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A comparative study of the P and Q representations of a feedback controlled two-qubit system

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ABSTRACT

In this Letter the Master equation of a two qubit system is transformed into Fokker–Planck equations in order to find the Glauber–Sudarshan P function representation. For the two qubit system examined in this Letter, the P representation is ill defined, which indicates the system is non-classical. A qualitative measure of the non-classical nature of the system is found by taking the semi-classical limit of the Fokker–Planck equation and obtaining a simplified Glauber–Sudarshan P representation. The agreement between the simplified P representation and the Q representation as well as the system stability are discussed when feedback is present and absent.

1. Introduction

A criteria to determine if a system has non-classical features is of increasing interest [1–10]. Two criteria that are commonly used are the Glauber and Sudarshan's P representation [11] and entanglement [12,13]. The P representation has been used as a criteria for determining if a system is non-classical in nature since it's introduction. Since the recent development of quantum information and computation entanglement has been used as a quantitative measure of the quantum nature of a system with at least two qubits or two modes.

In previous investigations feedback has been used to increase the steady-state entanglement between two two-level systems (qubits) by a factor of three [14]. These results were found using a Master equation approach. The density matrix solution to the Master equation is represented by the quantum Q representation. The Q representation is a positive quasi-probability distribution function which is a projection of the density matrix onto a coherent state. The other commonly used quasi-probability distribution function is Glauber and Sudarshan's P representation. The quasi-probability formulation of quantum mechanics was first suggested by Schrödinger [15], and has subsequently been solidified by Wigner with quantum density matrices [16]. Later development of this formulation resulted in the Q representation which is defined as the Fourier transformation of the anti-normal ordered characteristic function [17]. The Q representation can be defined as $\pi Q(\alpha) = \langle\alpha|\rho|\alpha\rangle$, where $|\alpha\rangle$ is the coherent state. During the same time period, Glauber and Sudarshan introduced the P representation for the probability density. The P representation is defined as the Fourier transform of the normal-ordered characteristic function. The P representation is known to be either negative or ill defined for states exhibiting non-classical behavior [11].

In prior work, steady-state entanglement creation was examined using a Q representation and the quantum Master equation technique [14]. The purpose of the current Letter is to study the semiclassical limit of the Fokker–Planck equation corresponding to the quantum Master equation and express the results using the Glauber and Sudarshan's P representation. One goal of this study is to see to what extent and under what situation the agreement between quantum Q function and the semi-classical P function is reasonably good in the two-qubit model. The result of this investigation will shed light on quantum–classical correspondences and the effect of feedback control on the system stability.

2. Semiclassical *P* representation approach to steady-state entanglement

The system to be considered is shown in Fig. 1, where two qubits are situated in a single-mode cavity. The cavity is pumped by an external driving laser. The output of the cavity is measured and the resulting photocurrent is fed back to modulate the driving laser such that the steady state entanglement is optimized.

A collective angular moment for the two qubits can be defined by arranging the Pauli matrices in the following way.

$$\hat{J}^{\pm} = \hat{\sigma}_1^{\pm} + \hat{\sigma}_2^{\pm}. \tag{1}$$

A comparison between the Bloch states found by the P and Q representations will be studied at the feedback and driving amplitudes that maximize the steady-state entanglement. This comparison requires using the P representation of two qubit system. The

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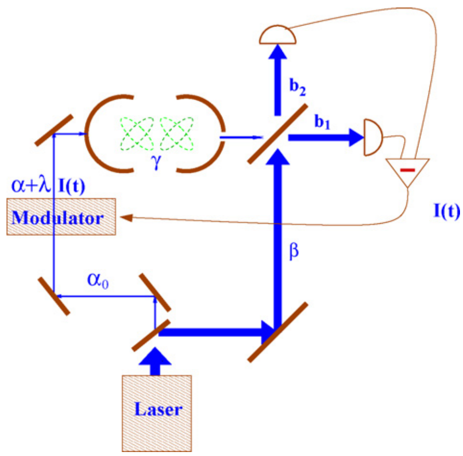

Fig. 1. Feedback diagram for dynamical creation of steady-state entanglement for two qubits. The laser beam is split to produce both the local oscillator β and the field α which is modulated using the current $I(t)$. The modulated beam, with amplitude proportional to $\alpha+\lambda I(t)$, drives two atoms at the center of the cavity formed by parabolic mirrors. Here λ is the feedback amplitude. The fluorescence thus collected is subject to homodyne detection using the local oscillator, and gives rise to the homodyne photocurrent $I(t)$.

