The Langlands Program Seminar

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Abstract

These are notes from the ongoing Langlands Program Seminar organized since Spring 2014. What is known today as the Langlands Program, originating in a letter from Langlands to Weil in 1967, is now a vast area of research. While it asks questions of largely number theoretic interest, its methods have required input from harmonic analysis, representation theory, algebraic geometry, algebraic number theory, and more. Regarding emphasis, I have used as a guide the expository writing of Langlands, and as a result we have focused on functoriality via the trace formula, and reciprocity in terms of motives. These notes are produced by a naive graduate student and there are mistakes scattered throughout, so use them at your own risk! Please direct feedback and comments to `twong at gradcenter.cuny.edu`.

Contents

1 Basic objects

1.1 Automorphic Representations

1.1. We begin with the classical theory of theta series:

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 z}$$

with functional equation $\theta(-1/z) = (-iz)^{1/2}\theta(z)$, proven by the Poisson summation. Riemann used this to prove the functional equation for Riemann's zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}.$$

Riemann showed that the completed zeta function $\xi(s)$ is a Mellin transform:

$$\xi(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \int_0^{\infty} \frac{\theta(it)-1}{2} t^{s-1} dt$$

and using the functional equation of $\theta(z)$, the functional equation for $\zeta(s)$ follows: $\xi(s) = \xi(1-s)$.

1.2. Next we define a **modular form** of weight k for $G = SL(2,\mathbb{R})$ with discrete subgroup $\Gamma = SL(2,\mathbb{Z})$ or some congruence subgroup to be a holomorphic $f(z)$ on the upper-half plane $\mathbf{H}$ with transformation law

$$f(\gamma(z)) = f\Big(\frac{az+d}{cz+d}\Big) = (cz+d)^k f(z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Thus we can think of modular forms as representations of $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$ by lifting $f(z)$ to $\phi_f(g)$, requiring ϕ_f to be left-invariant under $SL(2,\mathbb{Z})$, right-$SO(2,\mathbb{R})$ finite, an eigenfunction of the Casimir operator, and satisfying a certain slow-growth condition.

More generally, an **automorphic form** of weight k for a topological group G with discrete subgroup Γ to be a function $f(\gamma(z)) = j(\gamma,z)^k f(z)$ where $j(\gamma,z)$ is called a factor of automorphy, satisfying a similar growth, eigenfunction and invariance conditions. Denote $\mathcal{A}$ the space of automorphic forms.

1.3. Hecke generalized Riemann's proof as follows: let $\{a_n\}$ be a sequence of complex numbers with $a_n = O(n^c)$ for $c > 0$, and let $h > 0, k > 0, \lambda > 0, C = \pm 1$, and define

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/h}, \ \phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, f(z), \text{ and } \Phi(s) = (\frac{\lambda}{2\pi})^s \Gamma(s)\phi(s)$$

Theorem 1.1.1. *The following are equivalent:*
(i) $\Phi + a_0/s + C/(k-s)$ *is entire, bounded on vertical strips, and satisfies* $\Phi(k-s) = C\Phi(s)$
(ii) $f(-1/z) = C(z/i)^k f(z)$*, i.e.,* $f(z)$ *is an automorphic form of weight* k*.*

The next question asks when a Dirichlet series $\sum a_n n^{-s}$ has an Euler product $\prod_p L_p(s)$. Hecke showed that for Hecke operator T_p acting on the space $\mathcal{A}$, we have the following

Theorem 1.1.2. *Assume, for convenience, that* $f(z) = \sum a_n e^{2\pi i n z}$ *is a modular form for* $SL(2,\mathbb{Z})$ *and* $a_1 = 1$*. Then* f *is an eigenfunction of* T_p *for all* p*, i.e.,*

$$T_p f(z) = p^{k-1} \sum_{\Gamma\backslash M_m} j(\gamma,z)^k f(\gamma(z)) = a_p f(z)$$

if and only if the a_n *are multiplicative, i.e., if* $a_{mn} = a_m a_n$ *whenever* m, n *are relatively prime.*

As a corollary of the multiplicativity of the a_n, $\phi(s)$ will have an Euler product as follows:

$$\phi(s) = \prod_p L_p(s) = \prod_p (1 - a_p p^{-s} + p^{k-1-2s})^{-1}$$

1.4. Tate, in a different setting, proved the functional equation for a Hecke L-function using adelic methods. Recall the adele ring of a number field F to be $\mathbb{A}_F = \prod_v' F_v$, where F_v is a local field completed at the prime v, and product is the restricted direct product, where almost all factors are $\mathcal{O}_v$, the ring of integers of F_v. In particular, is $F = \mathbb{Q}$ we have $\mathbb{A}_\mathbb{Q} = \mathbb{R} \times \prod' \mathbb{Q}_p$, where p are finite primes. This analysis follows in the spirit of Hasse's local-global principle.

Now fix a Hecke character $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ (a one dimensional representation of $GL(1, \mathbb{A})$) such that $\chi(x) = \prod \chi_v(x_v)$ where χ_v is trivial on units (unramified) for almost all v. Also define a Schwartz-Bruhat function $f(x) = \prod f_v(x_v)$ where f_v is Schwartz if v is an infinite prime, and locally constant compactly supported when v is finite. In summary, Tate defined a local zeta function

$$\zeta(f_v, \chi_v, s) = \int_{F_v^\times} f_v(x)\chi_v(x)|x|^{s-1}dx$$

and using Poisson summation and an appropriately defined Fourier transform for functions on number fields, proved a functional equation and analytic continuation $\zeta(f_v, \chi_v, s) = \gamma(\chi_v, s)\zeta(\hat{f}_v, \chi_v^{-1}, 1-s)$ where $\gamma(\chi_v, s)$ (local gamma factor) is a meromorphic function of s. Then considering the global zeta function $\zeta(f, \chi, s) = \prod \zeta(f_v, \chi_v, s)$ Tate showed that Hecke's L-series

$$L(s, \chi) = \prod_v L(s, \chi_v) = \prod_{v \text{ unram}} (1 - \chi_v(\tilde{\omega})N(p)^{-s})^{-1}$$

has an analytic continuation and functional equation $L(\chi, s) = \varepsilon(s, \chi)L(1-s, \chi^{-1})$ with $\varepsilon(s, \chi) = \prod \varepsilon(s, \chi_v)$ and $\varepsilon(s, \chi_v) = 1$ when χ_v is unramified.

1.5. Now we define an automorphic representation by making $\mathcal{A}$ into a **Hecke algebra $\mathcal{H}$** module as follows. For each finite place v let $\mathcal{H}_v$ be the convolution algebra of complex valued locally constant functions compactly supported on$G(F_v)$, with Haar measure normalized to give $G(\mathcal{O}_v)$ measure 1, so that the characteristic function of $G(\mathcal{O}_v)$ is an idempotent I_v in $\mathcal{H}_v$. Now form the restricted tensor product $\mathcal{H}_f$ of $\mathcal{H}_v$ with respect to I_v for all finite places v, by taking product functions that are I_v at almost every place. Next let $\mathcal{H}_\infty$ be the convolution algebra of all K_∞ finite distributions on G_∞ supported on K_∞, a maximal compact subgroup. Then define $\mathcal{H} = \mathcal{H}_\infty \otimes \mathcal{H}_f$, so that $\mathcal{A}$ is a smooth right $\mathcal{H}$-module, and an **automorphic representation** of $\mathcal{H}$ is any irreducible subquotient of $\mathcal{A}$. If $\mathcal{A}^0$ is the space of cusp forms, then any irreducible subquotient of $\mathcal{A}^0$ as an $\mathcal{H}$-module is a **cuspidal automorphic representation.**

1.2 The L-group

2.1. Let k be a field, with algebraic closure $\bar{k}$. There is a canonical bijection between isomorphism classes of connected reductive $\bar{k}$-groups and isomorphism classes of root systems: associate to G the root datum $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ where T is a maximal torus in G, $X_*(T)$ the group of characters of T, Φ the set of roots of G with respect to T. The choice of a Borel subgroup is equivalent to that of a basis Δ of Φ, whence a bijection between isomorphism classes of (G, B, T) and isomorphism classes of based root data $\psi(G) = (X^*(T), \Delta, X_*(T), \Delta^\vee)$.

To the $\bar{k}$-group G we first associate the group $\check{G}$ over $\mathbb{C}$ such that $\psi(\check{G}) = \psi(G)^\vee$, the dual root datum of G. Similarly we obtain $\check{B}$ and $\check{T}$. Let $f : G \to G'$ be a k-morphism, whose image is a normal subgroup. Then f induces maps $\psi(f) : \psi(G) \to \psi(G')$ and $\psi^\vee(f) : \psi(G')^\vee \to \psi(G)^\vee$, and thus $f^\vee : G'^\vee \to \check{G}$.

For example, if $G = GL_n, SL_n, Sp_{2n}$ then $\check{G} = GL_n(\mathbb{C}), PGL_n(\mathbb{C}), S)_{2n+1}(\mathbb{C})$.

2.2. Given $\gamma \in \mathrm{Gal}(k_s/k) = \Gamma_k$ there is a $g \in G(k_s)$ such that $g^{\gamma}Tg^{-1} = T$ thus an automorphism of $\psi(G)$ depending only on γ. So we have a homomorphism $\mu_G : \Gamma_k \to \mathrm{Aut}\ \psi(G)$. If G' is a k-group isomorphic to G over $\bar{k}$, then $\mu_G = \mu_{G'}$ if and only if G, G' are inner forms of each other. We have canonically Aut $\psi(G) =$ Aut $\psi(G)^{\vee}$, hence we may view μ_G descends to a map μ'_G to Aut $\psi(G)^{\vee}$. There is a split exact sequence

$$1 \longrightarrow \mathrm{Int}\ G \longrightarrow \mathrm{Aut}\ G \longrightarrow \psi(G) \longrightarrow 1$$

where any two splittings differ by an inner automorphism. So choose a monomorphism of Aut $\psi(G)^{\vee}$ to Aut $(\check{G}, \check{B}, \check{T})$. Then define the **L-group** of G to be ${}^L(G/k) = {}^LG = \check{G} \rtimes \Gamma_k$ with respect to μ'_G, well defined up to inner automorphism. We have the split exact sequence

$$1 \longrightarrow \check{G} \longrightarrow^L G \longrightarrow \Gamma_k \longrightarrow 1$$

If G splits over k, then Γ_k acts trivially on $\check{G}$ and ${}^LG = \check{G} \times \Gamma_k$. There will be many variants of this notion, replacing Γ_k by a group more suited to the context at hand. A representation of ${}^LG$ will be a continuous homomorphism $r : {}^LG \to GL_n(\mathbb{C})$ whose restriction to $\check{G}$ is a morphism of complex Lie groups.

1.3 The Satake Isomorphism

3.1. Let G be a connected reductive group over a local field k. Fix a maximal k-split torus S with centralizer M, B the Borel subgroup containing S. Let $W = N_G(S)/Z_G(S)$ be the restricted Weyl group of S, U and K the unipotent radical and (hyperspecial) maximal compact subgroup of P.

Theorem 1.3.1. *(Satake) Define the* **Satake transform** *as a linear map $S : H(G, K) \to H(M, M \cap K)^W$ by*

$$Sf(m) = \delta(m)^{1/2} \int_N f(mn)dn$$

for f in $\mathcal{H}(G, K)$ and m in M, and $\delta = |\det|$ is the modulus character. Then S is an isomorphism.

We do not present a proof of the theorem, and note that most expositions of the isomorphism (except Cartier's) give only a sketch of the proof.

