# SAITO-KUROKAWA LIFTS OF SQUARE-FREE LEVEL

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ABSTRACT. Let $f \in S_{2\kappa-2}(\Gamma_0(M))$ be a Hecke eigenform with $\kappa \geq 2$ even and $M \geq 1$ and odd and square-free. In this paper we survey the construction of the Saito-Kurokawa lifting from the classical point of view. We also provide some arithmetic results on the Fourier coefficients of Saito-Kurokawa liftings. We then calculate the norm of the Saito-Kurokawa lift.

## 1. INTRODUCTION

It is well-established that one can prove deep theorems in arithmetic by studying liftings of automorphic forms from a reductive algebraic group to a larger reductive algebraic group. For instance, one can see Ribet's proof of the converse of Herbrand's theorem ([21]), Wiles' proof of the main conjecture of Iwasawa theory for totally real fields ([29]), or Skinner-Urban's proof of the main conjecture of Iwasawa theory for GL(2) ([27]) for three prominent examples of this philosophy. One such lifting that has figured prominently in several such results is the Saito-Kurokawa lifting that lifts a form from GL(2) to GSp(4). One can see [1, 4, 16, 26] for examples of arithmetic applications of Saito-Kurokawa liftings. It is these liftings that this paper focuses on.

The Saito-Kurokawa lifting in the full level case was established via a series of papers culminating in the work of Zagier ([30]). The lifting was established from an automorphic point of view via the work of Piatetski-Shapiro ([20]) and Schmidt ([23, 24]). For the arithmetic applications referenced above, one needs a classical construction of the Saito-Kurokawa lifting of square-free level. This lifting was claimed in a series of papers ([19, 17, 18]). Unfortunately, there are many omitted proofs in these papers and the generalized Maass lifting used in these papers is known to be given incorrectly. It was not until recently that a correct treatment of the generalized Maass lifting was given by Ibukiyama ([11]). This allows one to give a correct classical construction of the Saito-Kurokawa lifting of square-free level.

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In this paper after introducing some notation in §2, in §3 we survey the classical construction of a Saito-Kurokawa lift $F_f \in S_\kappa(\Gamma_0^{(2)}(M))$ from $f \in S_{2\kappa-2}(\Gamma_0(M))$ for $M$ square-free and $\kappa \geq 2$ even. In the same section we also show that with a suitable choice of scalar one can fix a lifting $F_f$ that has Fourier coefficients in the same ring as $f$'s Fourier coefficients. This is essential for arithmetic applications. In §4 we compute the norm of $F_f$. Such a calculation originally appeared in [3], but this was based on the incorrect Maass lifting mentioned above so is not correct. This norm is needed for the main result of [1].

## 2. Definitions and notation

In this section we fix basic definitions and notations we will use throughout the rest of the paper. Throughout this paper we let $M \geq 1$ denote a square-free integer and $\kappa \geq 2$ an even integer.

Given a ring $R$, we let $\mathrm{Mat}_n(R)$ denote the set of $n$ by $n$ matrices with entries in $R$. As usual we let $\mathrm{GL}(n,R) \subset \mathrm{Mat}_n(R)$ denote the group of invertible matrices and $\mathrm{SL}(n,R) \subset \mathrm{GL}(n,R)$ the matrices with determinant 1. We write $1_n$ for the identity matrix in $\mathrm{GL}_n(R)$ and $0_n$ for the zero matrix in $\mathrm{Mat}_n(R)$. Given $A \in \mathrm{Mat}_n(R)$, we denote the transpose of $A$ by ${}^tA$. Let $J = \begin{pmatrix} 0_2 & -1_2 \\ 1_2 & 0_2 \end{pmatrix}$. The symplectic group $\mathrm{GSp}(4,R)$ is defined by

$$\mathrm{GSp}(4,R) = \{g \in \mathrm{GL}(4,R) : {}^tgJg = \mu(g)J, \mu(g) \in \mathrm{GL}(1,R)\}.$$

We set $\mathrm{Sp}(4,R) = \ker(\mu)$. We let $\mathrm{PGSp}(4,R)$ denote the projective symplectic group. Let $\Gamma_0(M) \subset \mathrm{SL}(2,\mathbb{Z})$ have its usual meaning and set

$$\Gamma_0^{(2)}(M) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{Z}) : C \equiv 0 \pmod{M} \right\}.$$

We write $e(z) = e^{2\pi i z}$. We let $\mathfrak{h}^n = \{Z = X + iY \in \mathrm{Mat}_n(\mathbb{C}) : X, Y \in \mathrm{Mat}_n(\mathbb{R}), Y > 0\}$. We let $S_\kappa(\Gamma_0(M))$ denote the cusp forms of weight $\kappa$ and level $\Gamma_0(M)$. Let $f \in S_\kappa(\Gamma_0(M))$ be a normalized eigenform with Fourier expansion

$$f(z) = \sum_{n \geq 1} a_f(n) e(nz).$$

Given a ring $R$, we write $S_\kappa(\Gamma_0(M); R)$ to denote the space of cusp forms that have Fourier coefficients in $R$. We define the Peterson product on $S_\kappa(\Gamma_0(M))$ by setting

$$\langle f_1, f_2 \rangle = \frac{1}{[\mathrm{SL}(2,\mathbb{Z}) : \Gamma_0(M)]} \int_{\Gamma_0(M)\backslash \mathfrak{h}} f_1(z)\overline{f_2(z)} y^{\kappa-2} dx dy$$

for $f_1, f_2 \in S_\kappa(\Gamma_0(M))$.

We denote the cuspidal automorphic representation associated to an eigenform $f$ by $\pi_f = \otimes' \pi_{f,p}$. Recall that $\pi_{f,\infty}$ is the discrete series representation with lowest weight vector of weight $\kappa$ and for $p \nmid M$ the representation $\pi_{f,p}$ is an unramified principal series representation. The local representations for $p \mid M$ are determined by the Atkin-Lehner eigenvalues of $f$. For $p \mid M$, recall the Atkin-Lehner operator at $p$ is the matrix $W_p = \begin{pmatrix} pa & b \\ Mc & pd \end{pmatrix}$ with $p^2ad - Mbc = p$. If $f \in S_\kappa^{\text{new}}(\Gamma_0(M))$, we let $\epsilon_p \in \{\pm 1\}$ denote the Atkin-Lehner eigenvalue of $f$ at $p$, i.e., $W_p f = \epsilon_p f$. If $\epsilon_p = -1$, then $\pi_{f,p} = \mathrm{St}_{\mathrm{GL}(2)}$ and if $\epsilon_p = 1$ then $\pi_{f,p} = \xi\,\mathrm{St}_{\mathrm{GL}(2)}$ where $\mathrm{St}_{\mathrm{GL}(2)}$ is the Steinberg representation and $\xi$ is the unique non-trivial unramified quadratic character of $\mathbb{Q}_p^\times$.

We will also need $L$-functions attached to $f$ and $\pi_f$. For each prime $p \nmid M$ there exists a character $\sigma_p$ so that $\pi_{f,p} = \pi(\sigma_p, \sigma_p^{-1})$ (see [5, Section 4.5].) The $p$-Satake parameter of $f$ is given by $\alpha_0(p; f) = \sigma_p(p)$. The $L$-function associated to $\pi_{f,p}$ is

$$L(s, \pi_{f,p}) = (1 - \alpha_0(p; f)p^{-s})^{-1}(1 - \alpha_0(p; f)^{-1}p^{-s})^{-1}.$$

For $p \mid M$

$$L(s, \pi_{f,p}) = (1 + \epsilon_p p^{-s-\frac{1}{2}})^{-1}.$$

We set

$$L_\infty(s, \pi_{f,\infty}) = (2\pi)^{-(s+(\kappa-1)/2)}\Gamma\left(s + (\kappa - 1)/2\right).$$

The $L$-function associated to $\pi_f$ is

$$L(s, \pi_f) = \prod_p L(s, \pi_{f,p}).$$

The functional equation for $L(s, \pi_f)$ is given by

$$L(s, \pi_f) = \varepsilon(s, \pi_f)L(1 - s, \pi_f)$$

where $\epsilon(s, \pi_f) = \prod_p \epsilon_p(s, \pi_{f,p})$ and

$$\varepsilon_p(s, \pi_{f,p}) = \begin{cases} (-1)^{\kappa/2} & \text{if } p = \infty, \\ -p^{\frac{1}{2}-s} & \text{if } \epsilon_p = -1, p < \infty, \\ p^{\frac{1}{2}-s} & \text{if } \epsilon_p = 1, p < \infty. \end{cases}$$

In particular, the sign of the functional equation is given by $\varepsilon(\frac{1}{2}, \pi_f) \in \{\pm 1\}$. The classical $L$-function of $f$ is given by

$$L(s, f) = \prod_{p<\infty} L\left(s + \frac{1}{2} - \kappa/2, \pi_{f,p}\right).$$

Let $S_\kappa(\Gamma_0^{(2)}(M))$ denote the space of Siegel modular forms of weight $\kappa$ and level $\Gamma_0^{(2)}(M)$. A form $F \in S_\kappa(\Gamma_0^{(2)}(M))$ has a Fourier expansion

$$F(z) = \sum_{T \in \Lambda_2} a_F(T)e(\mathrm{Tr}(Tz))$$

where $\Lambda_2$ is the set of 2 by 2 half integral positive definite symmetric matrices. As above, for a ring $R$ we write $S_\kappa(\Gamma_0^{(2)}(N); R)$ to denote the forms with Fourier coefficients in $R$.