P representation is found by deriving and then solving the Fokker–Planck equations corresponding to the Master equation.

To derive the Fokker–Planck equation, the density matrix is rewritten as:

$$\hat{\rho}(t)=\int P\left(\zeta,\zeta^{\dagger},t\right)\hat{\Lambda}\left(\zeta,\zeta^{\dagger}\right)d^2\zeta. \tag{2}$$

Here $P(\zeta,\zeta^{\dagger},t)$ is the Glauber–Sudarshan representation of the density operator [11], and the projection operator $\hat{\Lambda}(\zeta,\zeta^{\dagger})$ is defined as:

$$\hat{\Lambda}\left(\zeta,\zeta^{\dagger}\right)=|\zeta\rangle\langle\zeta|, \tag{3}$$

where $|\zeta\rangle$ is defined as an atomic coherent state [18,19]

$$|\zeta\rangle=\left(1+\left|(\zeta)\right|^2\right)^{-j}\exp\left[\zeta\hat{J}^{+}\right]|j,-j\rangle. \tag{4}$$

In the following $|j,m\rangle$ are the usual basis vectors which are eigenvectors of $\hat{J}_z=\sigma_{1z}+\sigma_{2z}$ and $\hat{J}^2=\frac{1}{2}(\hat{J}_+\hat{J}_-+\hat{J}_-\hat{J}_+)+\hat{J}_z^2$:

$$\hat{J}_z|j,m\rangle=m|j,m\rangle,$$
$$\hat{J}^2|j,m\rangle=j(j+1)|j,m\rangle. \tag{5}$$

The effect of $\hat{J}^{+}$ and $\hat{J}^{-}$ on the above states is given by

$$\hat{J}^{+}|j,m\rangle=\sqrt{(j-m)(j+m+1)}\,|j,m+1\rangle,$$
$$\hat{J}^{-}|j,m\rangle=\sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle. \tag{6}$$

As can be seen by this definition, $\hat{J}^{+}$ and $\hat{J}^{-}$ are the collective raising and lowering operators for the two qubit system.

To get the Fokker–Planck equation for the P representation, $\hat{\rho}(t)$ from Eq. (2) is substituted for ρ_c in the feedback controlled Master equation given below [14].

$$d\rho_c=dt\,\gamma\mathcal{D}\left[J^{-}\right]\rho_c-i\,dt\,[H_\alpha,\rho_c]-i\,dt\left[F,-iJ^{-}\rho_c+i\rho_cJ^{+}\right]$$
$$+dt\,\frac{1}{\gamma}\mathcal{D}[F]\rho_c+dW(t)\mathcal{H}\left[-i\sqrt{\gamma}J^{-}-i\lambda J_x\right]\rho_c. \tag{7}$$

Here the subscript c means conditioned. This conditioning arises due to the feedback modifying the system evolution based on the previous photocurrent record. The driving Hamiltonian is defined as $H_\alpha=\alpha J_x$, where α is the driving amplitude and $J_x=J^{+}+J^{-}$. The feedback Hamiltonian is $H_{\mathrm{fb}}=I(t)F$, where $F=\lambda J_x$ is the feedback operator, λ is the feedback amplitude. The operator $\mathcal{H}[A]B$ is defined as $\mathcal{H}[A]B\equiv AB+BA^{\dagger}-\mathrm{Tr}[AB+BA^{\dagger}]$. The homodyne photocurrent $I(t)$ is defined as:

$$I(t)=\langle J_x\rangle(t)+\xi(t)/\sqrt{\eta}. \tag{8}$$

The term $\xi(t)$ represents Gaussian white noise, such that

$$\xi(t)\,dt=dW(t), \tag{9}$$

is an infinitesimal Wiener increment defined by [20]

$$dW(t)^2=dt, \tag{10}$$
$$\mathrm{E}\left[dW(t)\right]=0. \tag{11}$$

Here the symbol E represents ensemble average. The deterministic term $\langle J_x\rangle(t)$ in Eq. (8) is a result of the interference effect between beam b and the local oscillator β. The spontaneous emission terms are written in the Lindblad form [21]. The Lindblad superoperator $\mathcal{D}$ is defined for arbitrary operators A and B as

$$\mathcal{D}[A]B\equiv ABA^{\dagger}-\frac{1}{2}\left\{A^{\dagger}A,B\right\}. \tag{12}$$