Proposition 1.3.2. *If G is split, then $\mathcal{H}(M, M \cap K) \simeq \mathbb{C}[X_*(M)]$.*

Proof. Define $\gamma(m)$ to be the cocharacter such that $(\gamma(m), \chi) = \mathrm{ord}(\chi(m))$ for all χ in $X^*(M)$. This gives an exact sequence

$$1 \longrightarrow M(\mathcal{O}) \longrightarrow M(k) \longrightarrow X_*(M) \longrightarrow 0$$

To get a splitting of the sequence, we first note that each function in $\mathcal{H}(G, K)$ is constant on double K cosets, it is also compactly supported and hence a finite linear combination of characteristic functions 1_{KgK}, so that the 1_{KgK} are a basis for $\mathcal{H}(G, K)$. Given λ in $X_*(M)$ and ϖ uniformizing element of $\mathcal{O}$, then $\lambda(\varpi)$ belongs to $T(k)$, and since $\lambda(\mathcal{O}^{\times}) \subset T(\mathcal{O}) \subset K$ the double coset $K\lambda(\varpi)K$ does not depend on choice of ϖ. Then the mapping λ to $1_{K\lambda(\varpi)K}$ gives the splitting $M(k)/M(\mathcal{O}) \simeq X_*(M)$. Then by considering the group algebra $\mathbb{C}[X_*(M)] = \mathbb{C}[M(k)/M(\mathcal{O})] = \mathbb{C}[M/M \cap K]$ as the convolution algebra of functions from $M/M \cap K$ to $\mathbb{C}$, hence $\mathcal{H}(M, M \cap K)$.

Proposition 1.3.3. *There is a canonical identification $X_*(M) \simeq X^*(M^{\vee})$.*

Proof. Let $M^{\vee}(\mathbb{C}) = \mathrm{Hom}(M(k)/M(\mathcal{O}), \mathbb{C}^{\times})$. Then from the above splitting we have the identification $M^{\vee}(\mathbb{C}) = \mathrm{Hom}(X_*(M), \mathbb{C}^{\times})$. Also, given a complex torus $M(\mathbb{C})$ we can identify it with $\mathrm{Hom}(X^*(M), \mathbb{C}^{\times})$ by duality. So $\mathrm{Hom}(X^*(M^{\vee}), \mathbb{C}^{\times}) \simeq \mathrm{Hom}(X_*(M), \mathbb{C}^{\times})$, hence $X_*(M) \simeq X^*(M^{\vee})$.

Theorem 1.3.4. *(Chevalley) If $\check{G}$ is a complex connected reductive group with maximal torus $M^\vee$ and Weyl group W, then the restriction regular functions on $\check{G}$ to $M^\vee$ induces an isomorphism $\mathbb{C}[G]^G \simeq \mathbb{C}[M^\vee]^W$.*

A basis for conjugation invariant rational functions on G is given by characters of irreducible highest weight algebraic representations, spanning a $\mathbb{Z}$-lattice inside $\mathbb{C}[G]^G$, which can be identified with the algebra of isomorphism classes of virtual representations of $\check{G}$. The positive roots in $X^*(M^\vee)$ index irreducible representations V_λ of $\check{G}$, viewing $\mathrm{Tr}(V_\lambda)$ as an element of $\mathbb{C}[X^*(M^\vee)] = \mathrm{Rep}(\check{G})$.

Proposition 1.3.5. *There is a bijection between characters of $\mathcal{H}(G,K)$ and unramified characters χ of M.*

A representation (π, V) of G is *smooth* if every vector is fixed by a sufficiently small compact open subgroup. A smooth representation is *spherical* or *unramified* with respect to K if it contains a nonzero K-fixed vector, i.e., if $V^K \neq 0$. Given an irreducible spherical representation π of G, π^K is an irreducible $\mathcal{H}(G,K)$ module. By the Satake isomorphism (or Gelfand's lemma), $\mathcal{H}(G,K)$ is commutative, hence by Schur's lemma $\dim \pi^K = 1$ and $\mathcal{H}(G,K)$ acts by a character.

Conversely, given an unramified character χ of M, we may view it as a character of B and induce the spherical principal series representation $I(\chi) = \mathrm{Ind}_B^G(\delta^{\frac{1}{2}}\chi)$, locally constant functions from G to $\mathbb{C}$ such that $f(bgk) = \delta^{\frac{1}{2}}(b)\chi(b)f(g)$. χ is trivial on $M \cap K$, so by Iwasawa's decomposition $\dim\chi^K = 1$. Since $(\cdot)^K$ is exact, $I(\chi)$ has a unique irreducible spherical sub quotient π_χ.

Theorem 1.3.6. *(Langlands) $M^\vee/W$ naturally corresponds to semisimple $\check{G}$ conjugacy classes in $\check{G} \rtimes \sigma$.*

This was proved by Langlands, which proof we omit here. Finally we have the following identification:

Corollary 1.3.7. *There is a bijection between characters of $\mathcal{H}(G,K)$ and semisimple $\check{G}$ conjugacy classes in $\check{G} \rtimes \sigma$.*

Concisely, $\mathrm{Hom}(\mathcal{H}(G,K),\mathbb{C}) = \mathrm{Hom}(\mathcal{H}(M, M\cap K)^W,\mathbb{C}) = \mathrm{Hom}(\mathbb{C}[X^*(M^\vee)]^W,\mathbb{C}) = M^\vee/W = (\check{G} \rtimes \sigma)_{ss}$. We will explain the last in the next section.

1.4 The Weil-Deligne group

Let F be a field, $\overline{F}$ the algebraic separable closure of F. For each finite extension E of F in $\overline{F}$, let $G_E = \mathrm{Gal}(\overline{F}/E)$. If G is a topological group, G^c is the closure of its commutator subgroup, and $G^{ab} = G/G^c$ is the maximal abelian Hausdorff quotient of G. We begin with the exact sequence for E, F finite Galois extensions of $\mathbb{Q}_p$:

$$1 \longrightarrow I_{E/F} \longrightarrow \mathrm{Gal}(E/F) \longrightarrow \mathrm{Gal}(k_E/k_F) \longrightarrow 0$$

which letting E be $\overline{F}$,

$$1 \longrightarrow I_F \longrightarrow \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{Gal}(\overline{k}_F/k_F) \simeq \hat{\mathbb{Z}} \longrightarrow 0$$

This is the setting for the local Weil group. We now state the existence theorem of local class field theory:

Theorem 1.4.1. *Let E be a finite Galois extension of F. The map $E \to F^\times/N_{E/F}E^\times$ is a bijection between finite abelian Galois extensions E of F with finite index open subgroups of $F^\times$.*

Corollary 1.4.2. *The local reciprocity map $\theta_F : F^\times \to W_F^{ab} \subset G_F^{ab}$ is a topological isomorphism.*

The **Weil group** of F is a topological group W_F with a homomorphism φ into G_F with dense image and isomorphism $r_E : C_F \to W_E^{ab}$, where $C_E = E^\times$ if E is local and $\mathbb{A}_E^\times/E^\times$ if E is global, satisfying

(1) For each E, the composition $C_E \xrightarrow{\sim} W_E^{ab} \xrightarrow{\varphi} G_E^{ab}$ is the reciprocity law homomorphism.

(2) For w in W_E and $\varphi(w)$ in W_E, conjugation by w or $\phi(w)$ commutes with r.

(3) For $E' \subset E$, the inclusion $C_{E'} \hookrightarrow C_E$ and transfer homomorphism $t : G^{ab} \to H^{ab}$ commute with r.

(4) For $E \subset F$, the natural map $W_F \to \varprojlim W_F/W_E^c$ is an isomorphism of topological groups.
where $W_E = \varphi^{-1}(G_E)$. Continuity of φ implies W_E is open in W_F and dense image means φ induces bijection of homogeneous spaces $W_F/W_E \xrightarrow{\sim} G_F/G_E \simeq \mathrm{Hom}_F(E,F)$, which is $\mathrm{Gal}(E/F)$ when E/F is Galois.

Proposition 1.4.3. *A unique W_F exists for every F.*

Now we discuss the Weil group for the following four cases of F:

(1) F local nonarchimedean. For each E let k_E be its residue field with degree q_E, and $\overline{k}$ the union of all k_E. We take W_F to be the dense subgroup of G_F inducing on $\overline{k}$ the map $x \to x^{q_E^n}$ for n integral, so W_F contains the inertia group I_F and $W_F/I_F = \mathbb{Z}$. Toplogize W_F such that I_F is open with the profinite topology.

(2) F global function field. This is as above with 'constan1t field' for 'residue field' and 'inertia group' for 'geometric Galois group'.

(3) F local archimidean. If F is complex then $W_F = \mathbb{C}^\times$, φ trivial and $r_F = \mathrm{id}$. If F is real then we can take $W_F = \mathbb{C}^\times \cup j\mathbb{C}^\times$ with $j^2 = -1$ and $j\sigma j^{-1} = \overline{\sigma}$, where σ is the nontrivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. φ takes $\mathbb{C}^\times$ to 1 and $j\mathbb{C}^\times$ to σ; while $r_F = \mathrm{id}$.

(4) F global number field. φ is surjective with kernel the connected component of 1 and isomorphic to the inverse limit of connected components of 1 in C_E under $N_{E/E'}$.

What is the motivation behind this? Local class field theory gives not only an isomorphism between the completions $\mathrm{Gal}(\overline{F}/F)^{ab}$ and $\overline{F}^\times$ but also between the distinguished dense subgroups W_F and $F^\times$. Using the isomorphism between C_F and W_F^{ab} we can identify characters of C_F with those of W_F^{ab}, and by the dense image of W_F of G_F we can identify the isomorphism classes of representations of G_F as a subset of those of W_F, called 'of Galois type'.

Furthermore, for a nonarchimidean local field F. We view W_F as a group scheme over $\mathbb{Q}$, the inverse limit W_F/J over all normal subgroups J of I. Define the **Weil-Deligne group (scheme)** W_F' as $W_F \ltimes \mathbb{G}_a$, where W_F acts by $wxw^{-1} = ||w||x$.

Let E be a field of characteristic 0. A **representation of** W_F' **over** E is a homomorphism $\rho : W_F \to GL(V)$ with kernel containing an open subgroup of I_F (continuous for the discrete topology of $GL(V)$), V a finite dimensional vector space over E, and a nilpotent endomorphism N of V such that $\rho(w)N\rho(w)^{-1} = ||w||N$. The representation is semi simple if the Frobenius acts semisimply on V.

Let l be a prime different from the residue characteristic of k and G be an algebraic group over $\mathbb{Q}$ An l**-adic representation of** W_F is a homomorphism $W_F \to GL(V)$ where V is a finite dimensional vector space over a finite extension E_l of $\mathbb{Q}_l$.

Theorem 1.4.4. *(Deligne) There is a natural bijection between l-adic representations of W_F and representations of W_F' over E_l.*

Every element x of W_F' has a unique Jordan decomposition $x = x_s x_u = x_u x_s$. Moreover x is unipotent if and only if it is in $\mathbb{G}_a$, semi simple if either x is inertial or $\varepsilon(x)$ is nonzero, where $\varepsilon : W_F' \to \mathbb{Z}$. An **admissible homomorphism** of W_F' to ${}^LG$ over G_F is a continuous homomorphism α such that $\alpha(\mathbb{G}_a)$ is unipotent in $\check{G}$, and if $\alpha(W_F')$ is contained in a Levi subgroup of a parabolic subgroup P, then P is relevant. When $G = GL_n$, α is a semisimple representation of W_F'. Denote $\Phi(G)$ the set of equivalence classes of α up to inner automorphism by elements of $\check{G}$.

