Recitation 6: Constraint Satisfaction Problems (CSP), Notes
October 21, 2005

1. Backtracking Search

The pseudocode for backtracking search (BT) follows. Note that backtracking search is simply depth-first search with backtracking on the search tree of partial assignments as its nodes. The input is a (partial) assignment \( A \) and a list of unassigned variables \( U \). The output is either failure or a (complete) assignment. We denote the domain of variable \( X \) as \( D(X) \). The algorithm is initially called with \( A \) the empty assignment and \( U \) the list of all the variables of the CSP.

\[
\text{BT}(A, U) \quad \text{if} \quad A \text{ is complete} \quad \text{then} \\
\quad \text{return} \ A \\
\text{end if} \\
\text{Remove a variable } X \text{ from } U \\
\text{for all values } x \in D(X) \text{ do} \\
\quad \text{if } X = x \text{ is consistent with } A \text{ according to the constraints then} \\
\quad \quad \text{Add } X = x \text{ to } A \\
\quad \quad \text{result} \leftarrow \text{BT}(A, U) \\
\quad \quad \text{if } \text{result} \neq \text{failure} \quad \text{then} \\
\quad \quad \quad \text{return result} \\
\quad \quad \text{end if} \\
\quad \quad \text{Remove } X = x \text{ from } A \\
\text{end if} \\
\text{end for} \\
\text{return failure}
\]

2. Backtracking Search with Forward Checking

The pseudocode for backtracking search with forward checking (BT+FC) follows. The algorithm uses an extra input \( D \) that corresponds to the current domains of the variables. Initially, \( D \) corresponds to the set of original domains.

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1These notes are primarily based on the recitation notes of Kimberle Koile and the slides of Tomás Lozano-Perez from previous terms, as well as the book by Russell and Norvig [2003].
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BT+FC(A, U, D)
if A is complete then
  return A
end if
Remove a variable X from U
for all values x ∈ D(X) do
  if X = x is consistent with A according to the constraints then
    Add X = x to A
    D′ ← D (Save the current domains)
    for all Y ∈ U (i.e., Y an unassigned variable), Y −−− X (i.e., Y a neighbor of X in
    the constrained graph) do
      Remove values for Y from D′(Y) that are inconsistent with A
    end for
    if for all Y ∈ U, Y −−− X, we have D′(Y) not empty then
      result ← BT+FC(A, U, D′)
      if result ≠ failure then
        return result
      end if
    end if
  end if
end for
Remove X = x from A
return failure

3. Constraint Propagation

The pseudocode for the arc consistency algorithm (AC), the most basic form of constraint
propagation, follows. \(^2\) The basic idea of the algorithm is to just check that all the arcs
in the constraint graph are consistent. If not, it eliminates values from the domain of the
variable that make the arc inconsistent, and insert to the queue arcs that are affected by that
change in domain values. We denote that there is an arc from variable X to variable Y in
the constraint graph by X ↛ Y.

\(^2\)In the CSP literature, this particular version is actually named AC-3, because this was the third version given
by its inventor in Mackworth [1977].
AC
Add all arcs in the constraint graph to $Q$

while $Q$ is not empty do
    Remove first arc $X \rightarrow Y$ from $Q$
    if REMOVE-INCONSISTENT-VALUES($X, Y$) then
        for all variables $Z \neq Y$ such that $Z \rightarrow X$ do
            Add $Z \rightarrow X$ to $Q$ (if not there already)
        end for
    end if
end while

REMOVE-INCONSISTENT-VALUES($X, Y$)
for all values $x \in D(X)$ do
    if there does not exist a value $y \in D(Y)$ such that $X = x, Y = y$ is consistent with the constraint between $X$ and $Y$ then
        Delete $x$ from $D(X)$
    end if
end for
if deleted values then
    return yes
else
    return no
end if

(a) (Optional) Time Complexity
The running time of the algorithm is $O(ed^3)$ where $e$ is the number of constraints and $d$ is the number of values for each variable (i.e., the size of the domain). The running time analysis is as follows. Every arc is inserted into the queue $Q$ at most $d$ times, since $d$ is the number of values for each variable and the algorithm eliminates at least one value from the domain of some variable at each round (otherwise, if no value is removed for the arcs in the queue $Q$, then $Q$ becomes empty and the algorithm stops). Therefore, the maximum number of iterations before $Q$ becomes empty is $O(ed)$. Checking that an arc is consistent at each iteration takes $O(d^2)$. Therefore, the total running time $O((ed)(d^2))$.

You should contrast this worst-case running time, which is linear in $n$, to that of backtracking search which is $O(d^n)$, which is exponential in $n$. Hence, we can apply constraint propagation to try to reduce the size of the domains before we do backtracking since it does not take that long to run. Alternatively, you can run constraint propagation at every partial assignment node of the search tree. Usually in this case, however, constraint propagation is run only when there is only one value left for the variable just extended.

4. (Optional) Alternative Search Formulation and Algorithms for CSPs
An alternative search formulation to solve CSPs is by first constructing a state-space graph. The nodes in the graph correspond to complete assignments. There is an edge between two nodes if their respective assignments differ by the value assignment of only one variable. This graph looks just like the graph for map problems we saw for basic and optimal search. The goal nodes correspond to solutions for the CSP. We start from some arbitrary node (i.e., some
complete assignment) and try to reach a goal node in the graph in a minimum number of steps.

References
