A comparative study of the $P$ and $Q$ representations of a feedback controlled two-qubit system

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1. Introduction

A criteria to determine if a system has non-classical features is of increasing interest [1–10]. Two criteria that are commonly used are the Glauber and Sudarshan's $P$ representation [11] and entanglement [12,13]. The $P$ representation has been used as a criteria for determining if a system is non-classical in nature since it’s introduction. Since the recent development of quantum information and computation entanglement has been used as a quantitative measure of the quantum nature of a system with at least two qubits or two modes.

In previous investigations feedback has been used to increase the steady-state entanglement between two two-level systems (qubits) by a factor of three [14]. These results were found using a Master equation approach. The density matrix solution to the Master equation is represented by the quantum $Q$ representation. The $Q$ representation is a positive quasi-probability distribution function which is a projection of the density matrix onto a coherent state. The other commonly used quasi-probability distribution function is Glauber and Sudarshan’s $P$ representation. The quasi-probability formulation of quantum mechanics was first suggested by Schrödinger [15], and has subsequently been solidified by Wigner with quantum density matrices [16]. Later development of this formulation resulted in the $Q$ representation which is defined as the Fourier transformation of the anti-normal ordered characteristic function [17]. The $Q$ representation can be defined as $\pi Q(\alpha) = \langle \alpha | \rho | \alpha \rangle$, where $| \alpha \rangle$ is the coherent state. During the same time period, Glauber and Sudarshan introduced the $P$ representation for the probability density. The $P$ representation is defined as the Fourier transform of the normal-ordered characteristic function. The $P$ representation is known to be either negative or ill defined for states exhibiting non-classical behavior [11].

In prior work, steady-state entanglement creation was examined using a $Q$ representation and the quantum Master equation technique [14]. The purpose of the current Letter is to study the semiclasical limit of the Fokker–Planck equation corresponding to the quantum Master equation and express the results using the Glauber and Sudarshan’s $P$ representation. One goal of this study is to see to what extent and under what situation the agreement between quantum $Q$ function and the semi-classical $P$ function is reasonably good in the two-qubit model. The result of this investigation will shed light on quantum–classical correspondences and the effect of feedback control on the system stability.

2. Semiclassical $P$ representation approach to steady-state entanglement

The system to be considered is shown in Fig. 1, where two qubits are situated in a single-mode cavity. The cavity is pumped by an external driving laser. The output of the cavity is measured and the resulting photocurrent is fed back to modulate the driving laser such that the steady state entanglement is optimized.

A collective angular moment for the two qubits can be defined by arranging the Pauli matrices in the following way.

$$\hat{j}^\pm = \hat{\sigma}_1^\pm + \hat{\sigma}_2^\pm.$$  (1)

A comparison between the Bloch states found by the $P$ and $Q$ representations will be studied at the feedback and driving amplitudes that maximize the steady-state entanglement. This comparison requires using the $P$ representation of two qubit system. The
The driving Hamiltonian is rewritten as:

\[ H_D = \alpha J^+ + \beta J^- \]

Here the subscript \( \gamma \) means conditioned. This conditioning arises from the feedback modulating the system evolution based on the previous photocurrent record. The driving Hamiltonian is defined as \( H_\alpha = \alpha J^+ \), where \( \alpha \) is the driving amplitude and \( J^+ \) is the feedback operator, \( \lambda \) is the feedback amplitude. The operator \( H(A|B) \) is defined as \( H(A|B) = AB + BA^\dagger - \text{Tr}(AB + BA^\dagger) \). The homodyne photocurrent \( I(t) \) is defined as:

\[ I(t) = (J_x)(t) + \xi(t)/\sqrt{\gamma}. \]

The term \( \xi(t) \) represents Gaussian white noise, such that

\[ \xi(t) \, dt = dW(t), \]

is an infinitesimal Wiener increment defined by [20]

\[ E[dW(t)] = 0. \]

Here the symbol \( E \) represents ensemble average. The deterministic term \( \langle J_x \rangle(t) \) in Eq. (8) is a result of the interference effect between beam \( b \) and the local oscillator \( \beta \). The spontaneous emission terms are written in the Lindblad form [21]. The Lindblad superoperator \( \mathcal{D} \) is defined for arbitrary operators \( A \) and \( B \) as

\[ \mathcal{D}[A|B] = AB - \frac{1}{2} \{ A^\dagger A, B \}. \]

After the substitution of Eq. (2) into Eq. (7), the operation of \( J^+ \) and \( J^- \) on \( \Lambda(\zeta, \zeta^* \rangle \) in terms of differential operators needs to be found. This can be derived by noting that

\[ \frac{d\Lambda(\zeta, \zeta^* \rangle}{d\zeta} = \left( \frac{-2j\zeta^*}{1 + |\zeta|^2} + j^+ \right) \Lambda(\zeta, \zeta^* \rangle = \left( \frac{2j}{\zeta(1 + |\zeta|^2)} - \frac{1}{\zeta^2} J^- \right) \Lambda(\zeta, \zeta^* \rangle. \]