1.5 Touchstone I: Local Langlands

5.1. We now arrive at the first milestone: the local Langlands correspondence, or the local Langlands reciprocity conjecture, or local Langlands for short. First, let's motivate the conjecture with Artin reciprocity:

Theorem 1.5.1. *(Artin) Let k'/k be a finite abelian Galois extension of number fields, $\theta_{k'/k}$ the reciprocity map of C_k to Gal(k'/k), σ a representation of Gal(k'/k) on $GL_{\mathbb{C}}(V)$, and $\chi = \sigma \circ \theta_{k'/k}$. Then the Artin L-function is equal to the Hecke L-function,*

$$L_{k'/k}(\sigma, s) = L(s, \chi)$$
$$\prod_{\mathfrak{p}} (\det(I - \sigma(\mathrm{Fr}_P)|_{V^{I_P}} N(\mathfrak{p})^{-s}))^{-1} = \prod_{\mathfrak{p}\ \text{ramified}} (1 - \chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}})|\mathfrak{o}_k/\mathfrak{p}|^{-s})^{-1}$$

where $\mathrm{Fr}|_P$ *is the Frobenius element of Gal*$(\mathfrak{o}_{k'}/P; \mathfrak{o}_k/\mathfrak{p}) \simeq G_P/I_P$.

This correspondence first suggests to look for a bijection between $GL_n(\mathbb{C})$-representations of Γ_k and certain (automorphic) representations of GL_n, such that Artin's L-function will now be equal to an automorphic L-function, which we will define below. But this is not right formulation; it is not even true for $n = 1$! The left side has only finite-order Galois characters, while the right side has all Hecke characters. The first correction is by replacing Γ_k with the *Weil group* W_k. Then for $n = 1$ this is again class field theory. But we are not done: for $n = 2$, there are still too many representations on the right side in the local nonarchimidean case! Deligne observed that if one used ℓ-adic rather than complex Galois representations one had extra structure, and so introduced the *Weil-Deligne group* W'_k at the nonarchimedean places.

Lastly, in the global setting there is a conjectural *Langlands group* L_k that is equal to the Weil and Weil-Deligne groups when k is local, and in principle should be an extension of W_k by compact group. On the other hand, automorphic representations can be defined for any reductive algebraic group, not just GL_n. To this end Langlands introduced the *L-group*, and we finally seek the representations $\varphi : L_F \to {}^LG$, which conjugacy classes are called the *Langlands parameter*. Thus we have gone from

$$\sigma : \Gamma_k \longrightarrow GL_n(V) \text{ to } (r \circ \varphi) : L_F \longrightarrow {}^LG \longrightarrow GL_n(V).$$

5.2. In this section let G be a connected reductive algebraic group and k a local field. Set $\Gamma_k = \mathrm{Gal}(\bar{k}/k)$. If k is nonarchimedean, denote $W'_k = W'_k \ltimes \mathbb{G}_a$ the Weil-Deligne group of k, and if k is archimedean, set $W'_k = W_k$, the Weil group of k. We describe two sets, the 'Galois side' and the 'automorphic side':

1. $\Phi(G(k))$: the set of admissible homomorphisms from L_F to ${}^LG$ up to $\check{G}$-conjugacy, that is, if there is a g in $\check{G}$ such that $g\varphi_1(w)g^{-1} = \varphi_2(w)$. Recall an *admissible homomorphism* φ from W'_k to ${}^LG$ is
   (a) a continuous homomorphism over Γ_k, i.e, φ commutes with maps into Γ_k,
   (b) $\varphi(\mathbb{G}_a)$ is unipotent, and φ maps semisimple elements to semi simple elements, and
   (c) if $\varphi(W'_k)$ is contained in a Levi subgroup of a proper parabolic subgroup P of ${}^LG$, then P is relevant. (For the definition of *relevant*, we refer the reader to Borel's paper.)
2. $\Pi(G(k))$: the set of irreducible admissible (automorphic) representations of $G(k)$ up to infinitesimal equivalence. Recall that an *automorphic representation* of G is a topological G-module such that the subspace of admissible vectors is an automorphic representation of $\mathcal{H}$, that is, an irreducible subquotient of the space of automorphic forms on G, viewed as a $\mathcal{H}$-module.

   A continuous representation of a reductive group is *admissible* if every element is fixed by an open subgroup (smooth) and the fixed-point set of every compact open subgroup is finite dimensional. A $(\mathfrak{g}, K)$-*module* is a $\mathfrak{g}$-module and K-module V such that (1) Kv is a finite-dimensional subspace of V with smooth K-action for all v in V, (2) $kXv = (\mathrm{Ad}(k)X)kv$ for all X in $\mathfrak{g}$, and (3) on any finite-dimensional K-invariant subspace, the differential of the action of K coincides with the restriction of $\mathfrak{k} = \mathrm{Lie}(K)$ to the $\mathfrak{g}$ action. If π is an admissible representation of G, then π^K is a $(\mathfrak{g}, K)$-module. Two admissible representations are *infinitesimally equivalent* if they induce isomorphic $(\mathfrak{g}, K)$-modules.

Now let us state the local Langlands conjecture for a connected reductive group G over a local field k.

Conjecture 1.5.2. *(Local Langlands reciprocity) Let k be a local field. There is a partition of* $\Pi(G(k))$ *into finite sets* Π_φ *indexed by* $\Phi(G(k))$*. If* $G = GL_n$*, the correspondence is one-to-one and* $L(s,\varphi) = L(s,\pi,r)$.

The sets Π_φ are known as *L-packets*, and representations π belonging to the same packet are called *L-indistinguishable*, since the attached L-functions cannot distinguish them. How the L-packets should be parametrized is essentially known, but we will not detail it here.

If one views the conjecture as using $\Pi(G(k))$ to obtain information about $\Phi(G(k))$, this may be thought of as a nonabelian local class field theory. Vice versa, this gives a parametrization of irreducible admissible representations of G by also giving a parametrization of the representations r.

5.3. We will briefly describe the local L-factors of an automorphic L-function. Let a local field k with q the order of its residue field. Let χ be a character of Γ_k. Identify χ with a character of $W_k \simeq C_k$ corresponding to the reciprocity law homomorphism. First following Tate, set the local abelian L-functions, nonvanishing and meromorphic in s:

$$L(s,\chi) = \begin{cases} \Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) & k \text{ archimedean real} \\ \Gamma_{\mathbb{C}} = 2(2\pi)^{-s}\Gamma(s) & k \text{ archimedean complex} \\ (1-\chi(\varpi)q^{-s})^{-1} & k \text{ nonarchimedean, } \chi \text{ unramified} \\ 1 & k \text{ nonarchimedean, } \chi \text{ ramified} \end{cases}$$

to each there is a function $\varepsilon(s,\chi,\psi)$ where ψ is a nontrivial additive character of k, corresponding to the local functional equation. Then following Artin, we have the local nonabelian L-functions for a representation V of W'_k,

$$L(s,V) = \begin{cases} L(s,\chi) & k \text{ archimedean} \\ \det(1-\mathrm{Fr}|_{V^I} q^{-s})^{-1} & k \text{ nonarchimedean} \end{cases}$$

since $W'_{\mathbb{C}} = \mathbb{C}^\times$, and for real k the irreducible V of dim> 1 are $V = \mathrm{Ind}_{\mathbb{C}/\mathbb{R}}\chi$. In this setting the ϵ-functions are shown to exist, but the explicit formula is not known. Now let π be in $\Pi_\varphi(G(k))$ for some φ in $\Phi(G(k))$, r a representations of ${}^LG$. Then the *Langlands local L-factor* $L(s,\pi,r)$ is defined simply as

$$L(s,\pi,r) = L(s, r\circ\varphi), \varepsilon(s,\pi,r,\psi) = \varepsilon(s, r\circ\varphi,\psi).$$

In the case where k is local nonarchimedean and unramified, the Satake isomorphism allows us to view π as a semisimple conjugacy class in $\check{G} \rtimes \sigma$, and thus $L(s,\pi,r) = \det(1 - r([g\rtimes\sigma])q^{-s})^{-1}$.

When $G = GL_n$ we have the principal L-functions of Godement-Jacquet attached to each π, which are equal to the Langlands L-function at almost every place. Further, we also have π' for GL_m, then we have the Selberg-Rankin convolution L-functions $L(s,\pi\times\pi',r)$.

We have outlined how Langlands' automorphic L-function generalizes Hecke's L-function, but we have not mentioned the generalization of Artin's L-function on the Galois side. In general one expects the motivic L-function, which include the Artin and Hasse-Weil L-functions as special cases.

5.4. The following are some of the the most complete results of the local conjecture:

1. Langlands (1973) proved the conjecture for archimedean k for reductive groups (Langlands classification).
2. Laumon, Rapoport and Stuhler (1993) proved the case for GL_n nonarchimedean positive characteristic, then Harris and Taylor (1999), Henniart (2000) and Scholze (2013) proved the case in nonarchimedean characterstic 0. The proofs use ℓ-adic representations on both sides, translated into complex analytic functions by isomorphism with $\mathbb{C}$.
3. The Satake classification (1963) solves the case for G quasiplit and split over a finite Galois extension of k nonarchimedean and unramified.

1.6 The automorphic L-function

6.1. Let k be a global field with ring of integers $\mathcal{O}$, G a connected reductive k-group. Let π be an irreducible admissible representation of $G(\mathbb{A})$, and r a representation of ${}^LG$. By a well-known theorem of Flath,

Theorem 1.6.1. *Any irreducible admissible representation π of $G(\mathbb{A})$ is uniquely isomorphic $\otimes'\pi_v$ where π_v is an irreducible admissible representation of $G(k_v)$.*

Also, by restriction r defines a representation r_v of ${}^LG(k_v)$. Now *assuming the local Langlands correspondence*, we have a unique φ_v in $\Phi(G(k_v))$ such that π_v is in $\Pi(G(k_v))$. Now we define the automorphic L-function as follows

$$L(s,\pi,r)=\prod_v L(s,\pi_v,r_v), \quad \varepsilon(s,\pi,r)=\prod_v \varepsilon(s,\pi_v,r_v,\psi_v)$$

where ψ_v is a nontrivial additive character of k_v, and the local factors are those defined in the previous section.

Theorem 1.6.2. *(Langlands) Let π be an irreducible admissible unitarizable representation of $G(\mathbb{A})$ and r a representation of ${}^LG$. Then $L(s,\pi,r)$ converges absolutely for Re(s) large enough.*

Now let π be a cuspidal automorphic representation of $G(\mathbb{A})$, hence unitary modulo the center. It is shown that every cuspidal representation is a constituent of a representation induced from a cuspidal representation of the Levi subgroup of G, yielding

Theorem 1.6.3. *(Langlands) Let π be a cuspidal automorphic representation of $G(\mathbb{A})$, and r a representation of ${}^LG$. Then $L(s,\pi,r)$ converges absolutely in some right half-plane.*

We say that φ in $\Pi(G)$ is unramified if it is trivial on $\mathbb{G}_a$ and the inertia group I, and we may assume that the image of φ lies in ${}^LT$, and by a property of admissible homomorphsms G is quasisplit. Then Langlands' interpretation of the Satake isomorphism associates a semisimple conjugacy class of ${}^LG$ to Hecke algebra character associated to π is represented by $t \rtimes \sigma$.

The following are some examples:

(a) $G = GL_n$, r the standard representation of $GL_n(\mathbb{C})$. Then $L(s,\pi,r)$ satisfies a functional equation, and for π cuspidal and r irreducible nontrivial, then $L(s,\pi,r)$ is entire. This is proved in $n=2$ by Jacquet and Langlands, and for $n \geq 2$ by Godement and Jacquet using 'standard' L-functions for GL_n.