After the substitution of Eq. (2) into Eq. (7), the operation of $\hat{J}^{+}$ and $\hat{J}^{-}$ on $\hat{\Lambda}(\zeta,\zeta^*)$ in terms of differential operators needs to be found. This can be derived by noting that

$$\frac{d\hat{\Lambda}(\zeta,\zeta^*)}{d\zeta}=\left(\frac{-2j\zeta^*}{1+|\zeta|^2}+\hat{J}^{+}\right)\hat{\Lambda}(\zeta,\zeta^*)$$
$$=\left(\frac{2j}{\zeta(1+|\zeta|^2)}-\frac{1}{\zeta^2}\hat{J^{-}}\right)\hat{\Lambda}(\zeta,\zeta^*). \tag{13}$$

This above equation leads to the solution:

$$\hat{\Lambda}(\zeta,\zeta^*)\hat{J}^{+}=\left(\frac{d}{d\zeta}+\frac{2j\zeta^*}{1+|\zeta|^2}\right)\hat{\Lambda}(\zeta,\zeta^*),$$
$$\hat{\Lambda}(\zeta,\zeta^*)\hat{J}^{-}=\left(\frac{2j\zeta}{1+|\zeta|^2}-\zeta^2\frac{d}{d\zeta}\right)\hat{\Lambda}(\zeta,\zeta^*). \tag{14}$$

Similarly we find

$$J^{+}\hat{\Lambda}(\zeta,\zeta^*)=\left(\frac{2j\zeta^*}{1+|\zeta|^2}-\zeta^{*2}\frac{d}{d\zeta^*}\right)\hat{\Lambda}(\zeta,\zeta^*),$$
$$J^{-}\hat{\Lambda}(\zeta,\zeta^*)=\left(\frac{d}{d\zeta^*}+\frac{2j\zeta}{1+|\zeta|^2}\right)\hat{\Lambda}(\zeta,\zeta^*). \tag{15}$$

Using the above derived operations, and rearranging the Master equation (7) with Eq. (2) substituted in, the partial integration places both the differentiation with respect to ζ and ζ^* and the differentiation with respect to time t on the left-hand side of $P(\hat{\Lambda},\lambda^*,t)$. Additionally, the P function's derivatives go to zero at the limits of the integration (± infinity). The analytic solution of the complete Fokker–Planck equation has not been found. The lack of a solution indicates that the system is non-classical. In order to get a solution the semiclassical limit of the equation is used to compare with the Master equation solution. The semiclassical limit is obtained by eliminating all the second order derivative terms such that the Fokker–Planck equation becomes

$$\frac{\partial P(\zeta,\zeta^*,t)}{\partial t}=\frac{\partial}{\partial\zeta}\left(A_\zeta(\zeta,\zeta^*)P(\zeta,\zeta^*)\right)$$
$$+\frac{\partial}{\partial\zeta^*}\left(A_{\zeta^*}(\zeta,\zeta^*)P(\zeta,\zeta^*)\right), \tag{16}$$

where

$$A_\zeta(\zeta,\zeta^*) = -i\alpha + i\alpha\zeta^2 + \gamma\zeta + \lambda\frac{2\zeta^3}{1+|\zeta|^2} + \lambda\frac{2\zeta^*}{1+|\zeta|^2} - \lambda\frac{2|\zeta|^2\zeta}{1+|\zeta|^2} - \lambda\frac{2|\zeta|^2\zeta}{1+|\zeta|^2} - \lambda\frac{2\zeta}{1+|\zeta|^2} - \frac{\lambda^2}{\gamma}\zeta^3 + \frac{\lambda^2}{\gamma}\zeta,$$

$$A_{\zeta^*}(\zeta,\zeta^*) = i\alpha - i\alpha\zeta^2 + \gamma\zeta^* + \lambda\frac{2\zeta^{*3}}{1+|\zeta|^2} + \lambda\frac{2\zeta}{1+|\zeta|^2} - \lambda\frac{2|\zeta|^2\zeta^*}{1+|\zeta|^2} - \lambda\frac{2|\zeta|^2\zeta^*}{1+|\zeta|^2} - \lambda\frac{2\zeta^*}{1+|\zeta|^2} + \frac{\lambda^2}{\gamma}\zeta^{*3} - \frac{\lambda^2}{\gamma}\zeta^*. \quad (17)$$

From this Fokker–Planck equation, the semiclassical equations of motion are found to be

$$\frac{\partial\zeta}{\partial t} = -A_\zeta(\zeta,\zeta^*),$$
$$\frac{\partial\zeta^*}{\partial t} = -A_{\zeta^*}(\zeta,\zeta^*). \quad (18)$$

Each of the above equations is the complex conjugate of the other, so it is sufficient to solve one of them. The steady-state solution on the Bloch sphere can be found by setting either A_ζ or A_{ζ^*} to zero. A numeric solution is found for the maximally entangled steady-state. The optimal driving and feedback amplitudes can be substituted into either of the above equations to get numerical results.