Given $g \in \mathrm{Sp}_4^+(\mathbb{Q})$, we write $T(g)$ to denote

$$\Gamma_0^{(2)}(M) g \Gamma_0^{(2)}(M).$$

We define the usual action of $T(g)$ on Siegel modular forms by setting

$$T(g)F = \sum_i F|_\kappa g_i$$

where $\Gamma_0^{(2)}(M) g \Gamma_0^{(2)}(M) = \coprod_i \Gamma_0^{(2)}(M) g_i$ and $F \in M_\kappa(\Gamma_0^{(2)}(M))$. Let $p$ be prime and define

$$T_S(p) = T(\mathrm{diag}(1_2, p1_2)).$$

In the case that $p \mid M$ we write $U_S(p)$ for $T_S(p)$. Set

$$T'_S(p) = pT(\mathrm{diag}(1, p, p^2, p)) + p(1 + p + p^2) T(\mathrm{diag}(p, p, p, p)).$$

Let $F \in S_\kappa(\Gamma_0^{(2)}(M))$ be an eigenform. Then $F$ generates a space of cuspidal automorphic forms on $\mathrm{GSp}_4(\mathbb{A})$ invariant under right translation. In general this space may not be irreducible, but does decompose into a finite number of irreducible, cuspidal, automorphic representations. Let $\pi_F$ be one of these irreducible pieces. The representation $\pi_F$ can be decomposed into local components $\Pi_F = \otimes \Pi_{F,p}$ with $\Pi_{F,p}$ a representation of $\mathrm{PGSp}(4, \mathbb{Q}_p)$. We refer the reader to [2, Section 3] for the details concerning the construction of cuspidal automorphic representations associated to Siegel cusp forms. For all but finitely many places $p$ the representation $\Pi_{F,p}$ will be an Iwahori spherical representation $\Pi(\sigma, \chi_1, \chi_2)$, which is isomorphic to the Langlands quotient of an induced representation of the form $\chi_1 \times \chi_2 \rtimes \sigma$ with $\chi_i$ and $\sigma$ unramified characters of $\mathbb{Q}_p^\times$. One can see [2, 22] for the definitions and details. For such $p$ the $p$-Satake parameters are defined by $b_0 = \sigma(p)$ and $b_i = \chi_i(p)$ for $i = 1, 2$. We define

$$L(s, \Pi_{F,p}, \mathrm{spin}) = \left((1 - b_0 p^{-s})(1 - b_0 b_1 p^{-s})(1 - b_0 b_2 p^{-s})(1 - b_0 b_1 b_2 p^{-s})\right)^{-1}$$

for $\Pi_{F,p} = \Pi(\sigma, \chi_1, \chi_2)$. We leave the local $L$-functions for $p = \infty$ and $p \mid M$ undefined as defining these would take us too far afield. On can see [24] for the definitions. Set

$$L(s, \Pi_F, \mathrm{spin}) = \prod_p L(s, \Pi_{F,p}, \mathrm{spin}).$$

The classical spinor $L$-function is given by

$$L(s, F, \mathrm{spin}) = \prod_{p < \infty} L(s - \kappa + 3/2, \Pi_{F,p}, \mathrm{spin}).$$

Given an Euler product $L(s) = \prod_p L_p(s)$, we write $L^M(s) = \prod_{p \nmid M} L_p(s)$.

## 3. Classical construction

In this section we gather known results and give a classical construction of the Saito-Kurokawa lifting from $S_{2\kappa-2}(\Gamma_0(M))$ to $S_\kappa(\Gamma_0^{(2)}(M))$ for $\kappa \geq 2$ an even integer and $M \geq 1$ an odd square-free integer. The existence of a Saito-Kurokawa lifting is known from a representation theory point of view via [24]. In particular, Theorem 5.2 of [24] gives that this lift is unique up to constant multiple. Using the classical construction given in this section we fix the scalar so that the resulting Saito-Kurokawa lift is more useful for arithmetic applications (see for example [1, 16]). In particular, given a newform $f \in S_{2\kappa-2}^{\mathrm{new}}(\Gamma_0(M))$, if we let $\mathcal{O}$ be a ring containing the Hecke eigenvalues of $f$, we show the Saito-Kurokawa lift of $f$ can be normalized so that it has Fourier coefficients lying in $\mathcal{O}$ as well.

The classical lifting is constructed via a composition of liftings, the first from integral to half-integral weight, then from half-integral weight to Jacobi forms, and finally from Jacobi forms to Siegel modular forms. We begin by recalling results on the lifting from integral to half-integral weight forms.

Let $S^+_{\kappa-\frac{1}{2}}(\Gamma_0(4M))$ denote Kohnen's $+$-space, i.e., the cusp forms of weight $\kappa - \frac{1}{2}$ and level $\Gamma_0(4M)$ whose $n$th Fourier coefficients at infinity vanish for $(-1)^{\kappa-1}n \equiv 2, 3 \pmod 4$. For a prime $p \nmid M$ we define the $p^2$th Hecke operator acting on $S^+_{\kappa-\frac{1}{2}}(\Gamma_0(4M))$ by

$$
\begin{aligned}
T_H(p) & \sum_{\substack{n\geq 1\\ (-1)^{\kappa-1}n\equiv 0,1(4)}} c(n)q^n \\
&= \sum_{\substack{n\geq 1\\ (-1)^{\kappa-1}n\equiv 0,1(4)}} \left( c(p^2n) + \left(\frac{(-1)^{\kappa-1}n}{p}\right) p^{\kappa-2}c(n) + p^{2\kappa-3}c(n/p^2)\right) q^n.
\end{aligned}
$$

We follow the literature and denote this by $T_H(p)$ instead of $T_H(p^2)$ since the $p$th Hecke operator on half-integral weight modular forms vanishes. For $p \mid M$ we set

$$
U_H(p) \sum_{\substack{n\geq 1\\ (-1)^{\kappa-1}n\equiv 0,1(4)}} c(n)q^n = \sum_{\substack{n\geq 1\\ (-1)^{\kappa-1}n\equiv 0,1(4)}} c(p^2n)q^n.
$$

The inner product on $S^+_{\kappa-\frac{1}{2}}(\Gamma_0(4M))$ is given by

$$
\langle g_1, g_2\rangle = \frac{1}{[\Gamma_0(4):\Gamma_0(4M)]} \int_{\Gamma_0(4M)\backslash \mathfrak{h}^1} g_1(z)\overline{g_2(z)} y^{\kappa-5/2} dxdy
$$

for $z = x + iy$.

Let $D < 0$ be a fundamental discriminant and let $\theta_{\kappa,D} : S_{2\kappa-2}(\Gamma_0(M)) \to S^+_{\kappa-\frac{1}{2}}(\Gamma_0(4M))$ be Shintani's lifting ([25]). One has that $\theta_{\kappa,D}$ gives a Hecke-equivariant isomorphism between $S_{2\kappa-2}^{\mathrm{new}}(\Gamma_0(M))$ and $S^{+,\mathrm{new}}_{\kappa-\frac{1}{2}}(\Gamma_0(4M))$ ([12,

§ 5, Theorem 2]) where $S^{+,\mathrm{new}}_{\kappa-\frac{1}{2}}(\Gamma_0(4M))$ denotes the subspace of newforms as defined in [12, § 5].

Let $\mathcal{O}$ be a ring so that an embedding of $\mathcal{O}$ into $\mathbb{C}$ exists. We choose such an embedding and identify $\mathcal{O}$ with its image in $\mathbb{C}$. Let $f \in S_{2\kappa-2}(\Gamma_0(M))$ be a normalized Hecke eigenform and assume $\mathcal{O}$ contains the Fourier coefficients of $f$. The Shintani lifting of $f$ is determined up to a scalar multiple. In [28] Stevens' constructs a cohomological version of the Shintani lifting as a step in producing a $\Lambda$-adic Shintani lifting. This cohomological Shintani lifting allows one to conclude the following result.