(b) $G = GL_m \times GL_n$ where $m < n$, $r = r_m \otimes r_n$ and $\pi \times \pi'$ a unitary cuspidal automorphic representation. Then $L(s,\pi\times\pi', r_m\otimes r_n)$ converges for $\mathrm{Re}(s)$ large enough, and extends to an entire function of s bounded in vertical strips, satisfying a functional equation.

Converse theorems. To what extent can analytic properties of a given L-function characterize a representation? THe first main result was due to Hecke, followed by Weil, and proven in the GL_2 setting by Jacquet Langlands, and subsequently for GL_3 by Jacquet, Piatetskii-Shapiro and Shalika.

1.7 Reductive groups

Let G be a connected reductive linear algebraic group, fix T a maximal torus of G. Define a **root system** $\Phi = \Phi(G,T)$ to be the set of nontrivial characters of T in diagonalizing the adjoint representation of T on $\mathrm{Lie}(G)$.

Associate to (G,T) the **root datum** $\psi(G,T) = (X^*(T), \Phi, \check{X}_*(T), \check{\Phi})$. For α in Φ, let T_α be the identity component of the kernel of α, which is a subtorus of codimension 1. Its centralizer in G is a connected reductive group with maximal torus T, whose derived group G_α is semisimple of rank 1 (isomorphic to SL_2 or PSL_2. There is a unique homomorphism $\breve{\alpha} : \mathbb{G}_m \to G_\alpha$ such that $T = (\mathrm{Im}\ \breve{\alpha})T_\alpha$ and $(\alpha, \breve{\alpha}) = 2$.

An **isogeny** of algebraic groups is a surjective rational homomorphism with finite kernel. Example: The canonical homomorphism of SL_2 to PSL_2, but in characteristic 2 this is an isomorphism of abstract but not algebraic groups.

Theorem 1.7.1. *(1) For any root datum Ψ with reduced root system there exists a reductive group G with maximal torus T in G such that $\Psi = \psi(G,T)$. The pair (G,T) is unique up to isomorphism.*

(2) Let $\Psi = \psi(G,T)$ and $\Psi' = \psi(G',T')$. If f is an isogeny of Ψ' into Ψ there exists a central isogeny ϕ of (G,T) onto (G',T') with $f(\phi) = f$ Two such ϕ differ by an automorphism Int(t) of G, where $t \in T$.

Now let k be a field. A linear algebraic group defined over k is a k-group, and $G(k)$ its k-rational points. Let G, G' be k-groups. G' is a ***k*-form** of G if G and G' are isomorphic over the algebraic closure. Example: U_n is an $\mathbb{R}$-form of GL_n.

If $G \to GL_n$ is a k-isomorphism of G onto a closed subgroup of GL_n, then $G(k^s)$ determines G up to k-isomorphism. The k-forms of G, up to isomorphism, are as follows: there is a continuous function $c : s \to c_s$ of $\mathrm{Gal}(k^s/k)$ to the k^s-automorphisms of G such that $c_{st} = c_s s(c_t)$. G' is an **inner form** of G if all c_s are inner automorphisms, and G' is k-isomorphic to G if and only if there is an automorphism c such that $c_s = c^{-1}sc$.

The continuous functions c_s above are **1-cocycles** of $\mathrm{Gal}(k^s/k)$ with values in G. Modulo the last relation above these form the cohomology $H^1(k,G)$.

Let G be a connected reductive k-group. G is **quasisplit** if it contains a Borel subgroup defined over k, and **split** (over k) if it contains a maximal k-split torus. Example: $\mathrm{SO}(Q)$ is quasiplit but not split if and only if the dimension n of the underlying vector space is even and the index is $n/2 - 1$. G is **anisotropic** (over k) if it has no nontrivial k-split subtorus. Examples:

(i) Let Q be a nondegenerate quadratic form on a k-vector space (char$\neq 2$), then the identity component $\mathrm{SO}(Q)$ of $\mathrm{O}(Q)$ is anisotropic if an only if Q is not zero over k.

(ii) If k is locally compact and not discrete, then G is anisotropic over k if and only if $G(k)$ is compact.

(iii) If k is any field then G is anisotropic if and only if $G(k)$ has no nontrivial unipotent elements and the group is its k-rational characters $\mathrm{Hom}_k(G, \mathbb{G}_m)$ is trivial.

A **parabolic** subgroup P of an algebraic group G is a closed subgroup such that G/P is a projective scheme. Equivalently, P is parabolic if P contains a Borel subgroup of G (a maximal Zariski closed and connected solvable algebraic subgroup.).

Let G be a connected reductive k-group an P a parabolic k-subgroup, with unipotent radical N. A **Levi** subgroup of P is a k-subgroup L such that $P = L \rtimes N$. If A is a maximal k-split torus in the center of L then $L = Z(A)$.

The **Tits building** $\mathcal{B}$: Let G be a connected reductive k-group. The vertices of $\mathcal{B}$ are the maximal nontrivial parabolic k-subgroups of G. A set of vertices determine a simplex of if and only if their intersection is parabolic. A simplex is a face of another simplex if and only if the its parabolic is contained in the other. Maximal simplices (chambers) correspond to minimal parabolics. A codimension-1 face of a chamber is a wall. $G(k)$ acts on $\mathcal{B}$.

Example: GL_n is a reductive k-group. The subgroup A of diagonal matrices is a maximal k-split torus which is also a maximal torus of G. The e_i mapping $a \in A$ onto the ith diagonal element form a basis for $X^*(A)$. The root system consist of $e_i - e_j$, $i \neq j$, and one has the root datum $(\mathbb{Z}^n, \{e_i - e_j\}, \mathbb{Z}^n, \{\check{e}_i - \check{e}_j\})$. The subgroup B of upper triangular matrices is a minimal parabolic k-subgroup, also Borel. The unipotent radical has ones on the diagonal, upper triangular. The Weyl group is isomorphic to S_n. The parabolic subgroups $P \supset B$ are the upper triangular block matrices, A_{ii} nonsingular, with unipotent radical when $A_{ii} = I$. The subgroup of P with $A_{ij} = 0$ for $i < j$ is a Levi subgroup of P.

Let $V = \bar{k}^n$. A **flag** in V is a sequence $0 = V_0 \subset \cdots \subset V_s = V$ of distinct subspaces. A k-flag has each V_i defined over k. GL_n acts on the set of all flags; the parabolic subgroups are the isotropy groups of flags, and there is a bijection between parabolic subgroups of G and flags. G/P can be viewed as the flags of the same type as P, i.e., their subspaces have constant dimension.

If G is an $\mathbb{R}$ group, then $G(\mathbb{R})$ is canonically a real Lie group. If G is a $\mathbb{C}$-group then $G(\mathbb{C})$ is a complex Lie group, and $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} G(\mathbb{R})$ is defined by $G(\mathbb{C})$.

1.8 Touchstone II: Functoriality principle

The principle of functoriality is simple to state, but its implications are many and deep. We begin with a description of functoriality.

Let G and G' be reductive groups, and $\rho : {}^L G \to {}^L G'$ an L-homomorphism, i.e., commutes with projection on to the Galois group and is complex analytic over $\check{G}$. Then there should be a correspondence of automorphic representations $\pi \to \pi'$ such that $t(\pi_v)$ and $\rho(t(\pi'_v))$ are conjugate for almost all v. Moreover, one expects for any finite dimensional representation r' or ${}^L G'$ the equality

$$L(s, \pi', r') = L(s, \pi, r' \circ \rho).$$

We list several key consequences of the conjecture to illustrate its power:

1. G arbitrary, $G' = GL_n$, reduces the theory for reductive groups to that of GL_n. In particular one gets analytic control of their automorphic L-functions through the Godement-Jacquet L-function.
2. G trivial, $G' = GL_n$, would imply the reciprocity conjecture, yielding the Artin conjecture.
3. $G = GL_2$, $G' = GL_n$, would imply the Ramanujan conjecture in utmost generality.
4. $G = GL_n \times GL_m$, $G' = GL_{nm}$ endows the category of automorphic representations of GL_k for all $k > 0$ the structure of a monoidal category.

Functoriality, conjectured in 1979, has only been established in a very limited number of cases, and the complete picture is not expected for a long time still. The following are the strongest results towards functoriality:

1. $G = GL_n(F)$, $G' = \mathrm{Res}_{E/F} GL_n$ where $\mathrm{Gal}(E/F)$ is cyclic. This result is due to Arthur-Clozel (1989) using the stable trace formula.
2. $G = GL_2$, $G' = GL_3, GL_4, GL_5$, symmetric square, cube and fourth by Gelbart-Jqcquet (1978), Kim-Shahidi (2002) and Kim (2003) using converse theorems.
3. $G = GL_2 \times GL_2$, $G' = GL_4$, tensor product representation, due to Ramakrishnan (2000) using the converse theorem of Cogdell and PS.
4. $G = GL_2 \times GL_3$, $G' = GL_6$, tensor product representation, due to Kim-Shahidi (2002) again by converse theorem.

In the theory of automorphic forms, one usually carries out a *comparison* of trace formulas, corresponding to two groups. Roughly speaking, a trace formula is an identity

$$\text{(irreducible characters of } G \text{ on } f) = \text{(orbital integrals of } f),$$

respectively, the spectral and geometric sides. Here f is a properly chosen *test function*, allowing one to *match* the orbital integrals on both groups, giving cancellation and in the end an identity of characters. In general one has

$$\text{(cuspidal)} + \text{(one-dimensional)} + \text{(continuous)} + \text{(residual)}$$

$$= \text{(central)} + \text{(elliptic)} + \text{(hyperbolic)} + \text{(unipotent)}$$

2 The Arthur-Selberg trace formula

2.1 A first look

Let Γ be a discrete subgroup of a locally compact unimodular topological group G. Let R be the unitary representation of G by right translation on $L^2(\Gamma\backslash G)$. Choose a *test function* f in $C_c^\infty(H)$, and define the operator

$$R(f) = \int_G f(y)R(y)dy$$

acting on $L^2(\Gamma\backslash G)$ by

$$R(f)\varphi(x) = \int_G f(y)\varphi(xy)dy = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y).\varphi(y)dy$$

so $R(f)$ is an integral operator with kernel

$$K(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y),$$

where the sum is finite, taken over the intersection of the discrete Γ with the compact subset $x\text{supp}(f)y^{-1}$.

If Γ is cocompact, the kernel is square integrable, so $R(f)$ is a trace class operator and by the spectral theorem R decomposes discretely into irreducible representations π with finite multiplicity. Taking trace leads to the *geometric expansion*

$$\text{tr}R(f) = \int\limits_{\Gamma\backslash G} K(x,x)dx = \int\limits_{\Gamma\backslash G} \sum_{\{\Gamma\}} \sum_{\Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)dx = \sum_{\{\Gamma\}} \text{vol}(\Gamma_\gamma\backslash G_\gamma) \int\limits_{G_\gamma\backslash G} f(x^{-1}\gamma x)dx,$$

and the *spectral expansion*

$$\text{tr}R(f) = \sum_\pi m(\pi)\text{tr}(\pi(f)).$$

In the notation of Arthur, one has a first identity

$$\sum_\pi a_\Gamma^G(\pi) f_G(\pi) = \sum_\gamma a_\Gamma^G(\gamma) f_G(\gamma).$$

When $G = \mathbb{R}$, $\Gamma = \mathbb{Z}$, this is the Poisson summation formula, with irreducible representations $e^{2\pi inx}$. In the following we will almost exclusively consider the setting of $G = G(\mathbb{A})$, $\Gamma = G(F)$ where G is a reductive group and F a number field, or simply $\mathbb{Q}$. In general, $Z_\mathbb{A} G_\mathbb{Q}\backslash G_\mathbb{A}$ is not compact, causing $R(f)$ to not be trace class and R not to decompose discretely. (The quotient is noncopact when G contains a proper parabolic subgroup P defined over $\mathbb{Q}$.)

