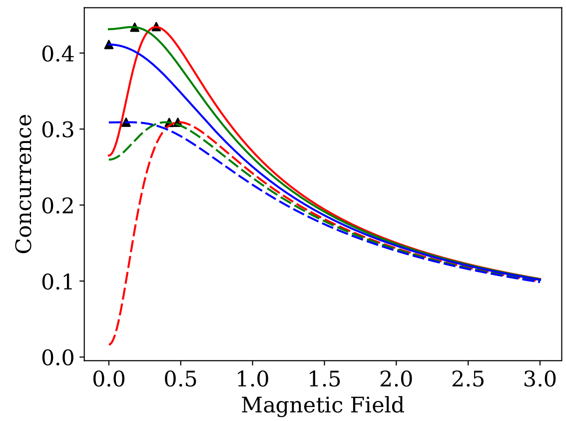

Fig. 2. Steady-state concurrence as a function of B for both collective (solid line) and individual decays (dashed line). The parameters used in the system are $J = 0.8$, $r = 0.4$, and $\gamma = 1/3$, $2/3$, and 1 are colored red, green, and blue, respectively. The maximum concurrence occurs from left to right with $\gamma = 1$, 2/3 and 1/3. The triangles on each line represent the maximum entanglement over the range of applied magnetic field strengths.

In Fig. 3, the dependence of the entropy of the system on the magnetic field and on the collective and individual decay rate γ is plotted. It should be noted that the steady-state entropy follows a simple dependence on γ. The lower the value of γ, the higher the entropy, and vice versa. Entropy consistently decreases as the external magnetic field is increased. This is due to the qubits' spins are increasingly aligned with the external field which causes the system to be more pure. The concurrence compared to entropy is anti-correlated for low magnetic field strength and transitions to be correlated for high magnetic field strength. The anticorrelation at low magnetic field strength is due to the external field aligning the qubits and enhancing the interaction between the qubits which creates more entanglement. The reduced correlation at high magnetic field strength is due to the strong interaction strength between the external magnetic field and the qubits being much greater than the interaction between the qubits. Similarly, the concurrence also increases then decreases as the decay rate γ continuously increases. This effect is likely due to the photons emitted due to decay facilitates coherence between the qubits and enhances the entanglement. However,

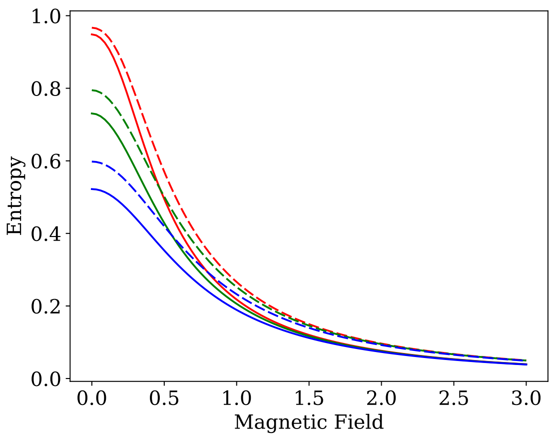

Fig. 3. Steady-state entropy as a function of B for both collective (solid line) and individual decays (dashed line). The parameters and line descriptions are the same as in Fig. 2.

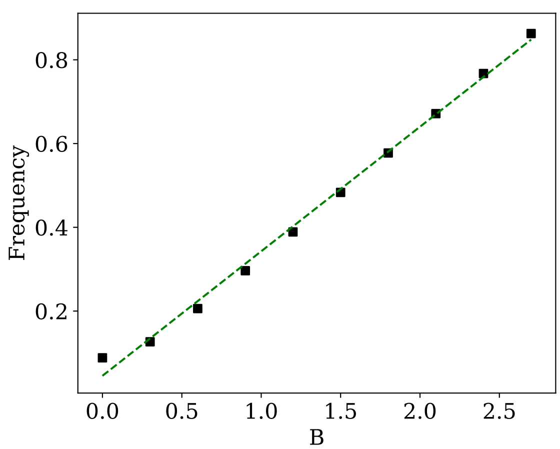

Fig. 4. Concurrence dynamics is simulated with collective decay for B-field values ranging from 0 to 2.7 with 0.3 increments for parameters: $J = 0.6$, $r = 0.5$, and $\gamma = 0.3$. The frequency of the concurrence oscillations versus the B-field value is plotted. The squares indicate the frequency obtained from the peak to peak separation in each time evolution. The dashed line represents a linear curve fit to the frequency versus B field results. The slope of the line is 0.297 and the intercept is 0.046.

if the photons leave the system at a too high of a rate the system will decohere to the ground state with no possibility of entanglement.

One interesting question to ask is what determines the transient oscillation frequency of the entanglement before it reaches the steady state. For a single qubit, the nuclear magnetic resonance (NMR) precession frequency of an atomic spin is linearly dependent on the magnetic field and the gyro-magnetic constant of the atom [15]. From the numerical results shown in Fig. 4, the transient entanglement oscillation frequency (as shown in Fig. 1) varies linearly with the applied magnetic field strength. The linear dependence on the magnetic field strength implies the oscillations are due to Larmor precession $\omega = \alpha B$. This suggests the two quantum level qubits are synchronized by a classical magnetic field, an effect suggested in prior work [16]. In other words, quantum entanglement can be created by a classical synchronization mechanism. Irradiation of spin chains at the resonant Larmor frequency for one or more of the spins in the spin chain is a potential means for controlling the state of the system and implementing CNOT gates in quantum computing [17].

IV. Conclusion

The dynamic entanglement in Heisenberg spin chains as quantified by the measure of concurrence involve damped oscillations. It was found that collective decay results in more entanglement than individual decay. The interaction between an external magnetic field and entanglement was investigated as well as the similar effect of decay rate. The transient oscillation frequency of the entanglement was found to vary linearly with the applied magnetic field. This behavior may be related to the proportionality between the Larmor precession frequency in nuclear spins in the presence of an applied magnetic field.

Acknowledgment

The authors would like to thank the Office of the Vice President Research, University of Michigan, for the research support.

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