**Theorem 3.1.** *([28, Prop. 2.3.1]) Let $f \in S_{2\kappa-2}(\Gamma_0(M))$ be a Hecke eigenform. Let $D < 0$ be a fundamental discriminant. If the Fourier coefficients of $f$ are in $\mathcal{O}$, there exists a Shintani lifting $\theta^{\mathrm{alg}}_{\kappa,D}(f)$ with Fourier coefficients in $\mathcal{O}$.*

This gives what we need for the first part of the construction. We now consider the lifting from half-integral weight forms to Jacobi forms. Let $\Gamma_0(M)^J = \Gamma_0(M) \ltimes \mathbb{Z}^2$, i.e., the set of pairs $(\gamma, X)$ with $\gamma \in \Gamma_0(M)$ and $X \in \mathbb{Z}^2$ with the group law given by $(\gamma_1, X_1)(\gamma_2, X_2) = (\gamma_1\gamma_2, X_1\gamma_2 + X_2)$. We denote the space of Jacobi cusp forms of weight $\kappa$, index 1, and level $\Gamma_0(M)^J$ by $J^c_{\kappa,1}(\Gamma_0(M)^J)$. Given a positive integer $m$, define

$$\Delta_{M,0}(m) = \left\{ \begin{pmatrix} a & b \\ Mc & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bcM = m, \gcd(a, M) = 1 \right\}.$$

We make use of the Hecke operators on $J^c_{\kappa,1}(\Gamma_0(M)^J)$ defined by

$$T_J(p)\phi = p^{\kappa-4} \sum_{\gamma} \sum_{(\lambda,\mu)\in\mathbb{F}_p^2} \phi|_{\kappa,1}(\gamma/p)|_1(\lambda, \mu)$$

where $\gamma$ runs over

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & p^2 \end{pmatrix} : b = 0, \ldots, p^2 - 1 \right\} \cup \left\{ \begin{pmatrix} p & b \\ 0 & p \end{pmatrix} : b = 0, \ldots, p - 1 \right\} \cup \left\{ \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

In the case that $p \mid M$, we define $U_J(p)$ as we defined $T_J(p)$ with the only difference being here we only take $\gamma$ running over

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & p^2 \end{pmatrix} : b = 0, \ldots, p^2 - 1 \right\}.$$

The inner product on $J^c_{\kappa,1}(\Gamma_0(M)^J)$ is defined by

$$\langle \phi_1, \phi_2 \rangle = \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(M)]} \int_{\Gamma_0(M)^J \backslash \mathfrak{h}^1 \times \mathbb{C}} \phi_1(\tau, z)\overline{\phi_2(\tau, z)} v^{\kappa-3} e^{-4\pi y^2/v} dx dy du dv$$

for $\tau = u + iv$ and $z = x + iy$.

We recall the isomorphism between $J^c_{\kappa,1}(\Gamma_0(M)^J)$ and $S^+_{\kappa-\frac{1}{2}}(\Gamma_0(4M))$ for $M$ an odd integer.

Let $g \in S^+_{\kappa-\frac{1}{2}}(\Gamma_0(4M))$. Define $g_j(\tau)$ by

$$g_j(\tau) = \sum_{\substack{n>0 \\ n\equiv j \pmod 4}} a_g(n)e(n\tau/4)$$

for $j = 0, 1$ where the $a_g(n)$ are the Fourier coefficients of $g$. It is clear that $g(\tau) = g_0(4\tau) + g_1(4\tau)$. Define

$$\vartheta_j(\tau, z) = \sum_{n\in\mathbb{Z}} e\left(\frac{2n - j^2}{4}\tau + (2n - j)z\right).$$

Define a map $\mathcal{J}$ by

$$\mathcal{J}(g)(\tau, z) = g_0(\tau)\vartheta_0(\tau, z) + g_1(\tau)\vartheta_1(\tau, z)$$

for $g \in S^+_{\kappa-\frac{1}{2}}(\Gamma_0(4M))$. The following theorem provides the necessary connection between half-integral weight forms and Jacobi forms. Note that the compatibility with Hecke operators follows immediately from the definitions and the map given.

**Theorem 3.2.** *(*[15, Corollary 3]*) The map $\mathcal{J}$ gives a Hecke-equivariant isomorphism between $S^+_{\kappa-\frac{1}{2}}(\Gamma_0(4M))$ and $J^c_{\kappa,1}(\Gamma_0(M)^J)$. Moreover, if $g \in S^+_{\kappa-\frac{1}{2}}(\Gamma_0(4M))$ has Fourier coefficients in $\mathcal{O}$, so does $\mathcal{J}$.*

Finally, we recall the Maass lifting from the Jacobi forms to Siegel forms. Let

$$V_m : J^c_{\kappa,t}(\Gamma_0(M)^J) \to J^c_{\kappa,mt}(\Gamma_0(M)^J)$$

be the index shifting operator defined by

$$(V_m\phi)(\tau, z) = m^{\kappa-1} \sum_{g\in\Gamma_0(M)\backslash\Delta_{M,0}(m)} (\phi|_{\kappa,t}g)(\tau, z).$$

If the Fourier expansion of $\phi \in J^c_{\kappa,1}(\Gamma_0(M)^J)$ is given by

$$\phi(\tau, z) = \sum_{\substack{D<0, r\in\mathbb{Z} \\ D\equiv r^2 \pmod 4}} c(D, r)e\left(\frac{r^2 - D}{4}\tau + rz\right),$$

then

$$(V_m\phi)(\tau, z) = \sum_{\substack{D<0, r\in\mathbb{Z} \\ D\equiv r^2 \pmod{4m}}} \left( \sum_{\substack{d|\gcd(r,m) \\ \gcd(d,M)=1 \\ D\equiv r^2 \pmod{4md}}} d^{\kappa-1}c\left(\frac{D}{d^2}, \frac{r}{d}\right)\right) e\left(\frac{r^2 - D}{4m}\tau + rz\right).$$

Define a function on $\mathfrak{h}^2$ by

$$(\mathcal{V}_M\phi)(Z) = \sum_{m=1}^{\infty} (V_m\phi)(\tau, z)e(m\tau')$$

where $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$. One has the following result of Ibukiyama.

**Theorem 3.3.** *([11, Theorems 3.2, 4.1]) The map $\mathcal{V}_M$ is an injective linear map from $J^c_{\kappa,1}(\Gamma_0(M)^J)$ to $S_\kappa(\Gamma_0^{(2)}(M))$. Moreover one has for $p \nmid M$*

$$T_S(p)(\mathcal{V}_M\phi) = \mathcal{V}_M \left(T_J(p)\phi + (p^{\kappa-1} + p^{\kappa-2})\phi\right),$$
$$T_S^{'}(p)(\mathcal{V}_M\phi) = \mathcal{V}_M \left((p^{\kappa-2} + p^{\kappa-1})T_J(p)\phi + (2p^{2\kappa-3} + p^{2\kappa-4})\phi\right).$$

*If $p \nmid M$, then*

$$U_S(p)(\mathcal{V}_M\phi) = \mathcal{V}_M(U_J(p)\phi)$$

*where $T_J(p)$ and $U_J(p)$ are as defined above and the Siegel Hecke operators $T_S(p)$ and $U_S(p)$ were defined in the previous section.*

An immediate consequence of the calculation of the Fourier coefficients of $\mathcal{V}_M$ carried out in [11] is the following corollary.

**Corollary 3.4.** *Let $\phi \in J^c_{\kappa,1}(\Gamma_0(M)^J, \mathcal{O})$ for some ring $\mathcal{O}$. Then $\mathcal{V}_M\phi \in S_\kappa(\Gamma_0^{(2)}(M), \mathcal{O})$.*

The following theorem gives the existence of a Saito-Kurokawa lifting by combining the result on the Shintani lifting with Theorems 3.2 and 3.3

**Theorem 3.5.** *Let $\kappa \geq 2$ be an even integer and $M \geq 1$ an odd square-free integer. Let $f \in S_{2\kappa-2}(\Gamma_0(M))$ be a normalized Hecke eigenform. Then there exists a nonzero cuspidal Siegel eigenform $F_f \in S_\kappa(\Gamma_0^{(2)}(M))$ satisfying*

$$L^M(s, F_f, \mathrm{spin}) = \zeta^M(s-\kappa+1)\zeta^M(s-\kappa+2)L^M(s,f).$$

*Moreover, if $\mathcal{O}$ is a ring that can be embedded into $\mathbb{C}$ and $f$ has Fourier coefficients in $\mathcal{O}$, the lift $F_f$ can be normalized to have Fourier coefficients in $\mathcal{O}$. If $\mathcal{O}$ is a DVR, $F_f$ can be normalized to have Fourier coefficients in $\mathcal{O}$ with at least one Fourier coefficient in $\mathcal{O}^\times$.*