Selberg introduced the trace formula for $G = SL_2(\mathbb{R})/SO_2$, Γ a subgroup of $SL_2(\mathbb{Z})$. He studied this in the context of the Laplacian on a Riemann surface (or 'weakly symmetric spaces')

2.2 Coarse expansion

Previously we showed that tr$R(f)$ leads to the identity

$$\sum_{\pi} a_{\Gamma}^{G}(\pi) f_G(\pi) = \sum_{\gamma} a_{\Gamma}^{G}(\gamma) f_G(\gamma)$$

in the case of compact quotient. When the quotient is noncompact, parallel problems arise: continuous spectrum appears in L^2 and some measures become infinite. The first fix is to study the quotient mod center, i.e., $Z_{\mathbb{A}} G_{\mathbb{Q}} \backslash G_{\mathbb{A}}$, or what is nearly equivalent, $(G_{\mathbb{Q}} \backslash G_{\mathbb{A}})^1$, the elements of norm one. We will use both. (See Knapp's *Theoretical Aspects of the Trace Formula* (1997) for a discussion on their precise relationship). Now we will develop Arthur's coarse expansion for the case $G = GL_2$. The most general formulas will be left for the end.

$\mathbf{GL_2}$ example. Recall the decomposition of $GL_2 = PK$ where P is a parabolic subgroup and K a maximal compact subgroup, then $P = MN$, respectively Levi subgroup and nilpotent subgroup. For example,

$$P = \begin{pmatrix} a & * \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & *' \\ 0 & 1 \end{pmatrix}, K = SO_2, \quad a, b \in \mathbb{A}^{\times}$$

unless stated otherwise we fix the subgroups P, M, N, K.

Rewrite the kernel. Manipulating the integral operator

$$R(f) = \int_{Z_{\mathbb{A}} \backslash G_{\mathbb{A}}} f(x^{-1}y)dy = \int_{Z_{\mathbb{A}} N_{\mathbb{A}} M_{\mathbb{Q}} \backslash G_{\mathbb{A}}} \Big(\sum_{\gamma \in Z_{\mathbb{Q}} \backslash M_{\mathbb{Q}}} \int_{N_{\mathbb{A}}} f(x^{-1}\gamma n y)dn \Big) dy := \int_{Z_{\mathbb{A}} N_{\mathbb{A}} M_{\mathbb{Q}} \backslash G_{\mathbb{A}}} K_P(x,y)dy$$

so the kernel $K_P(x, y)$ is that of $R(f)$ acting in $L^2(Z_{\mathbb{A}} N_{\mathbb{A}} M_{\mathbb{Q}} \backslash G_{\mathbb{A}})$ in place of $L^2(Z_{\mathbb{A}} G_{\mathbb{Q}} \backslash G_{\mathbb{A}})$ before. Parallel to this, one also rewrites $K(x, y)$ using Eisenstein series, but we leave this for the third section.

Modify the kernel. The modified kernel is the following:

$$k^T(x, f) = K(x, x) - \sum_{P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}} K_P(\delta x, \delta x) \hat{\tau}_P(H(\delta x) - T)$$

where $H_P : M_{\mathbb{A}} \to \mathfrak{a}_M = \hom(X(M), \mathbb{R}) \simeq \mathbb{R}^2$ is the height function sending $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ to $(\log |a|_{\mathbb{A}}, \log |b|_{\mathbb{A}})$, and T is a point on a positive cone $\mathfrak{a}_M^+$ (which is roughly the first quadrant). Essentially the characteristic function is 1 at heights large enough, so the modification subtracts the parts 'at infinity' from the kernel. Consequently, when the quotient is compact, or when there are no proper parabolics, $k^T(x, f) = K(x, x)$.

Theorem 2.2.1. *(Arthur) The integral*

$$\int_{Z_{\mathbb{A}} G_{\mathbb{Q}} \backslash G_{\mathbb{A}}} k^T(x, f)dx$$

converges absolutely, is a polynomial in T and a distribution on $f \in C_c^{\infty}(G_{\mathbb{A}})$

The proof of this long, involving careful analysis of the roots and domain of integration. An important consequence of this theorem is that it gives *a priori* control on the growth of the integral, avoiding more complicated analysis in later computations.

Coarse geometric expansion. Any element of G has a Jordan decomposition into unipotent and semisimple parts $\gamma = \gamma^s \gamma^u$, and we will say two elements are equivalent if their semisimple components are $G_{\mathbb{Q}}$-conjugate. Denote such an equivalence class $\mathfrak{o}$, and $\mathcal{O}$ the set of $\mathfrak{o}$.

If $\mathfrak{o}$ does not intersect P, we will call it *elliptic*. (Arthur calls these *anisotropic*, and reserves the former for a smaller subset.) If $\mathfrak{o}$ intersects P then there is a representative

$$p = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

If $a = b = 1$ then $\mathfrak{o}$ contains the trivial class and the *unipotent* classes. If a or b are nontrivial then $\mathfrak{o}$ is a *hyperbolic* class. When $G = GL_2$ conjugacy means their characteristic polynomials are equal, and the three classes correspond to the sign of the discriminant. Then we decompose

$$K_P(x, y) = \sum_{\mathfrak{o} \in \mathcal{O}} K_{P,\mathfrak{o}}(x, y) = \sum_{\mathfrak{o} \cap Z_{\mathbb{Q}} \backslash M_{\mathbb{Q}}} \int_{N_{\mathbb{A}}} f(x^{-1} \gamma n y) dn$$

And $k_{\mathfrak{o}}^T(x, f)$ is defined similarly.

Coarse spectral expansion. By Langlands' work on Eisenstein series, $L^2(Z_{\mathbb{A}} G_{\mathbb{Q}} \backslash G_{\mathbb{A}})$ decomposes into right $G_{\mathbb{A}}$ invariant subspaces indexed by $\mathfrak{X}$, the set of cuspidal automorphic data $\chi = \{(P, \sigma)\}$, where P is a standard parabolic subgroup and σ a irreducible cuspidal representation of $Z_{\mathbb{A}} \backslash M_{\mathbb{A}}$. Recall that a cuspidal representation is subspace of $L^2(Z_{\mathbb{A}} \backslash G_{\mathbb{A}})$ where the functions satisfy

$$\int_{N_{\mathbb{A}}} f(nx)\ dx = 0$$

(Parallel to Fourier series with constant coefficient zero). For GL_2, we have only either $P = G$, in which case σ is a cuspidal representation of $Z_{\mathbb{A}} \backslash G_{\mathbb{A}}$, where

$$K_\chi(x, y) = \sum_{\text{o.n set of } \pi} R(f) \varphi(x) \overline{\varphi(y)},$$

or $P =$ upper triangular matrices, where $\sigma(\begin{pmatrix} a & * \\ 0 & b \end{pmatrix}) = \mu(ab^{-1})$ where μ is a character of $(\mathbb{Q}^\times \backslash \mathbb{A}^\times)^1$. Here

$$K_\chi(x, y) = \sum_{\text{o.n set of } R(s,\mu)} \int_{i\mathbb{R}} E(x, R(s, \mu)(f) \varphi, s) \overline{E(x, \varphi, s)} ds$$

where $R(s, \mu)$ is right translation in the induced representation space $\mathrm{Ind}_{P_{\mathbb{A}}}^{G_{\mathbb{A}}} \mu(\cdot) |\cdot|^s$ of μ from P to G, and $E(x, \varphi, s)$ is the Eisenstein series

$$\sum_{\gamma \in P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}} \varphi(x) e^{s\alpha(H_P(x))},$$

with α in the $\mathbb{Z}$ dual $\mathfrak{a}_M^*$. Then $K_{P,\chi}(x, y)$ and $k_\chi^T(x, f)$ are defined similarly. We have immediately

$$J_\chi^T(f) := \sum_{\chi \in \mathfrak{X}} k_\chi^T(x, f) = \sum_{\mathfrak{o} \in \mathcal{O}} k_{\mathfrak{o}}^T(x, f) := J_{\mathfrak{o}}^T(f).$$

The integrals of either side over $Z_{\mathbb{A}} G_{\mathbb{Q}} \backslash G_{\mathbb{A}}$ again converge absolutely, as a nontrivial result of Arthur.

Explicit $\mathbf{GL}_2$ formula. For reference, we include the resulting computations of the integrals for each class, omitting details. This can be done explicitly for GL_2, but quickly becomes intractable in higher rank, hence Arthur's theory.

The geometric side:

$$\begin{aligned}
J_{\text{ellip}}(f) &= \text{vol}(Z_{\mathbb{A}}G_{\mathbb{Q}}\backslash G_{\mathbb{A}}) \int_{G_{\gamma\mathbb{A}}\backslash G_{\mathbb{A}}} f(x^{-1}\gamma x)dx \\
J^T_{\text{hyperb}}(f) &= (T_1 - T_2)\text{vol}((\mathbb{Q}^\times\backslash\mathbb{A}^\times)^1) \int_{M_{\mathbb{A}}\backslash G_{\mathbb{A}}} f(x^{-1}\gamma x)dx \\
&\quad - \frac{1}{2}\text{vol}((\mathbb{Q}^\times\backslash\mathbb{A}^\times)^1) \int_{M_{\mathbb{A}}\backslash G_{\mathbb{A}}} f(x^{-1}\gamma x)\alpha(H(wx)+H(x))dx, \\
J_{\text{unip}}(f) &= \text{vol}(Z_{\mathbb{A}}G_{\mathbb{Q}}\backslash G_{\mathbb{A}})f(1) + \underset{s=1}{\text{fp}}\,(\zeta(F,s)) + (T_1 - T_2)\text{vol}((\mathbb{Q}^\times\backslash\mathbb{A}^\times)^1)\int_{\mathbb{A}} F(y)dy \\
&\quad \text{where } F(y) = \int_K f(k^{-1}\begin{pmatrix}1 & a \\ 0 & 1\end{pmatrix} k)\ dk \text{ and } \zeta(F,s) = \int_{\mathbb{A}^\times} F(a)|a|^s\ da, \\
J^T_{\text{cusp}}(f) &= \text{tr}\pi(f) \\
J^T_{\mu\neq\mu^{-1}}(f) &= (T_1 - T_2)\int_{i\mathbb{R}} \text{tr}(R(s,\mu)(f))\ ds + \int_{i\mathbb{R}} \text{tr}(M(-s)M'(s)R(s,\mu)(f))\ ds \\
J^T_{\mu=\mu^{-1}}(f) &= (T_1 - T_2)\int_{i\mathbb{R}} \text{tr}(R(s,\mu)(f))\ ds + \int_{i\mathbb{R}} \text{tr}(M(-s)M'(s)R(s,\mu)(f))\ ds \\
&\quad + \frac{1}{4}\text{tr}(M(0)R(0,\mu)(f)) + \text{vol}(Z_{\mathbb{A}}G_{\mathbb{Q}}\backslash G_{\mathbb{A}})\mu(f)
\end{aligned}$$

Note that the first three and the last three are in principle parallel.