In the absence of feedback ($\lambda = 0$), a maximum entanglement of 0.11 as measured by concurrence is found with a driving amplitude ($\alpha = 0.38\gamma^{-1}$), the two fixed points on the block sphere are found to be

$$\theta_1 = 0.72\pi, \qquad \phi_1 = -\frac{\pi}{2},$$
$$\theta_2 = 0.27\pi, \qquad \phi_2 = -\frac{\pi}{2}. \quad (19)$$

The double solution to the equation indicates the semiclassical system is unstable at the point of maximum entanglement when feedback is absent. This result is consistent with the result found in [22]. In their model, the qubits are trapped ions driven simultaneously by a laser and coupled collectively to a heat bath.

In the presence of feedback, the optimized steady-state entanglement as measured by concurrence is 0.31 for driving and feedback amplitudes of $\alpha = 0.4\gamma^{-1}$, $\lambda = -0.8\gamma^{-1}$. After substituting this into Eq. (2) there is only one solution on the block sphere as indicated below

$$\theta_1 = 0.94\pi, \qquad \phi_1 = -\frac{\pi}{2}. \quad (20)$$

In the presence of feedback only one solution exists which indicates that the stability of the semiclassical system at the point of maximum entanglement is increased. This is due to the ability of feedback to actively cancel quantum noise to stabilize the system [23]. The above results with and without feedback are from the semi-classical solution of the Fokker–Planck equation using the P representation.

Now these semi-classical results are compared with the results from the quantum Q representation. The Q representation is the projection of the density matrix into a coherent state represented in polar coordinates, and is defined as

$$Q(\theta,\phi) = \langle j,\theta,\phi|\rho|j,\theta,\phi\rangle,$$

$$|j,\theta,\phi\rangle = \sqrt{\frac{(2j)!}{(j+m)!(j-m)!}} \times \sum_{m=-j}^{j}\left[\cos^{j+m}\left(\frac{\theta}{2}\right)\sin^{j-m}\left(\frac{\theta}{2}\right)e^{-im\phi}\right]|j,m\rangle, \quad (21)$$

where the state $|j,\theta,\phi\rangle$ is the spherical representation of a coherent state. Here j, θ and ϕ are the standard spherical polar coordinates defined by $\zeta = \mathrm{ctan}\,\frac{\theta}{2}e^{-i\phi}$. The atomic coherent state $|\zeta\rangle$ is given in Eq. (4). The magnitude of the Q function in a particular direction (θ and ϕ) is represented by the distance measured from the origin. The density matrix of which the Q representation is based on is obtained from solving the Master equation given in Eq. (7).

The value of θ and ϕ for which entanglement is maximized corresponds to where $Q(\theta,\phi)$ has a maximum as found previously [14]. When feedback is off $\lambda = 0$, the maximum value of $Q(\theta,\phi)$ occurs for a driving amplitude of $\alpha = 0.38\gamma^{-1}$. This maximum value on the Q representation sphere is

$$\theta_1 = 0.8\pi, \qquad \phi_1 = -\frac{\pi}{2}. \quad (22)$$

As can be seen the above result agrees reasonably well with that from the first fixed point in the P representation. However, when feedback is on ($\lambda = -0.8\gamma^{-1}$), with a driving amplitude of ($\alpha = 0.4\gamma^{-1}$), the peak point on the Q representation sphere is

$$\theta_1 = 0.4\pi, \qquad \phi_1 = -\frac{\pi}{2}. \quad (23)$$

Comparing this result with that from the P representation, there is a notable disagreement. This disagreement when the feedback is switched on and entanglement between the two qubits is maximized indicates the system does not have a well behaved P representation. To quantitatively see how classical the system is the semi-classical limit of the Fokker–Planck equation is found. The difference between the solutions indicates that without feedback the system is much closer to a classical system then when feedback is on. The difference between the semiclassical P and Q function can be used as a measure of “quantumness” similar to measures of entanglement.

3. Discussions and conclusions

A couple of related papers that use the P function to determine if a system is quantum in nature will now be discussed. The first paper did comparative studies on a squeezed vacuum state using different quasi-probability functions. The second paper used the P representation to experimentally show that a system was quantum in nature.