*Proof.* The only thing that remains to prove is that if $\mathcal{O}$ is a DVR, we can normalize $F_f$ so that it has Fourier coefficients in $\mathcal{O}$ with at least one in $\mathcal{O}^\times$. The point is that the Shintani lifting is determined only up to scalar multiple. By choosing the Shintani lifting $\theta^{\mathrm{alg}}_{\kappa,D}(f)$ to have Fourier coefficients in $\mathcal{O}$, it certainly follows that $F_f$ has Fourier coefficients in $\mathcal{O}$ from the above construction. Now suppose that $m = \min_{T\in\Lambda_2} \mathrm{val}_\varpi(a_F(T))$ where $\varpi$ is a uniformizer of $\mathcal{O}$. Thus, if we normalize the Shintani lifting by $\varpi^{-m}$ the resulting $F_f$ will still have Fourier coefficients in $\mathcal{O}$, but will have at least one coefficient in $\mathcal{O}^\times$. □

## 4. Norm of $F_f$

We now calculate the norm of $F_f$ in terms of the norm of $f$. This forms a key step in the main result of [1], but is also of independent interest. We do this by relating the norm of the image of each lift to the norm of the form being lifted for each of the three lifts composed to give the Saito-Kurokawa lift. Again, we fix $\kappa \geq 2$ to be an even integer and $M \geq 1$ to be odd and square-free.

Let $f \in S^{\text{new}}_{2\kappa-2}(\Gamma_0(M))$ be a newform. For each prime $\ell \mid M$, recall $W_\ell$ is the Atkin-Lehner involution on $S^{\text{new}}_{2\kappa-2}(\Gamma_0(M))$. Define $\epsilon_\ell \in \{\pm 1\}$ by

$$f \mid W_\ell = \epsilon_\ell f.$$

The following theorem relates the norm of $f$ to that of $\theta^{\text{alg}}_{\kappa,D}(f)$.

**Theorem 4.1.** [13, Corollary 1] *Let $f \in S^{\text{new}}_{2\kappa-2}(\Gamma_0(M))$ be a newform and let $D < 0$ be a fundamental discriminant. Suppose that $\left(\frac{D}{\ell}\right) = \epsilon_\ell$ for all primes $\ell \mid M$. Then*

$$\frac{\left|a_{\theta^{\text{alg}}_{\kappa,D}(f)}(|D|)\right|^2}{\langle \theta^{\text{alg}}_{\kappa,D}(f), \theta^{\text{alg}}_{\kappa,D}(f)\rangle} = 2^{\nu(M)} \frac{(\kappa-2)!}{\pi^{\kappa-1}} |D|^{\kappa-3/2} \frac{L(\kappa-1, f, \chi_D)}{\langle f, f\rangle} \tag{1}$$

*where $\chi_D = \left(\frac{D}{\cdot}\right)$ and $\nu(M)$ is the number of prime divisors of $M$.*

One should note that if there is a prime $\ell \mid M$ so that $\left(\frac{D}{\ell}\right) \neq \epsilon_\ell$, then one has $a_{\theta^{\text{alg}}_{\kappa,D}(f)}(|D|) = 0$.

Now let $g \in S^+_{\kappa-\frac{1}{2}}(\Gamma_0(4M))$ and $\mathcal{J}(g)$ the associated form in $J^c_{\kappa,1}(\Gamma_0(M)^J)$. Let $g(z) = \displaystyle\sum_{n=1}^{\infty} a_g(n) e(nz)$ be the Fourier expansion of $g$. Consider the summation $\displaystyle\sum_{n=1}^{\infty} \frac{a_g(n)^2}{n^{s+\kappa-3/2}}$. Applying the Rankin-Selberg method to this summation we have for sufficiently large $s$:

$$\begin{aligned} \frac{\Gamma(s+\kappa-3/2)}{(4\pi)^{s+\kappa-\frac{1}{2}}} \sum_{n=1}^{\infty} \frac{a_g(n)^2}{n^{s+\kappa-3/2}} &= \int_{\mathfrak{h}^1/\Gamma_\infty} |g(z)|^2 y^{s+\kappa-5/2} dx dy \\ &= \int_{\mathfrak{h}^1/\Gamma_0(4M)} y^{\kappa-\frac{1}{2}} |g(z)|^2 E^{4M}_s(z) \frac{dx dy}{y^2} \end{aligned}$$

where $E^{4M}_s(z) = \displaystyle\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4M)} (\operatorname{Im}(\gamma z))^s$ and $\Gamma_\infty$ the stabilizer of $\infty$. In other words,

$$\sum_{n=1}^{\infty} \frac{a_g(n)^2}{n^{s+\kappa-3/2}} = \frac{(4\pi)^{s+\kappa-\frac{1}{2}}}{\Gamma(s+\kappa-3/2)} \int_{\Gamma_0(4M)\backslash \mathfrak{h}^1} E^{4M}_s(z) g(z) \overline{g(z)} y^{\kappa-\frac{1}{2}} \frac{dx dy}{y^2}. \tag{2}$$

Taking residues at $s = 1$ we obtain

$$\operatorname*{res}_{s=1}\left(\sum_{n=1}^{\infty}\frac{a_g(n)^2}{n^{s+\kappa-3/2}}\right) = \frac{(4\pi)^{1+\kappa-\frac{1}{2}}\,[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(4M)]}{\Gamma(\kappa-\frac{1}{2})}\langle g,g\rangle \operatorname*{res}_{s=1}E_s^{4M}(z)$$
$$= \frac{3\cdot 2^{\kappa-1}(4\pi)^{\kappa-\frac{1}{2}}}{\pi^{3/2}(2\kappa-3)!!}\langle g,g\rangle$$

where

$$n!! = \begin{cases} n(n-2)\dots 5\cdot 3\cdot 1 & n>0,\text{ odd}\\ n(n-2)\dots 6\cdot 4\cdot 2 & n>0,\text{ even}\end{cases}$$

and we have used that

$$\operatorname*{res}_{s=1}E_s^{4M}(z) = \frac{1}{[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(4M)]}\operatorname*{res}_{s=1}E_s(z)$$
$$= \frac{1}{[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(4M)]}\left(\frac{3}{\pi}\right)$$

where $E_s(z) = E_s^1(z)$ and one can see [9] for the calculation of the residue of the Eisenstein series.

Solving the above residue calculation for $\langle g,g\rangle$ we obtain

(3) $$\langle g,g\rangle = \frac{(2\kappa-3)!!}{3\cdot 2^{3\kappa-2}\pi^{\kappa-2}}\operatorname*{res}_{s=1}\left(\sum_{n=1}^{\infty}\frac{a_g(n)^2}{n^{s+\kappa-3/2}}\right).$$

Recall the two half-integral weight modular forms $g_0$ and $g_1$ defined in § 3. Applying the same process to $g_0$ and $g_1$ we obtain

$$\langle g_j,g_j\rangle = \frac{(2\kappa-3)!!}{3\cdot 2^{3\kappa-2}\pi^{\kappa-2}}\cdot 2^{2\kappa-1}\operatorname*{res}_{s=1}\left(\sum_{n\equiv j}\frac{a_g(n)^2}{n^{s+\kappa-3/2}}\right).$$

Thus

(4) $$\langle g_0,g_0\rangle + \langle g_1,g_1\rangle = 2^{2\kappa-1}\langle g,g\rangle.$$

We need a slight generalization of Theorem 5.3 in [8]. In [8], the formula given only deals with the case $M=1$. However, the proof carries through verbatim to the general case.

**Theorem 4.2.** *([8], Theorem 5.3) For $\mathcal{J}(g)$ and $g_j$ as defined above, one has*

(5)
$$\langle\mathcal{J}(g),\mathcal{J}(g)\rangle = \frac{1}{2\,[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(M)]}\int_{\Gamma_0(M)\backslash\mathfrak{h}^1}\sum_{j=0}^{1}g_j(z)\overline{g_j(z)}v^{\kappa-3/2}\frac{du\,dv}{v^2}.$$

Combining Equations 4 and 5 we obtain the following result.

**Lemma 4.3.** *For $\mathcal{J}(g)$ and $g$ defined as above,*

(6) $$\langle\mathcal{J}(g),\mathcal{J}(g)\rangle = \frac{2^{2\kappa-2}}{[\Gamma_0(M):\Gamma_0(4M)]}\langle g,g\rangle.$$

In light of Theorem 4.1 and Proposition 4.3, it only remains to calculate the ratio of $\langle\phi,\phi\rangle$ and $\langle\mathcal{V}_M\phi,\mathcal{V}_M\phi\rangle$ for $\phi\in J^c_{\kappa,1}(\Gamma_0(M)^J)$. We follow the arguments used in [14] where this ratio is computed when $M=1$. One should note this inner product was originally given in [3]. However, that result cited a theorem in [6] which in turn was based on the incorrect definition of the $\mathcal{V}_M$ map used in [19]. Thus, we give the computation here using the corrected definition given in [11]. The argument given in [3, § 4] is correct up until the point the result of [6] is invoked, however we include the complete argument here to have it given in one place.