Finally, we give for completeness the general form of $K_{P,\chi}(x,y)$, the spectral form of the kernel:

$$\sum_{P'\subset P} (n^P_{P'})^{-1} \int_{i\mathfrak{a}^*_M} \sum_{\varphi\in\mathcal{B}_{P',\chi}} E^P_{P'}(x, R(s,\mu)(f)\varphi, s)\overline{E^P_{P'}(x,\varphi,s)}\ ds$$

where $(n^P_{P'})^{-1}$ is a certain sum of Weyl elements associating P to P', $B_{P',\chi}$ is a orthonormal basis of the Hilbert space $B_{P',\chi}$ of complex functions on $Z_{\mathbb{A}}N_{\mathbb{A}}M_{\mathbb{Q}}\backslash G_{\mathbb{A}}$, and $E^P_{P'}$ is a relative form of $E = E^G_P$. For computations involving the above product one often uses the *Arthur truncation operator* Λ^T, similar to the modified kernel defined previously as a sum of integrals multiplied by a characteristic function.

References

[Ar] Arthur, J., *An Introduction to the trace formula*, 2005.

[Ge] Gelbart, S., *Lectures on the Arthur-Selberg trace formula*, 1995.

[GJ] Gelbart, S., and Jacquet, H., *Forms of GL(2) from the analytic point of view*, 1979.

[Kn] Knapp, T., *Theoretical aspects of the trace formula*, 1997.

Note: [Ar] is the master reference, giving an overview (close to 300 pages!) of his work on the trace formula. The coarse expansion is summarized in [Ge], explicit GL_2 computations are done in [GJ] (not using Arthur's paradigm!), while [Kn] gives the most gentle introduction to [GJ]. I recommend the uninitiated to start from the bottom of the list up.

2.3 Jacquet Langlands correspondence

Before developing the trace formula further, let's discuss a first application of the formula. In fact, the correspondence is *the* first application of the adelic trace formula.

Let $G = GL_2$ and $G' = D^\times$, where D is a quaternion algebra over $\mathbb{Q}$ with S the (finite) set of places where G' does not split. Then $G_v \simeq G'_v$ for all $v \notin S$. The Jacquet Langlands correspondence for GL_2 is the following:

Theorem 2.3.1. *There is a bijection between automorphic representations π' of $G'_{\mathbb{A}}$ with dim $\pi' > 1$ and cuspidal automorphic representations of $G_{\mathbb{A}}$, such that for all $v \notin S$, $\pi \simeq \pi'_v$ and for $v \in S$, π'_v is irreducible admissible, π_v is square-integrable mod center.*

We outline the proof of the theorem as follows: we develop the trace formulas for G and G', then compare them by match orbital integrals to obtain a spectral identity. From the identity one deduces character relations, and characterizes the image of the map.

Simple trace formula. First note that the quotient $Z_{\mathbb{A}}G'_{\mathbb{Q}}\backslash G'_{\mathbb{A}}$ is compact, so for $D^\times$ we have already

$$\mathrm{tr}R'_0(f')+\sum_{\mu^2\equiv 1}\mathrm{vol}(Z_{\mathbb{A}}G'_{\mathbb{Q}}\backslash G'_{\mathbb{A}})\mu(f') = \mathrm{vol}(Z_{\mathbb{A}}G'_{\mathbb{Q}}\backslash G'_{\mathbb{A}})f'(1)+\sum_{\gamma}\mathrm{vol}(Z_{\mathbb{A}}G'_{\gamma,\mathbb{Q}}\backslash G'_{\gamma,\mathbb{A}})\int\limits_{G'_{\gamma,\mathbb{A}}\backslash G'_{\mathbb{A}}} f'(x^{-1}\gamma x)\,dx.$$

On the other hand, the same quotient for G is not compact, but we have the following simplification:

Proposition 2.3.2. *Choose $f = \prod f_v$ such that for at least two places of v the local hyperbolic integrals*

$$\int_{G_v} f_v(g^{-1}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} g)\ d^\times g$$

vanish. Then one has

$$trR_0(f) + \sum_{\mu^2\equiv 1} vol(Z_{\mathbb{A}}G_{\mathbb{Q}}\backslash G_{\mathbb{A}})\mu(f) = vol(Z_{\mathbb{A}}\backslash G_{\mathbb{A}})f(1) + \sum_{ellip} J_{\mathfrak{o}}(f)$$

The hyperbolic integrals vanish immediately; the unipotent orbital integrals vanish under the above assumption, since

$$\int_{G_v} f_v(g^{-1}\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix} g)\ dg = \lim_{a\to 1}|1-a^{-1}|\int_{G_v} f_v(g^{-1}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} g)\ dg.$$

Also, the spectral contributions $\mathrm{tr}R(s,\mu)(f_v)$ vanish immediately while the $\mathrm{tr}(M(0)R(0,\mu))(f) = 0$ since $M(0)$ intertwines $R(0,\mu)$ with itself, hence acts by scalar.

Matching orbital integrals. We say f_v in $C_c^\infty(Z_v\backslash G_v)$ *matches* f'_v in $C_c^\infty(Z'_v\backslash G'_v)$, written $f'_v \sim f_v$, if

1. $f_v(1) = f'_v(1)$.
2. The regular hyperbolic integrals vanish identically as above.
3. for corresponding tori T_v and T'_v,

$$\int_{T_v\backslash G_v} f_v(g^{-1}tg)dg = \int_{T'_v\backslash G'_v} f'_v(g^{-1}t'g)d'g$$

Comparison. Now one shows that for matching f and f',

$$\mathrm{tr}R_0(f) = \mathrm{tr}R'_0(f')$$

by comparing the three corresponding terms. In fact, expanding the sum, one can show for fixed unramified representations τ_v for each $v \notin S$,

$$\sum_{\pi}\prod_{v\in S}\mathrm{tr}\ \pi_v(f_v) = \sum_{\pi'}\prod_{v\in S}\mathrm{tr}\ \pi'_v(f'_v),$$

with $f' \sim f$ and summed over π', π such that $\pi'_v, \pi_v \simeq \tau_v$ for all $v \notin S$. Now we can prove the theorem as follows: By strong multiplicity one the left hand side contains at most one term. Now suppose no such π corresponds to a given π', so that

$$\sum_{\pi'} \prod_{v \in S} \text{tr } \pi'_v(f'_v) = 0$$

which contradicts 'linear independence of characters'.

Characterizing the image. Define an inner product on the space of class functions on $Z_v \backslash G_v$ by

$$\langle f_1, f_2 \rangle = \sum_{T_v} \frac{1}{2} \int_{Z_v \backslash T_v^{\text{reg}}} \delta(t) f_1(t) \overline{f_2(t)}\ dt$$

the sum taken over conjugacy classes of compact tori of G_v. For G'_v, characters of irreducible representations of $Z'_v \backslash G'_v$ span a complete orthonormal set, while for G_v the characters of square integrable irreducible representations of $Z_v \backslash G_v$ are at least an orthonormal set.

Then given any square integrable π_v with $v \in S$, there is an irreducible π''_v onG'_v such that $\langle \chi_{\pi_v}, \chi_{\pi''_v} \rangle = a_{\pi_v} \neq 0$. Now fix $f'_v = \bar{\chi}_{\pi''_v}$, so that

$$\text{tr } \pi'_v(f') = \begin{cases} 1 & \text{if } \pi'_v \simeq \pi''_v \\ 0 & \text{otherwise} \end{cases}.$$

Then for a matching $f_v \sim f'_v$ we have for $v \in S$,

$$\text{tr } \pi_v(f_v) = \int_{Z_v \backslash G_v} f(g) \chi_{\pi_v}(g) dg = \sum \frac{1}{2} \int_{Z_v \backslash T_v^{\text{reg}}} \delta(t) \chi_{\pi_v}(t) \int_{T_v \backslash G_v} f_v(g^{-1}tg) dg = \langle \chi_{\pi_v}, \chi_{\pi''_v} \rangle = a_{\pi_v}$$

Then for $\tau_v \simeq \pi_v$ for $v \notin S$ this the trace formula identity reduces to

$$\prod_{v \in S} a_{\pi_v} = \sum_{\pi'_v} 1$$

where $\pi'_v \simeq \pi''_v$ for $v \in S$ and $\pi'_v \simeq \pi'_v$ for $v \notin S$. Finally, we claim that the right hand sight has exactly one term in it. If there were none, the left hand side would be zero, contradicting the assumption on a_{π_v}; if there were more than one, we would have $\prod |a_{\pi_v}| \geq 2$, which contradicts $|a_{\pi_v}| = |\langle \chi_{\pi_v}, \chi_{\pi''_v} \rangle| \leq 1$.

2.4 Refinement

The coarse expansion of the trace formula has two initial defects: first, its terms are often not explicit enough to use in applications and second, the distributions are generally not invariant under conjugation. To illustrate the importance of invariance we note that the Jacquet-Langlands correspondence in fact involved the matching of *invariant* orbital integrals.

Given a Levi subgroup M in G, we denote by $\mathcal{L}(M)$ the set of Levi subgroups containing M, and $\mathcal{P}(M)$ the set of parabolic subgroups with Levi component M.

Weighted orbital integrals. There is a finite set $S_{\mathfrak{o}}$ containing S_∞ such that for any finite set S containing $S_{\mathfrak{o}}$ and any f in $C_c^\infty(G^1_{F_S})$,

$$J_{\mathfrak{o}}(f) = \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in(M_F\cap\mathfrak{o})_{M,S}} a^M(S,\gamma)J_M(\gamma,f)$$

where $(M_F\cap\mathfrak{o})_{M,S}$ is the finite set of (M,S)-equivalence classes $M_F\cap\mathfrak{o}$ and $J_M(\gamma,f)$ is the general weighted orbital integral of f,

$$J_M(\gamma,f) = |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F_S)\backslash G(F_S)} f(x^{-1}\gamma x)v_M(x)dx$$

where $v_M(x)$ is the volume of the convex hull in $\mathfrak{a}_M^G$ of th projection of points $\{-H_P(x)|P\supset M \text{ fixed}\}$ and $D(\gamma)$ is the generalized Weyl discriminant

$$\prod_{v\in S} \det(1-\mathrm{Ad}(\sigma_v))_{\mathfrak{g}/\mathfrak{g}_{\sigma_v}}$$

where σ_v is the semisimple part of γ_v and $\mathfrak{g}_{\sigma_v}$ is the Lie algebra of G_{σ_v}.

Weighted characters. For any f in $\mathcal{H}(G)$,

$$J_\chi(f) = \sum_{M\in\mathcal{L}}\sum_{L\in\mathcal{L}(M)}\sum_\pi\sum_s |W_0^M||W_0^G|^{-1}|\det(s-1)_{\mathfrak{a}_M^G}|^{-1}\int_{i\mathfrak{a}_M^*/i\mathfrak{a}_G^*} \mathrm{tr}(\mathcal{M}_L(\lambda,P)M_P(s,0)R_\pi(\chi,\lambda)(f))d\lambda$$

where π runs over equivalence classes of irreducible unitary representations of $M^1_{\mathbb{A}}$, s is a regular element of

$$W^L(M)_{\mathrm{reg}} = \{t\in W^L(M) : \ker(t) = \mathfrak{a}_L\},$$

R is the induced representation $\mathrm{Ind}_{P_{\mathbb{A}}}^{G_{\mathbb{A}}}(\sigma\otimes e^{\lambda(H_P(\cdot))})$ with matrix coeffecient π and $\chi=(P,\sigma)$, $M_P(s,0)$ is the intertwining operator $M_{Q|P}(s,\lambda+\Lambda) : \mathcal{H}_Q\to\mathcal{H}_P$ with $P=Q$, and

$$\mathcal{M}_L(\Lambda,\lambda,P) = M_{Q|P}(\lambda)^{-1}M_{Q|P}(\lambda+\Lambda)$$

(G,M) **families.** Arthur's notion of (G,M) families unifies the computations required in refining both sides. Suppose for each $P\in\mathcal{P}(M)$, $c_P(\lambda)$ is a smooth function on $i\mathfrak{a}_M^*$. Then the collection $\{c_P(\lambda)\}$ is a *(G,M)-family* if $c_P(\lambda)=c_{P'}(\lambda)$ for adjacent P,P' and any λ in the hyperplane spanned by the common wall of the chambers of P and P'.