In the first paper, the squeezed vacuum probability density is found using the quasi-probabilities of the Wigner, Q, and P representations [24]. The paper found that when a quadrature variance is less than the minimum uncertainty limit, the squeezed thermal state of the super-Poissonian statistics does not have a well-behaved P representation and cannot be described by classical theory, while the Wigner functions and Q representation are obtained for all of the various squeezed states. This paper illustrates that when a system is quantum in nature the P representation may not have a solution.

A second paper created a solvable P representation of an experiment involving a single photon-added thermal state that exhibits a classically impossible negative probability function [25]. The P representation was constructed from experimental results, and exhibited the expected negative probability density. The conclusion of the paper indicated that the P function is a good indicator of non-classical behavior when other functions such as the Wigner function fail to differentiate between classical and non-classical effects. This paper exemplifies the use of the P representation to determine whether a system is quantum in nature as a result of finding a negative probability.

In summary, this Letter explored using the semiclassical P representation to determine the amount of quantumness a system exhibits. The results are compared with the quantum Q function solution. Close agreement was found between the two methods when feedback is not present. However, when feedback was present a notable difference between the two solutions was found. The difference between the P and Q solutions can be used as measure of the quantum nature of the system. In addition feedback can stabilize the system from the perspective of the semi-classical P representation. The author conjectures the increased entanglement in a two qubit system is due to the ability of feedback to stabilize the system.

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References

[1] J. Janszky, Min Gyu Kim, M.S. Kim, Phys. Rev. A 53 (1996) 502.
[2] Werner Vogel, Phys. Rev. Lett. 9 (2000) 1849.
[3] Th. Richter, W. Vogel, Phys. Rev. Lett. 89 (2002) 283601.
[4] M.S. Kim, W. Son, V. Buzek, P.L. Knight, Phys. Rev. A 65 (2002) 032323.
[5] J.K. Asboth, J. Calsamiglia, H. Ritsch, Phys. Rev. Lett. 94 (2005) 173602.
[6] E. Shchukin, Th. Richter, W. Vogel, Phys. Rev. A 71 (2005) 011802.
[7] J.K. Korbicz, J.I. Cirac, Jan Wehr, M. Lewenstein, Phys. Rev. Lett. 94 (2005) 153601.
[8] T. Kiesel, W. Vogel, V. Parigi, A. Zavatta, M. Bellini, Phys. Rev. A 78 (2008) 021804.
[9] T. Kiesel, W. Vogel, Phys. Rev. A 79 (2009) 022122.
[10] Petr Marek, M.S. Kim, Lee Jinhyoung, Phys. Rev. A 79 (2009) 052315.
[11] R.J. Glauber, Phys. Rev. 131 (1963) 2766;
E.C.G. Sudarshan, Phys. Rev. Lett. 10 (1963) 277.
[12] M. Nielsen, I.L. Chuang, Quantum Computation and Quantum Information, Cambridge Univ. Press, Cambridge, 2000.
[13] D. Bouwmeester, A. Ekert, A. Zeilinger (Eds.), The Physics of Quantum Information, Springer, Berlin, 2000.
[14] Jin Wang, H.M. Wiseman, G.J. Milburn, Phys. Rev. A 71 (2005) 042309.
[15] E. Schrödinger, Naturwissenschaften 14 (1926) 664.
[16] E.P. Wigner, Phys. Rev. 40 (1932) 749.
[17] K. Husimi, Proc. Phys. Math. Soc. Jpn. 22 (1940) 264.
[18] J.M. Radcliffe, J. Phys. A 4 (1971) 313.
[19] F.T. Arecchi, E. Courtens, R. Gilmore, H. Thomas, Phys. Rev. A 6 (1972) 2211.
[20] C.W. Gardiner, Handbook of Stochastic Methods, Springer, Berlin, 1985.
[21] G. Lindblad, Commun. Math. Phys. 48 (1976) 199.
[22] S. Schneider, G.J. Milburn, Phys. Rev. A 65 (2002) 042107.
[23] Jin Wang, H.M. Wiseman, Phys. Rev. A 64 (2001) 063810.
[24] M.S. Kim, F.A.M. De Oliveira, P.L. Knight, Phys. Rev. A 40 (1989) 2494.
[25] T. Kiesel, W. Vogel, V. Parigi, A. Zavatta, M. Bellini, Phys. Rev. A 78 (2008) 021804.