Let $F,G\in S_\kappa(\Gamma_0^{(2)}(M))$ be eigenforms with Fourier-Jacobi expansions given by

$$F(Z)=\sum_{N\geq 1}\phi_N(\tau,z)e(N\tau')$$

and

$$G(Z)=\sum_{N\geq 1}\psi_N(\tau,z)e(N\tau').$$

Define a Dirichlet series attached to $F$ and $G$ by

$$D_{F,G}(s)=\zeta^M(2s-2\kappa+4)\sum_{N\geq 1}\langle\phi_N,\psi_N\rangle N^{-s}$$

and set

$$D^*_{F,G}(s)=(2\pi)^{-2s}\Gamma(s)\Gamma(s-\kappa+2)D_{F,G}(s). \tag{7}$$

It is shown in [10] that $D^*_{F,G}(s)$ has meromorphic continuation to $\mathbb{C}$, is entire if $\langle F,G\rangle=0$ and otherwise has a simple pole at $s=\kappa$. The first step in calculating the ratio of inner products we desire is calculating the residue of $D_{F,G}$ at $s=\kappa$. We do this by writing $D_{F,G}$ as the Petersson product of $F(Z)\overline{G(Z)}|Y|^\kappa$ against a certain non-holomorphic Klingen Eisenstein series $E_{s,M}(Z)$. Define a Klingen Eisenstein series

$$E_{s,M}(Z)=\sum_{\gamma\in C_{2,1}(M)\backslash\Gamma_0^{(2)}(M)}\left(\frac{\det(\mathrm{Im}\gamma Z)}{\mathrm{Im}(\gamma Z)_1}\right)^s$$

where $(\gamma Z)_1$ denotes the upper left entry of $\gamma Z$ and

$$C_{2,1}(M)=\left\{\begin{pmatrix}a&0&b&\mu\\ \lambda'&1&\mu'&\kappa\\ c&0&d&-\lambda\\ 0&0&0&1\end{pmatrix}\in\Gamma_0^{(2)}(M)\right\},\ (\lambda',\mu')=(\lambda,\mu)\begin{pmatrix}a&b\\ c&d\end{pmatrix}.$$

Set

$$E^*_{s,M}(Z)=\pi^{-s}\Gamma(s)\zeta(2s)\prod_{p|M}(1-p^{-2s})E_{s,M}(Z). \tag{8}$$

One has that $E^*_{s,M}(Z)$ has mermomorphic continuation to $\mathbb{C}$ with possible simple poles at $s=0,2$ ([10]). It is known that $\operatorname{res}_{s=2}E^*_{s,1}(Z)=1$ ([14]). Note that this is independent of $Z$, so $\operatorname*{res}_{s=2} E^*_{s,1}(NZ) = 1$ for all positive integers $N$. Equation (8) gives $\operatorname*{res}_{s=2} E_{s,1}(Z) = \dfrac{90}{\pi^2}$. As above, this residue is independent of $Z$ so $\operatorname*{res}_{s=2} E_{s,1}(NZ) = \dfrac{90}{\pi^2}$ for all positive integers $N$. The following formula is given in [10]:

$$E_{s,1}(MZ) = \frac{1}{M^s} \sum_{d|M} d^{2s} \prod_{p|d} (1 - p^{-2s}) E_{s,d}(Z).$$

This formula allows one to calculate the residue of $E_{s,M}(Z)$ inductively in terms of $E_{s,d}(Z)$ for $d \mid M$. In fact, for $M = p_1^{m_1} \dots p_n^{m_n}$,

$$\operatorname*{res}_{s=2} E_{s,M}(Z) = \left(\frac{90}{\pi^2}\right) h(p_1, \dots, p_n) \prod_{i=1}^{n} \left(\frac{1}{p_i^{2m_i-2}(p_i^4 - 1)}\right) \tag{9}$$

where $h$ is a polynomial with coefficients in $\mathbb{Z}$ uniquely determined by $M$. For example, if $M = p^n$ for a prime $p$, then

$$h(p) = p^2 - 1$$

and if $M = p_1 \dots p_n$ is a product of distinct primes, then

$$h(p_1, \dots, p_n) = \prod_{i=1}^{n} (p_i^2 - 1).$$

We will be mainly interested in the case that $M = p_1 \dots p_n$ is odd and square-free. Appealing to (8) we obtain

$$\operatorname*{res}_{s=2} E^*_{s,M}(Z) = \prod_{i=1}^{n} \left(\frac{1 - p_i^{-4}}{p_i^2 + 1}\right). \tag{10}$$

We now turn our attention back to calculating the residue of $D_{F,G}(s)$ at $s = \kappa$. We make use of the following equation (see [10]):

$$\pi^{-\kappa+2} [\mathrm{Sp}_4(\mathbb{Z}) : \Gamma_0^{(2)}(M)] \langle F E^*_{s-\kappa+2,M}, G \rangle = M^s D^*_{F,G}(s).$$

Taking the residue of this equation at $s = \kappa$ and solving for $\operatorname*{res}_{s=\kappa} D^*_{F,G}(s)$ we obtain

$$\begin{aligned}
\operatorname*{res}_{s=\kappa} D^*_{F,G}(s) &= \frac{\pi^{2-\kappa} \left[\mathrm{Sp}_4(\mathbb{Z}) : \Gamma_0^{(2)}(M)\right]}{M^\kappa} \operatorname*{res}_{s=2} E^*_{s,M}(Z) \langle F, G \rangle \\
&= \frac{\pi^{2-\kappa} \left[\mathrm{Sp}_4(\mathbb{Z}) : \Gamma_0^{(2)}(M)\right]}{M^\kappa} \prod_{p|M} \left(\frac{1 - p_i^{-4}}{p_i^2 + 1}\right) \langle F, G \rangle \\
&= \frac{\pi^{2-\kappa} \left[\mathrm{Sp}_4(\mathbb{Z}) : \Gamma_0^{(2)}(M)\right]}{M^\kappa \zeta_M(4)} \prod_{p|M} \left(\frac{1}{p_i^2 + 1}\right) \langle F, G \rangle.
\end{aligned}$$

On the other hand, taking the residue at $s = \kappa$ of (7) gives

$$\operatorname*{res}_{s=\kappa} D^*_{F,G}(s) = (2\pi)^{-2\kappa}(\kappa-1)!\operatorname*{res}_{s=\kappa} D_{F,G}(s).$$

Combining these two results and solving for $\operatorname*{res}_{s=\kappa} D_{F,G}(s)$ we obtain

$$\operatorname*{res}_{s=\kappa} D_{F,G}(s) = \frac{2^{2\kappa}\pi^{\kappa+2}\left[\mathrm{Sp}_4(\mathbb{Z}) : \Gamma_0^{(2)}(M)\right]}{M^{\kappa}\zeta_M(4)(\kappa-1)!\prod_{p\mid M}(p_i^2+1)}\langle F, G\rangle. \tag{11}$$

Following [14], our next step is to calculate the adjoint of the operator $V_m$. We will need the following lemma.