For example, a collection of points $\{X_P\}$ in $\mathfrak{a}_M$ such that for adjacent P,P', $X_P-X_{P'}$ is perpendicular to the hyperplane spanned by the common wall of the chambers gives a (G,M) family $\{c_P(\lambda) = e^{\lambda(X_P)}\}$. In particular, $X_P=-H_P(x)$ is such an orthogonal set, and we write $v_P(\lambda,x_v)=e^{-\lambda(H_P(x_v))}$.

Now define a homogeneous polynomial $\theta_P(\lambda) = \text{vol}(\mathfrak{a}_M^G/\mathbb{Z}(\Delta_P^\vee))^{-1}\prod_{\alpha\in\Delta_P}\lambda(\alpha^\vee)$. Then we have the

Lemma 2.4.1. *For any (G,M) family $\{c_P(\lambda)\}$, the sum*

$$c_M(\lambda) = \sum_{P\in\mathcal{P}(M)} c_P(\lambda)\theta_P(\lambda)^{-1}$$

initially defined away from singular hyperplanes $\lambda(\alpha^\vee) = 0$ extends to a smooth function on $i\mathfrak{a}_M^*$.

In our example, v_M can be shown to be the Fourier transform of the characteristic function of the convex hull of $\{X_P\}$. As a second example, $d_Q(\Lambda) = M_{Q|P}(\lambda)^{-1}M_{Q|P}(s,\lambda')$ with $\Lambda = \lambda - \lambda'$ is also a $G, M)$-family. It follows that (G, M)-families arise naturally as weight factors in the orbital integrals and characters, leading to a certain uniformity in the parallel refinements.

2.5 Invariance

Our strategy for making the fine trace formula invariant will be to subtract from weighted orbital integrals sums of weighted characters. In nice situations one can do this by applying Poisson summation to the distributions. We say that a continuous, invariant linear form I is *supported on characters* if $I(f) = 0$ for any $f \in \mathcal{H}(G_{F_S})$ with $f_G = 0$ as a complex function of tempered representations. If I is supported on characters, then there is a continuous linear form $\hat{I}$ on $\mathcal{I}(G_{F_S})$ such that $\hat{I}(f_G) = I(f)$

Now let $\mathcal{H}_{\text{ac}}(G_{F_S})$ be the space of functions f on G_{F_S} with 'almost compact support', and define

$$\phi_M(f) = J_M(\pi, X, f) = \int_{i\mathfrak{a}_{M,S}^*/i\mathfrak{a}_{G,S}^*} J_M(\pi_\lambda, f^Z)e^{-\lambda(X)}d\lambda$$

where Z is the image of X in $\mathfrak{a}_{G,S}$, f^Z is the restriction of f to $\{x \in G_{F_S} : H_G(x) = Z\}$ and π_λ is an orbit in $\Pi_{\text{unit}}(G_{F_S})$. In fact, $\phi_M : \mathcal{H}_{\text{ac}}(G_{F_S}) \to \mathcal{I}_{\text{ac}}(M_{F_S})$ is a continuous linear map.

Theorem 2.5.1. *There exist invariant linear forms supported on characters*

$$I_M(\gamma, f) = J_M(\gamma, f) - \sum_{L\neq G\in\mathcal{L}(M)} \hat{I}_M^L(\gamma, \phi_L(f))$$

with $\gamma \in M_{F_S}$, $f \in \mathcal{H}_{ac}(G_{F_S})$ and

$$I_M(\pi, X, f) = J_M(\pi, X, f) - \sum_{L\neq G\in\mathcal{L}(M)} \hat{I}_M^L(\pi, X, \phi_L(f))$$

with $\pi \in \Pi(M_{F_S})$, $X \in \mathfrak{a}_{M,S}$, $f \in \mathcal{H}_{ac}(G_{F_S})$.

Then defining $I(f)$ inductively by setting

$$I(f) = J(f) - \sum_{L\neq G\in\mathcal{L}(M)} |W_0^M||W_0^G|^{-1}\hat{I}^L(\phi_L(f)),$$

the (refined) invariant trace formula follows:

Theorem 2.5.2. *For any $f \in \mathcal{H}(G)$ one has*

$$I(f) = \lim_S \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in\Gamma(M)_S} a^M(\gamma)I_M(\gamma, f) = \lim_T \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(M)^T} a^M(\pi)I_M(\pi, f)d\pi.$$

where W_0^G is the relative Weyl group, S is a finite set containing ramified places, $\Gamma(M)_S$ is the set of (M, S)-equivalence classes, and $d\pi$ is a measure on $\Pi(M)^T$, the union over $0 \le t \le T$ of $\Pi_t(M)$.

2.6 The unramified case

We illustrate the formulas of 11.1 in the simplest case. Call $\mathfrak{o}$ *unramified* if it contains only semisimple elements, and if for $\gamma \in \mathfrak{o}$ that is an anisotropic conjugacy class of $M_P(\mathbb{Q})$, the centralizer $G(\mathbb{Q})_\gamma$ is contained in $M_P(\mathbb{Q})$. Also, call χ *unramified* if for every pair (P, π) in χ the stabilizer of π in the Weyl group $W(P)$ is trivial. For example, in GL_3

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ is an unramified class but } \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \text{ is not.}$$

For the geometric classes, one begins with

$$J_{\mathfrak{o}}^T(f) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} k_{\mathfrak{o}}^T(x,f)dx = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \sum_P (-1)^{\dim A_P/A_G} \sum_{P(\mathbb{Q})\backslash G(\mathbb{Q})} K_{P,\mathfrak{o}}(\delta x, \delta x)\tau_P(H_P(\delta x) - T)\, dx$$

then unfolding the integral one obtains

Proposition 2.6.1. *Let $\mathfrak{o}$ be an unramified class with anisotropic rational datum represented by (P, α). Then*

$$J_{\mathfrak{o}}(f) = \mathrm{vol}(M_P(\mathbb{Q})_\gamma \backslash M_P(\mathbb{A})^1_\gamma) \int_{G(\mathbb{A})_\gamma \backslash G(\mathbb{A})} f(x^{-1}\gamma x) v_P(x) dx$$

where γ is any element in the $M_P(\mathbb{Q})$-conjugacy class α, and $v_P(x)$ is the volume of the projection onto $\mathfrak{a}_P^G$ of $\{-w^{-1}H_P\}$

For spectral classes, one begins with

$$J_\chi^T(f) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \Lambda_2^T K_\chi(x,x)dx = \sum_P n_P^{-1} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \int_{i\mathfrak{a}_P^*} \sum_\phi E(x, R_P(\lambda,\sigma)(f)\phi, \lambda) \overline{\Lambda^T E(x,\phi,\lambda)} d\lambda$$

Proposition 2.6.2. *Let $\chi = \{P, \sigma\}$ be an unramified class. Then*

$$J_\chi(f) = \sum_P n_P^{-1} \int_{i\mathfrak{a}_P^*} \mathrm{tr}(\mathcal{M}_P(\sigma) R_P(\sigma,\lambda)(f)) d\lambda$$

where

$$\mathcal{M}_P(\sigma) = \lim_{\zeta \to 0} \sum_Q \sum_w \theta_Q(s\lambda)^{-1} M(w,\lambda,\sigma)^{-1} M(w, \lambda + \zeta, \sigma)$$

is an operator on the Hilbert space $Ind_P(\sigma, \lambda)$, and $M(w, \lambda, \sigma)$ is the intertwining operator.

where Λ^T is the Arthur truncation operator, which formally looks similar to the modified kernel $k^T(x, f)$. The computation of the inner product of truncated Eisenstein series is known as the Maass-Selberg relation, worked out by Langlands in this generality.

3 Base change

3.1 GL(2) base change

First let's describe base change in general. Let G be a reductive groups over a number field F, and $G' = R_{E/F}G$ the group obtained by restriction of scalars from a finite extension E/F. The $R_{E/F}$ functor defines an isomorphism $G'(F) \simeq G(E)$. The principle of functoriality asserts that the diagonal embedding $\varphi : {}^LG \to {}^LG'$ should correspond to a lift of automorphic representations of $G(F)$ to $G(E)$.

Langlands' proof of base change for GL_2 uses the invariant trace formula, and a twisted version introduced by Shintani. When $G = GL_2$, then

$${}^LG = GL_2(\mathbb{C}) \times \mathrm{Gal}(\bar{k}/k) \text{ and } {}^LG' = (GL_2(\mathbb{C}) \times \cdots \times GL_2(\mathbb{C})) \rtimes \mathrm{Gal}(\bar{k}/k)$$

with one copy of GL_2 for each embedding of E into $\overline{F}$, and $\mathrm{Gal}(\bar{k}/k)$ acts by permuting the embeddings. We will work with E/F a cyclic extension of prime degree.

Conjugacy classes. Let E be a local field and F a cyclic extension of prime degree l. $\mathrm{Gal}(E/F) := \Gamma_{E/F}$ acts on classes of irreducible representations of $G(E)$ by $\Pi^\sigma(g) = \Pi(\sigma(g))$. Fix a generator σ of $\Gamma_{E/F}$ and define the norm N of $g \in G(E)$ to be the intersection of the conjugacy class of $g\sigma(g)\dots\sigma^{l-1}(g)$ with $G(F)$. If $x = g^{-1}y\sigma(g)$ for some $g, x, y \in G(E)$, we say x and y are σ-conjugate.

Lemma 3.1.1. *Now let F be a global field, and $u \in G(F)$. Then $u = Nx$ has a solution in $G(E)$ if and only if it has a solution in $G(E_v)$ for every v.*

Spherical functions. Now let F be a nonarchimedean local field, and $\mathcal{H}_F$ be the algebra of $G(\mathfrak{o})$-bi-invariant functions on $G(F)$ with compact support mod $N_{E/F}Z(E)$, which transform by $f(zg) = \xi^{-1}(z)f(g)$ where ξ is a character of $N_{E/F}Z(E)$. Define $\mathcal{H}_E$ similarly with $Z(E)$ in place of $N_{E/F}Z(E)$. The map ${}^LG(F) \to {}^LG(E)$ induces a homomorphism $\mathcal{H}_E \to \mathcal{H}_F$.

Lemma 3.1.2. *Suppose $\phi \in \mathcal{H}_E$ maps to $f \in \mathcal{H}_F$. If $\gamma = N\delta$ then*

$$\int_{G_\delta^\sigma(E)\backslash G(E)} \phi(g^{-1}\delta\sigma(g))dg = \xi(\gamma)\int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)dg$$

where $\xi(\gamma) = -1$ if γ is central and δ is not σ-conjugate to a central element, and $\xi(\gamma) = 1$ otherwise. If $\gamma \in G(F)$ is not the norm of an element in $G(E)$ then

$$\int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)dg = 0.$$

Orbital integrals. Now let F be a local field of characteristic 0. Let $f \in C_c^\infty(G(F))$ and $\gamma \in T_{\mathrm{reg}}$, a Cartan subgroup of $G(F)$, we define the following HCS family $\{\Phi_f(\gamma, T)\}$ and $\{\Psi_\phi(\gamma, T)\}$ Shintani family:

$$\Phi_f(\gamma, T) := \int_{T(F)\backslash G(F)} f(g^{-1}\gamma g)dg; \quad \Psi_\phi(\gamma, T) := \int_{G_\delta^\sigma(E)\backslash G(E)} \phi(g^{-1}\delta\sigma(g))dg$$

for all $\gamma = N\delta$, when there is no solution $\Psi_\phi(\gamma, T) = 0$. Now if $\gamma = N\delta \in G(F)$ then $G_\delta^\sigma(E) = G_\gamma^\sigma(F)$, and we may use this to carry measures from $G_\gamma(F)$ to $G_\delta^\sigma(E)$, which is $T(F)$ for our choice of γ.