This above equation leads to the solution:

\[ \Lambda(\zeta, \zeta^* \rangle \Lambda(\zeta, \zeta^* \rangle = \left( \frac{d}{d\zeta} + \frac{2j\zeta^*}{1 + |\zeta|^2} \right) \Lambda(\zeta, \zeta^* \rangle. \]

Similarly we find

\[ J^+ \Lambda(\zeta, \zeta^* \rangle = \left( \frac{2j\zeta^*}{1 + |\zeta|^2} - \zeta^2 \frac{d}{d\zeta} \right) \Lambda(\zeta, \zeta^* \rangle. \]

Using the above derived operations, and rearranging the Master equation (7) with Eq. (2) substituted in, the partial integration equation (7) with Eq. (2) substituted in, the partial integration

\[ \frac{\partial P(\zeta, \zeta^* \rangle, t)}{\partial t} = \frac{\partial}{\partial \zeta^*} (A(\zeta, \zeta^* \rangle P(\zeta, \zeta^* \rangle) \]

where
\[ A_\xi(\zeta, \zeta^*) = -i\alpha + i\alpha\zeta^2 + \gamma \zeta + \lambda \frac{2\zeta^2}{1 + |\zeta|^2} + \frac{2\zeta^*}{1 + |\zeta|^2} - \frac{\lambda^2}{\gamma} \zeta^3 + \frac{\lambda^2}{\gamma} \zeta^*, \]
\[ A_{\xi^*}(\zeta, \zeta^*) = i\alpha - i\alpha\zeta^2 + \gamma \zeta + \lambda \frac{2\zeta^2}{1 + |\zeta|^2} + \frac{2\zeta^*}{1 + |\zeta|^2} - \frac{\lambda^2}{\gamma} \zeta^3 + \frac{\lambda^2}{\gamma} \zeta^*. \]

From this Fokker–Planck equation, the semiclassical equations of motion are found to be

\[ \frac{\partial \zeta}{\partial t} = -A_\xi(\zeta, \zeta^*), \]
\[ \frac{\partial \zeta^*}{\partial t} = -A_{\xi^*}(\zeta, \zeta^*). \]

Each of the above equations is the complex conjugate of the other, so it is sufficient to solve one of them. The steady-state solution on the Bloch sphere can be found by setting either \( A_\xi \) or \( A_{\xi^*} \) to zero. A numeric solution is found for the maximally entangled steady-state. The optimal driving and feedback amplitudes can be substituted into either of the above equations to get numerical results.

In the absence of feedback (\( \lambda = 0 \)), a maximum entanglement of 0.11 as measured by concurrence is found with a driving amplitude (\( \alpha = 0.38\gamma^{-1} \)), the two fixed points on the block sphere are found to be

\[ \theta_1 = 0.72\pi, \quad \phi_1 = -\frac{\pi}{2}, \]
\[ \theta_2 = 0.27\pi, \quad \phi_2 = -\frac{\pi}{2}. \]

The double solution to the equation indicates the semiclassical system is unstable at the point of maximum entanglement when feedback is absent. This result is consistent with the result found in [22]. In their model, the qubits are trapped ions driven simultaneously by a laser and coupled collectively to a heat bath.

In the presence of feedback, the optimized steady-state entanglement as measured by concurrence is 0.31 for driving and feedback amplitudes of \( \alpha = 0.4\gamma^{-1} \), \( \lambda = -0.8\gamma^{-1} \). After substituting this into Eq. (2) there is only one solution on the block sphere as indicated below

\[ \theta_1 = 0.94\pi, \quad \phi_1 = -\frac{\pi}{2}. \]

In the presence of feedback only one solution exists which indicates that the stability of the semiclassical system at the point of maximum entanglement is increased. This is due to the ability of feedback to actively cancel quantum noise to stabilize the system [23]. The above results with and without feedback are from the semi-classical solution of the Fokker–Planck equation using the \( P \) representation.

Now these semi-classical results are compared with the results from the quantum \( Q \) representation. The \( Q \) representation is the projection of the density matrix into a coherent state represented in polar coordinates, and is defined as

\[ Q(\theta, \phi) = \langle j, \theta, \phi | j, \theta, \phi \rangle. \]
function fail to differentiate between classical and non-classical effects. This paper exemplifies the use of the $P$ representation to determine whether a system is quantum in nature as a result of finding a negative probability.

In summary, this Letter explored using the semiclassical $P$ representation to determine the amount of quantumness a system exhibits. The results are compared with the quantum $Q$ function solution. Close agreement was found between the two methods when feedback is not present. However, when feedback was present a notable difference between the two solutions was found. The difference between the $P$ and $Q$ solutions can be used as measure of the quantum nature of the system. In addition feedback can stabilize the system from the perspective of the semi-classical $P$ representation. The author conjectures the increased entanglement in a two qubit system is due to the ability of feedback to stabilize the system.

Acknowledgements

The author would like to acknowledge valuable discussions with H.W. Wiseman and G.J. Milburn.

References