**Lemma 4.4.** *Let* $\Delta^*_{M,0}(m) \subset \Delta_{M,0}(m)$ *be the matrices* $\begin{pmatrix} a & b \\ Mc & d \end{pmatrix}$ *with* $\gcd(a,b,c,d) = 1$*. The map*

$$\varphi_M : \Gamma_0(M,m)\backslash\Gamma_0(M) \to \Gamma_0(M)\backslash\Delta^*_{M,0}(m)$$
$$\begin{pmatrix} a & b \\ Mc & d \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} a & b \\ Mc & d \end{pmatrix}$$

*gives a bijection where*

$$\Gamma_0(M,m) = \left\{\begin{pmatrix} a & mb \\ Mc & d \end{pmatrix} \in \Gamma_0(M)\right\}.$$

*Proof.* The proof amounts to showing the cardinality of $\Gamma_0(M,m)\backslash\Gamma_0(M)$ is the same as the cardinality of $\Gamma_0(M)\backslash\Delta^*_{M,0}(m)$ and then showing the map $\varphi_M$ is injective by using the explicit coset representatives given for $\Gamma_0(M,m)\backslash\Gamma_0(M)$. Write $M = \prod_{i=1}^r p^{e_i}$ and $m = \prod_{i=1}^r p^{f_i}$ where the $e_i$ and $f_i$ are nonnegative integers. One has that

$$\#\left(\Gamma_0(M)\backslash\Delta_{M,0}(m)\right) = \prod_{i=1}^r \#\left(\Gamma_0(M)\backslash\Delta_{M,0}(p_i^{f_i})\right)$$

and so

$$\#\left(\Gamma_0(M)\backslash\Delta^*_{M,0}(m)\right) = \prod_{i=1}^r \#\left(\Gamma_0(M)\backslash\Delta^*_{M,0}(p_i^{f_i})\right).$$

Thus, we only need to calculate

$$\#\left(\Gamma_0(M)\backslash\Delta^*_{M,0}(p^f)\right)$$

for a prime $p$. This breaks into two cases depending upon whether $p$ divides $M$ or not. The main input is the fact that

$$\Gamma_0(M)\backslash\Delta_{M,0}(m) = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad = m, \gcd(a,M) = 1, 0 \leq b \leq d-1\right\}.$$

Using this, one sees that if $p \mid M$ then

$$\Gamma_0(M)\backslash\Delta^*_{M,0}(p^f) = \left\{\begin{pmatrix} 1 & b \\ 0 & p^f \end{pmatrix} : 0 \leq b < p^f\right\},$$

and so there are $p^f$ elements. If $p \nmid M$, one shows by induction on $f$ and counting as above that

$$\#\left(\Gamma_0(M)\backslash\Delta^*_{M,0}(p^f)\right) = p^f + p^{f-1}.$$

Recall the map

$$\lambda_T : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/T\mathbb{Z})$$

is surjective with kernel $\Gamma(T)$. From this, one sees that there is a bijection between $\Gamma_0(M,m)\backslash\Gamma_0(M)$ and $\lambda_T(\Gamma_0(M,m))\backslash\lambda_T(\Gamma_0(M))$ where $T = \mathrm{lcm}(M,m)$. Moreover, one has

$$\mathrm{SL}_2(\mathbb{Z}/T\mathbb{Z}) \cong \prod_{p^f \| T} \mathrm{SL}_2(\mathbb{Z}/p^f\mathbb{Z}).$$

Thus, one only needs to work with prime powers to compute the cosets. Write $M = p^e$ and $m = p^f$ for $e, f$ nonnegative integers. We break into cases for this:

(1) Suppose $e = 0$. In this case $\lambda_T(\Gamma_0(M,m))\backslash\lambda_T(\Gamma_0(M)) = \lambda_T(\Gamma^0(m))\backslash\lambda_T(\mathrm{SL}_2(\mathbb{Z}))$. In this case the coset representatives are given by $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for $0 \leq b < p^f$ and $\begin{pmatrix} 1 & 1 \\ (1+b)p-1 & (1+b)p \end{pmatrix}$ for $0 \leq b < p^{f-1}$ (see [7, Section 3.7]).
(2) Suppose $e$ is positive. In this case the representatives are given by $\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : 0 \leq b < p^f \right\}$.

Thus, the cardinalities matching up and the map $\varphi_M$ is clearly injective. □

**Proposition 4.5.** *Let $V_m^* : J^c_{\kappa,m}(\Gamma_0(M)^J) \to J^c_{\kappa,1}(\Gamma_0(M)^J)$ be the adjoint of $V_m$ with respect to the Petersson inner product. Let $\psi \in J^c_{\kappa,m}(\Gamma_0(M)^J)$ with*

$$\psi(\tau, z) = \sum_{\substack{D<0, r\in\mathbb{Z} \\ D\equiv r^2(4m)}} c(D,r) e\left(\frac{r^2 - D}{4m}\tau + rz\right).$$

*The action of $V_m^*$ on Fourier coefficients is given by*

$$V_m^*\psi(\tau,z) = \sum_{\substack{D<0, r\in\mathbb{Z} \\ D\equiv r^2(4)}} \left( \sum_{\substack{d|m \\ \gcd(d,M)=1}} d^{\kappa-2} \sum_{s\in S(r,d,D)} c\left(\frac{m^2}{d^2}D, \frac{m}{d}s\right)\right) e\left(\frac{r^2-D}{4}\tau + rz\right)$$

*where $S(r,d,D) = \{s \pmod{2d} : s \equiv r \pmod 2, s^2 \equiv D \pmod{4d}\}$.*

*Proof.* The proof here is analogous to the one given in [14] for the level 1 case. We include a proof for $M \geq 1$ and square-free here with more details for the reader's convenience. Let $\phi \in J^c_{\kappa,1}(\Gamma_0(M)^J)$. Given $a \in \mathbb{C}$, write $\phi_a(\tau,z)$ for the function $\phi(\tau, az)$. If $m'|m$ we write $m/m' = \square$ to denote that $m/m'$ is a perfect square. One has immediately from the definition and the lemma above that

$$\begin{aligned} V_m\phi &= m^{\kappa/2-1} \sum_{g\in\Gamma_0(M)\backslash\Delta_{M,0}(m)} \phi_{\sqrt{m}}|_{\kappa,m}\left(\frac{g}{\sqrt{m}}\right) \\ &= m^{\kappa/2-1} \sum_{\substack{m'|m \\ m/m'=\square}} \sum_{g\in\Gamma_0(M)\backslash\Delta^*_{M,0}(m)} \phi_{\sqrt{m}}|_{\kappa,m}\left(\frac{g}{\sqrt{m'}}\right) \\ &= m^{\kappa/2-1} \sum_{\substack{m'|m \\ m/m'=\square}} \sum_{g\in\Gamma_0(M,m)\backslash\Gamma_0(M)} \phi_{\sqrt{m}}|_{\kappa,m}\begin{pmatrix} \frac{1}{\sqrt{m'}} & 0 \\ 0 & \sqrt{m'} \end{pmatrix} g. \end{aligned}$$

Note that $\phi_{\sqrt{m}}|_{\kappa,m}\begin{pmatrix} \frac{1}{\sqrt{m'}} & 0 \\ 0 & \sqrt{m'} \end{pmatrix} \in J^c_{\kappa,m}(\Gamma_0(M,m)^J)$.

Given $\phi \in J^c_{\kappa,1}(\Gamma_0(M)^J)$ and $\psi \in J^c_{\kappa,m}(\Gamma_0(M)^J)$,

$$\begin{aligned} \langle V_m\phi,\psi\rangle &= m^{\kappa/2-1} \sum_{\substack{m'|m \\ m/m'=\square}} \sum_{g\in\Gamma_0(M,m)\backslash\Gamma_0(M)} \left\langle \phi_{\sqrt{m}}|_{\kappa,m}\begin{pmatrix} \frac{1}{\sqrt{m'}} & 0 \\ 0 & \sqrt{m'} \end{pmatrix} g,\psi \right\rangle \\ &= m^{\kappa/2-1} \sum_{\substack{m'|m \\ m/m'=\square}} [\Gamma_0(M):\Gamma_0(M,m)] \left\langle \phi_{\sqrt{m}}|_{\kappa,m}\begin{pmatrix} \frac{1}{\sqrt{m'}} & 0 \\ 0 & \sqrt{m'} \end{pmatrix},\psi \right\rangle \end{aligned}$$

where we have used $\langle \phi|\gamma,\psi\rangle = \langle \phi,\psi|\gamma^{-1}\rangle$ and $\psi|_{\kappa,m}g = \psi$ as $\psi$ has level $\Gamma_0(M)^J$. Note that $\psi_{\frac{1}{\sqrt{m}}}|_{\kappa,1}\begin{pmatrix} \sqrt{m'} & 0 \\ 0 & \frac{1}{\sqrt{m'}} \end{pmatrix} \in J^c_{\kappa,1}(\Gamma_0(M,m)^J)$. Moreover,

$$\left\langle \phi_{\sqrt{m}}|_{\kappa,m}\begin{pmatrix} \frac{1}{\sqrt{m'}} & 0 \\ 0 & \sqrt{m'} \end{pmatrix},\psi \right\rangle = \left\langle \phi,\psi_{\frac{1}{\sqrt{m}}}|_{\kappa,1}\begin{pmatrix} \sqrt{m'} & 0 \\ 0 & \frac{1}{\sqrt{m'}} \end{pmatrix} \right\rangle.$$

Observe we can write

$$\begin{aligned} &\left\langle \phi_{\sqrt{m}}|_{\kappa,m}\begin{pmatrix} \frac{1}{\sqrt{m'}} & 0 \\ 0 & \sqrt{m'} \end{pmatrix},\psi \right\rangle = \left\langle \phi,\psi_{\frac{1}{\sqrt{m}}}|_{\kappa,1}\begin{pmatrix} \sqrt{m'} & 0 \\ 0 & \frac{1}{\sqrt{m'}} \end{pmatrix} \right\rangle \\ &= \frac{1}{m'^2[\Gamma_0(M):\Gamma_0(M,m')]} \sum_{X\in(\mathbb{Z}/m'\mathbb{Z})^2} \sum_{g\in\Gamma_0(M,m')\backslash\Gamma_0(M)} \left\langle \phi,\psi_{\frac{1}{\sqrt{m}}}|_{\kappa,1}\begin{pmatrix} \sqrt{m'} & 0 \\ 0 & \frac{1}{\sqrt{m'}} \end{pmatrix} gX \right\rangle \end{aligned}$$

Thus, by essentially reversing the above argument we obtain

$$\langle V_m\phi,\psi\rangle = \left\langle \phi, m^{\kappa/2-3} \sum_{X\in(\mathbb{Z}/m\mathbb{Z})^2} \sum_{g\in\Gamma_0(M,m)\backslash\Gamma_0(M)} \psi_{\frac{1}{\sqrt{m}}}|_{\kappa,1}\left(\frac{gX}{\sqrt{m}}\right) \right\rangle.$$

One then checks that in fact

$$m^{\kappa/2-3} \sum_{X \in (\mathbb{Z}/m\mathbb{Z})^2} \sum_{g \in \Gamma_0(M) \backslash \Delta^*_{M,0}(m)} \psi_{\frac{1}{\sqrt{m}}} |_{\kappa,1} \left( \frac{gX}{\sqrt{m}} \right) \in J^c_{\kappa,1}(\Gamma_0(M)^J),$$

and so we obtain the formula for $V^*_m : J^c_{\kappa,m}(\Gamma_0(M)^J) \to J^c_{\kappa,1}(\Gamma_0(M)^J)$.

Thus, it just remains to compute the Fourier expansion of $V^*_m \psi$.

Let $\psi \in J^c_{\kappa,m}(\Gamma_0(M)^J)$ with Fourier expansion given by

$$\psi(\tau, z) = \sum_{\substack{D<0, r \in \mathbb{Z} \\ D \equiv r^2(4)}} c(D, r) e\left( \frac{r^2 - D}{4m} \tau + rz \right).$$

We use the same representatives for $\Gamma_0(M) \backslash \Delta_{M,0}(m)$ as above. Thus,

$$\begin{aligned} V^*_m \psi(\tau, z) &= m^{\kappa/2-3} \sum_{\lambda, \mu(m)} \sum_{\substack{ad=m \\ \gcd(a,M)=1 \\ b(d)}} \left( \frac{\sqrt{m}}{d} \right)^{\kappa} e(\lambda^2 \tau + 2\lambda z) \psi \left( \frac{a\tau + b}{d}, \frac{z + \lambda\tau + \mu}{d} \right) \\ &= m^{\kappa-3} \sum_{\lambda, \mu(m)} \sum_{\substack{ad=m \\ \gcd(a,M)=1 \\ b(d)}} d^{-\kappa} \sum_{\substack{D<0, r \in \mathbb{Z} \\ D \equiv r^2(4m)}} c(D, r) e\left( \left( \frac{r^2 - D}{4m} \frac{a}{d} + \frac{\lambda r}{d} + \lambda^2 \right) \tau \right) \\ &\quad \cdot e\left( \left( \frac{r}{d} + 2\lambda \right) z + \frac{r^2 - D}{4m} \frac{b}{d} + \frac{r\mu}{d} \right). \end{aligned}$$

Note that

$$\sum_{\substack{\mu(m) \\ b(d)}} e\left( \frac{r^2 - D}{4m} \frac{b}{d} + \frac{r\mu}{d} \right) = \begin{cases} md & \text{if } d \mid r \text{ and } d \mid \frac{r^2-D}{4m} \\ 0 & \text{otherwise.} \end{cases}$$

Setting $r_1 = dr$ and $D_1 = d^2 D$, we obtain

$$\begin{aligned} V^*_m \psi(\tau, z) = m^{\kappa-2} \sum_{\lambda(m)} \sum_{\substack{d \mid m \\ \gcd\left(\frac{m}{d}, M\right)=1}} d^{1-\kappa} \sum_{\substack{D_1<0, r_1 \in \mathbb{Z} \\ D_1 \equiv r_1^2(4m/d)}} c(d^2 D_1, dr_1) \\ \cdot e\left( \frac{(r_1 + 2\lambda)^2 - D_1}{4} \tau + (r_1 + 2\lambda) z \right). \end{aligned}$$

Letting $r_2 = r_1 + 2\lambda$,

$$\begin{aligned} V^*_m \psi(\tau, z) = m^{\kappa-2} \sum_{\lambda(m)} \sum_{\substack{d \mid m \\ \gcd\left(\frac{m}{d}, M\right)=1}} d^{1-\kappa} \sum_{\substack{D_1<0, r_2 \in \mathbb{Z} \\ D_1 \equiv (r_2 - 2\lambda)^2(4m/d)}} c(d^2 D_1, d(r_2 - 2\lambda)) \\ \cdot e\left( \frac{r_2^2 - D_1}{4} \tau + r_2 z \right). \end{aligned}$$

We can write

$$\lambda \equiv s + \frac{m}{d} s' \pmod{m}$$

where $s$ runs over $\mathbb{Z}/(m/d)\mathbb{Z}$ and $s'$ runs over $\mathbb{Z}/d\mathbb{Z}$. We immediately obtain

$$d(r_2 - 2\lambda) \equiv d(r_2 - 2s) \pmod{2m}, \quad D_1 \equiv (r_2 - 2s)^2 \pmod{4m/d}.$$

We now use the fact that the coefficients $c(D,r)$ depend only on the pair $(D,r)$ with $r \pmod{2m}$ and $D \equiv r^2 \pmod{4m}$ to write

$$V_m^*\psi(\tau,z) = m^{\kappa-2} \sum_{\substack{d|m \\ \gcd\left(\frac{m}{d},M\right)=1}} d^{2-\kappa} \sum_{s(m/d)} \sum_{\substack{D_1<0, r_1\in\mathbb{Z} \\ D_1\equiv(r_2-2s)^2(4m/d)}} c(d^2D_1, d(r_2-2s)) \cdot e\left(\frac{r_2^2-D_1}{4}\tau + r_2 z\right).$$

Finally, we change variables and replace $D_1$ by $D$, $d$ by $m/d$, and $r_2 - 2s$ by $s$ to obtain

$$V_m^*\psi(\tau,z) = \sum_{\substack{D<0, r\in\mathbb{Z} \\ D\equiv r^2(4)}} \left( \sum_{\substack{d|m \\ \gcd(d,M)=1}} d^{\kappa-2} \sum_{s\in S(r,d,D)} c\left(\frac{m^2}{d^2}D, \frac{m}{d}s\right)\right) e\left(\frac{r^2-D}{4}\tau + rz\right)$$

where $S(r,d,D) = \{s \pmod{2d} : s \equiv r \pmod{2}, s^2 \equiv D \pmod{4d}\}$. □

**Proposition 4.6.** *The map* $V_m^*V_m : J_{\kappa,1}^c(\Gamma_0(M)^J) \to J_{\kappa,1}^c(\Gamma_0(M)^J)$ *is given by*

$$V_m^*V_m = \sum_{\substack{d|m \\ \gcd(d,M)=1}} \varsigma(d) d^{\kappa-2} T_J\left(\frac{m}{d}\right)$$

*where* $T_J(n)$ *is the* $n$*th Hecke operator on* $J_{\kappa,1}^c(\Gamma_0(M)^J)$ *(we write* $U_J(p)$ *for* $T_J(p)$ *if* $p \mid M$*) and*

$$\varsigma(d) = d\prod_{p|d}\left(1+\frac{1}{p}\right).$$

*Proof.* The proof of this proposition follows along the same lines as the proof of the analogous result in the $M = 1$ case given in [14]. Note it is enough to check this fact on Fourier coefficients indexed by fundamental discriminants, as is pointed out in [14]. We show this for a representative case that is not too computationally cumbersome, but leave the proof of the general case to the reader.

Let $\phi \in J_{\kappa,1}^c(\Gamma_0(M)^J)$ and put $\psi = V_m\phi$, $\varphi = V_m^*\psi$. Write $c_\phi$ for the Fourier coefficients of $\phi$, $c_\psi$ for the Fourier coefficients of $\psi$, and $c_\varphi$ for the Fourier coefficients of $\varphi$.

We begin by recalling the definition of the action of $T_J(p)$ and $U_J(p)$ on the Fourier coefficients indexed by fundamental discriminants. Let $\phi \in J_{\kappa,1}^c(\Gamma_0(M)^J)$ with

$$\phi(\tau,z) = \sum_{\substack{D<0, r\in\mathbb{Z} \\ D\equiv r^2(4)}} c_\phi(D,r) e\left(\frac{r^2-D}{4}\tau + rz\right).$$

Then

$$c_{U_J(p)\phi}(D,r) = c_\phi(p^2D, pr)$$

and

$$c_{T_J(p)\phi}(D,r) = c_\phi(p^2D, pr) + \chi_D(p)p^{\kappa-2}c_\phi(D,r)$$

where $\chi_D$ is the quadratic character associated to the fundamental discriminant $D$.

Consider the case where $m = pq$ with $p \mid M$, $q \nmid M$. In this case

$$\sum_{\substack{d\mid m\\ \gcd(d,M)=1}} \varsigma(d)d^{\kappa-2}T_J\left(\frac{m}{d}\right) = T_J(m) + q^{\kappa-2}(q+1)U_J(p)$$

$$= T_J(q)U_J(p) + q^{\kappa-2}(q+1)U_J(p).$$

We have that the $(D,r)$th Fourier coefficient of $(T_J(m)+q^{\kappa-2}(q+1)U_J(p))\phi$ is given by

$$c_\phi(m^2D, mr) + q^{\kappa-2}(1+q+\chi_D(q))c_\phi(p^2D, pr).$$

We now calculate the $(D,r)$th Fourier coefficient of $\varphi$. Observe

$$c_\psi(D,r) = c_\phi(D,r) + q^{\kappa-1}c_\phi\left(\frac{D}{q^2}, \frac{r}{q}\right).$$

Using the above formula for $V_m^*$ on the Fourier coefficients we obtain

$$c_\varphi(D,r) = \sum_{\substack{d\mid m\\ \gcd(d,M)=1}} d^{\kappa-2} \sum_{s\in S(r,d,D)} c_\psi\left(\frac{m^2}{d^2}D, \frac{m}{d}s\right)$$

$$= \sum_{s\in S(r,1,D)} c_\psi(m^2D, ms) + q^{\kappa-2} \sum_{s\in S(r,q,D)} c_\psi(p^2D, ps).$$

Observe

$$\sum_{s\in S(r,1,D)} c_\psi(m^2D, ms) = c_\psi(m^2D, mr)$$

$$= c_\phi(m^2D, mr) + q^{\kappa-1}c_\phi(p^2D, pr).$$

For the second summation we have

$$\sum_{s\in S(r,q,D)} c_\psi(p^2D, ps) = \sum_{s\in S(r,q,D)} \left(c_\phi(p^2D, ps) + q^{\kappa-1}c_\phi\left(\frac{p^2D}{q^2}, \frac{ps}{q}\right)\right)$$

$$= \sum_{s\in S(r,q,D)} c_\phi(p^2D, ps)$$

where we used that $c_\phi(x,y) = 0$ unless $x$ and $y$ are both integers and for $\frac{p^2D}{q^2}$ to be an integer we must have $q^2 \mid D$, which cannot happen because $D$

is assumed to be a fundamental discriminant so cannot be divisible by the square of a prime. Hence

$$\begin{aligned}\sum_{s\in S(r,q,D)} c_\phi(p^2D,ps) &= c_\phi(p^2D,pr)\sum_{s\in S(r,q,D)} 1\\ &= (1+\chi_D(q))c_\phi(p^2D,pr)\end{aligned}$$

as the sum is counting whether $D$ is a square modulo $q$ or not. Thus, combining these

$$c_\varphi(D,r) = c_\phi(m^2D,mr) + q^{\kappa-2}\left(1+q+\chi_D(q)\right)c_\phi(p^2D,pr),$$

which is exactly what we were trying to prove. □

Let $F = \mathcal{V}_M\phi$ for $\phi \in J^c_{\kappa,1}(\Gamma_0(M)^J)$. Then

$$D_{F,F}(s) = \zeta^M(2s-2\kappa+4)\sum_{m\geq 1}\langle V_m\phi, V_m\phi\rangle m^{-s}.$$

Using the previous proposition we calculate

$$\begin{aligned}\langle V_m\phi, V_m\phi\rangle &= \langle V_m^* V_m\phi,\phi\rangle\\ &= \left\langle \sum_{\substack{d\mid m\\ \gcd(d,M)=1}} \varsigma(d)d^{\kappa-2}T_J\left(\frac{m}{d}\right)\phi,\phi\right\rangle\\ &= \sum_{\substack{d\mid m\\ \gcd(d,M)=1}} \varsigma(d)d^{\kappa-2}\lambda_f\left(\frac{m}{d}\right)\langle\phi,\phi\rangle\end{aligned}$$

where we recall that $T_J(n)\phi = \lambda_f(n)\phi$. Thus,

$$D_{F,F}(s) = \zeta^M(2s-2\kappa+4)\langle\phi,\phi\rangle\sum_{m\geq 1}\left(\sum_{\substack{d\mid m\\ \gcd(d,M)=1}} \varsigma(d)d^{\kappa-2}\lambda_f\left(\frac{m}{d}\right)\right)m^{-s}.$$

If we set

$$\begin{aligned}A(s) &= \sum_{\substack{d\geq 1\\ \gcd(d,M)=1}} a(d)d^{-s},\\ B(s) &= \sum_{t\geq 1} b(t)t^{-s}\end{aligned}$$

and

$$C(s) = \sum_{m\geq 1}\left(\sum_{\substack{dt=m\\ \gcd(d,M)=1}} a(d)b(t)\right)m^{-s},$$

then

$$C(s) = A(s)B(s).$$

We can apply this with $a(d) = \varsigma(d)d^{\kappa-2}$ and $b(t) = \lambda_f(t)$ to obtain

$$\begin{aligned} D_{F,F}(s) &= \zeta^M(2s-2\kappa+4)\langle\phi,\phi\rangle\left(\sum_{\substack{d\geq 1\\ \gcd(d,M)=1}} \varsigma(d)d^{-s+\kappa-2}\right)\left(\sum_{t\geq 1}\lambda_f(t)t^{-s}\right) \\ &= \zeta^M(2s-2\kappa+4)\langle\phi,\phi\rangle L(s,f)\left(\sum_{\substack{d\geq 1\\ \gcd(d,M)=1}} \varsigma(d)d^{-s+\kappa-2}\right). \end{aligned}$$

One can check immediately by expanding the right hand side that

$$\sum_{\substack{d\geq 1\\ \gcd(d,M)=1}} \varsigma(d)d^{-s} = \frac{\zeta^M(s-1)\zeta^M(s)}{\zeta^M(2s)}.$$

Thus,

$$D_{F,F}(s) = \zeta^M(s-\kappa+1)\zeta^M(s-\kappa+2)\langle\phi,\phi\rangle L(s,f).$$

In particular, taking the residue of each side at $s=\kappa$ we obtain

$$\begin{aligned} \operatorname*{res}_{s=\kappa} D_{F,F}(s) &= \operatorname*{res}_{s=1}\zeta^M(s)\zeta^M(2)\langle\phi,\phi\rangle L(\kappa,f) \\ &= \frac{\pi^2}{6}\left(\prod_{p\mid M}(1-p^{-1})^2(1+p^{-1})\right)\langle\phi,\phi\rangle L(\kappa,f). \end{aligned}$$

We now combine this with equations (1), (6), and (11) to obtain the following corollary.

**Corollary 4.7.** *Let $\kappa \geq 2$ be an even integer, $M$ an odd square-free integer, and $f \in S_{2\kappa-2}^{\mathrm{new}}(\Gamma_0(M))$ a newform. Let $F_f \in S_\kappa(\Gamma_0^{(2)}(M))$ be the Saito-Kurokawa lift of $f$. Then we have*

$$\langle F_f, F_f\rangle = \mathcal{A}_{\kappa,M}\frac{\left|a_{\theta^{\mathrm{alg}}_{\kappa,D}(f)}(|D|)\right|^2}{|D|^{\kappa-3/2}}\frac{L(\kappa,f)}{\pi L(\kappa-1,f,\chi_D)}\langle f,f\rangle$$

*where*

$$\mathcal{A}_{\kappa,M} = \frac{M^\kappa\zeta_M(4)\zeta_M(1)^2(\kappa-1)\left(\prod_{p\mid M}(1+p^2)(1+p^{-1})\right)}{2^{\nu(M)+3}[\Gamma_0(M):\Gamma_0(4M)][\mathrm{Sp}_4(\mathbb{Z}):\Gamma_0^{(2)}(M)]}.$$

One should note one can give a similar expression for general odd $M$ that depends upon the function $h(p_1,\ldots,p_n)$ that shows up in the calculation of the residue of the Eisenstein series $E_{s,M}(Z)$ above. As we will only be interested in the case of $M$ odd and square-free, we restrict our attention to this case.

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