Lemma 3.1.3. *A Shintani family is a HCS family. A HCS family is a Shintani family if and only if $\Phi_f(\gamma, T) = 0$ when $\gamma = N\delta$ has no solution.*

Using this we may associate to any $\phi \in C_c^\infty(G_E)$ an $f \in C_c^\infty(G_F)$ such that $\{\Phi_f(\gamma, T)\} = \{\Psi_\phi(\gamma, T)\}$. Note that f is not uniquely determined but its orbital integrals are, and we denote the correspondence $\phi \to f$. In particular, if E is unramified and $\phi = 1_{G(O_E)}\mathrm{vol}(G(O_E))$ we can take $f = 1_{G(O_F)}\mathrm{vol}(G(O_F))$ by the homomorphism of Hecke algebras.

Comparison. Let r denote the representation of $G(\mathbb{A}_F)$ on the discrete spectrum of the space of functions on $G(F)\backslash G(\mathbb{A}_F)$ that are square-integrable mod $Z(F)N_{E/F}Z(\mathbb{A}_E)$ and satisfy $f(zg) = \xi(z)f(g)$, where ξ is a unitary character of $N_{E/F}Z(\mathbb{A}_E)$ trivial on $Z(F)$. Choose a a smooth, compactly supported test function f which at almost every place is K_v-invariant and supported on $N_{E_v/F_v}(Z(E_v))K_v$

Let R be the representation of $G(\mathbb{A}_E)\times\Gamma_E$ on the direct sum of τ and l copies of r. Here $\tau = \frac{1}{2}\oplus_S \tau(\eta)$ where S is the set of quadratic characters and $\tau(\eta) = \tau(\eta,\sigma) = \rho(\sigma,\eta^\sigma)M(\eta)$. If the degree of E/F is not 2, then S is empty.

Suppose ϕ is as before. If v splits in E then ϕ_v is a product of l functions on $G(E_v) \simeq G(F_v)\times\cdots\times G(F_v)$, and send $\phi_v = f_1\times\cdots\times f_l$ to $f_1 * \cdots * f_l$. If v is not split in E, then map $\phi_v \to f_v$ following Lemma 11.3 if v is unramified and ϕ_v spherical or else Lemma 11.4.

Theorem 3.1.4. *Let $\phi \to f$ as above. Then the trace identity* $\mathrm{tr}(R(\phi)R(\sigma)) = \mathrm{tr}(r(f))$ *holds.*

We omit the proof of this, but include the trace formula below for comparison by the reader using the matching outlined above. The only nontrivial portion proven using some functional analysis is

$$\sum_v \frac{1}{2\pi}\int_{s(\eta)=0}\Big(lB(\phi_v,\eta_v) - \sum_{\eta'\to\eta} B(f_v,\eta'_v)\Big)\prod_{w\neq v}\mathrm{tr}(\rho(\phi,\eta_w)\rho(\sigma,\eta_w))ds = 0$$

Further, one refines the identity to the following. Let V be a finite set including infinite and ramified places, and for $v\in V$ fix an unramified representation Π_v that lifts from π_v and $\Pi_v^\sigma \simeq \Pi$. Then one shows:

$$\sum_\pi \prod_{v\in V}\mathrm{tr}\pi_v(f_v) = l\sum_\Pi\prod_{v\in V}\mathrm{tr}(\Pi_v(\phi_v)\Pi'_v(\sigma)) + \sum_{(\eta,\tilde\eta)}\prod_{v\in V}\mathrm{tr}(\tau(\phi_v,\eta_v)\tau(\sigma,\eta_v))$$

where π,Π,η are unramified outside of V, and $(\eta,\tilde\eta)$ are such that $\eta = (\mu,\nu) \neq \tilde\eta = \eta^\sigma$ and $\mu\nu = \xi_E$. The Π sum has at most one term by strong multiplicity one, and by a different argument so does the η sum. In fact, one shows that one of the two sums must always be empty.

From this last identity, part of the main result obtained is the following

Theorem 3.1.5.
Local results: every π_v has a unique lifting Π_v. Π_v is a lifting if and only if $\Pi_v^\sigma \simeq \Pi_v$ for all $\sigma\in\Gamma_{E_w/F_v}$. Local lifting is independent of choice of σ.

Global results: Every π has a unique lifting Π. If Π is cuspidal then it is a lifting if and only if $\Pi^\sigma \simeq \Pi$ for all $\sigma\in\Gamma_{E/F}$, furthermore, if Π_v is a lift of π_v for almost all v then Π is a lifting.

Trace formula. We introduce the trace formulas on the two groups without proof.

$$\mathrm{tr}f(f) = \sum_{\gamma} \epsilon(\gamma)\mathrm{vol}(N_{E/F}Z(\mathbb{A}_E)G_\gamma(F)\backslash G_\gamma(\mathbb{A}_F)) \int_{G_\gamma(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} f(g^{-1}\gamma g)dg \tag{3.1}$$

$$-\frac{1}{4}\sum_{\nu=(\mu,\mu)} \mathrm{tr}M(\eta)\rho(f,\eta) \tag{3.2}$$

$$\frac{1}{4\pi}\int_{D^0} m^{-1}(\eta)m'(\eta)\mathrm{tr}\rho(f,\eta)ds \tag{3.3}$$

$$\sum_{a\in N_{E/F}Z(E)\backslash Z(F)} l\lambda_0 \prod_v L(1,1_{F_v})^{-1}\int_{G_n(F_v)\backslash G(F_v)} f(g^{-1}ang)dg \tag{3.4}$$

$$-l\lambda_{-1}\sum_v \sum_{a\in N_{E/F}Z(E)\backslash A(F)} A_2(\gamma,f_v)\prod_{w\neq v}\Delta_w(\gamma)\int_{A(F_w)\backslash G(F_w)} f_w(g^{-1}\gamma g)dg \tag{3.5}$$

$$\frac{1}{2\pi}\int_{D^0}\sum_v B(f_v,\eta_v)\prod_{w\neq v}\mathrm{tr}\rho(f_w,\eta_w)ds \tag{3.6}$$

Now the twisted formula:

$$\mathrm{tr}r(\phi)r(\sigma) = \sum_{\{\gamma\}\text{ elliptic}} \epsilon(\gamma)\mathrm{vol}(Z(\mathbb{A}_E)G^\sigma_\gamma(E)\backslash G^\sigma_\gamma(\mathbb{A}_E))\int_{Z(\mathbb{A}_E)G^\sigma_\gamma(\mathbb{A}_F)\backslash G(\mathbb{A}_E)} \phi(g^{-1}\gamma\sigma(g))dg \tag{3.7}$$

$$\sum_{\{\gamma\}\text{ central}} \epsilon(\gamma)\mathrm{vol}(Z(\mathbb{A}_E)G^\sigma_\gamma(E)\backslash G^\sigma_\gamma(\mathbb{A}_E))\int_{Z(\mathbb{A}_E)G^\sigma_\gamma(\mathbb{A}_F)\backslash G(\mathbb{A}_E)} \phi(g^{-1}\gamma\sigma(g))dg \tag{3.8}$$

$$\frac{1}{4\pi}\int_{s(\eta)=0} m_E^{-1}(\eta)m_E'(\eta)\mathrm{tr}(\rho(\phi,\eta)\rho(\sigma,\eta)) \tag{3.9}$$

$$\lambda_0\Theta(0,\phi) \tag{3.10}$$

$$-\frac{\lambda_{-1}}{l}\sum_{A^{1-\sigma}(E)Z(E)\backslash A(E)}\sum_v A_2(\gamma,\phi_v)\prod_{w\neq v}F(\gamma,\phi_w) \tag{3.11}$$

$$-\frac{1}{4}\sum_{\eta^\sigma=\tilde{\eta}}\mathrm{tr}(\rho(\phi,\eta)\rho(\sigma,\eta^\sigma)M(\eta)) \tag{3.12}$$

$$\frac{1}{2\pi}\int_{s(\eta)=0}\sum_v B(\phi_v,\eta_v)\prod_{w\neq v}\mathrm{tr}(\rho(\phi,\eta_w)\rho(\sigma,\eta_w))ds \tag{3.13}$$

The matching is as follows: $l[(7)+(8)] = (1), l(9) = (3), l(10) = (4), l(11) = (5), l(12) = (2)$ and $l(13) - (6)$ remains.

The first terms, as always, are elliptic or central conjugacy classes. The terms containing $m(\eta)$ or $M(\eta)$, which are intertwining operators, are from the continuous spectrum. The terms containing λ_0 and λ_{-1} are the first nonzero coefficients in the Laurent expansion of Tate's zeta function at $s = 1$, coming from the unipotent classes. The surviving terms come from the hyperbolic classes, appearing as traces after applying Poisson summation to yield invariant distributions.

References

Langlands, *Base change for GL(2).*

3.2 Aside I: Weil restriction of scalars

Most of the material from this section is taken from Peter Clark's notes online.

Multiplicative group. First example: $\mathbb{G}_m(\mathbb{C})$, the algebraic group of multiplicative units has coordinate ring $\mathbb{C}[x_1, x_2]/(x_1x_2 - 1)$. Using the basis $\{1, i\}$ for $\mathbb{C}/\mathbb{R}$, write $x_1 = y_1 + iy_2$ and $.x_2 = y_3 + iy_4$. Then the relation $x_1x_2 - 1$ gives two relations over $\mathbb{R}$:

$$y_1y_3 - y_2y_4 - 1 = 0, \qquad y_1y_4 + y_2y_3 = 0$$

We claim that this variety $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G})$ is isomorphic to $\mathbb{C}^\times$. If $y_1 \neq 0$, we can solve for y_3 and y_4, so y_2 can be arbitrary. If $y_1 = 0$, then y_2 can be arbitrary. So we have a variety isomorphic to the pair of real numbers $(y_1, y_2) \neq (0, 0)$, or $\mathbb{R}[x_1, x_2][(x_1^2 + x_2^2)^{-1}]$.

Affine variety. Now let's describe the construction for an affine variety X over a finite separable extension E of F. Then $X(E)$ can be embedded in $\mathbb{A}^k_E$ as the zero set of polynomials $P_i(x_1, \ldots, x_k) \in E[x_1, \ldots, x_k]$. Viewing $\mathbb{A}^k_E$ as an kd-dimensional space over F, where $d = [E : F]$, choose a basis $\alpha_1, \ldots, \alpha_d$ of E/F. Then using a 'change of basis' $x_i = \sum_{j=1}^d \alpha_j y_j$ we can view each equation $P_i = 0$ over E as d equations over F. The resulting system of polynomials cuts out a closed subvariety of $\mathbb{A}^{kd}_F$, which will be $\mathrm{Res}_{E/F}X$.

Representable functor. Now for the first abstract definition: Let X be a scheme over E. Then $\mathrm{Res}_{E/F}X$ is a functor of F-schemes to sets defined by $\mathrm{Res}_{E/F}X : S \mapsto X(S \times_F E)$. In particular, if $S =$ Spec E then $\mathrm{Res}_{E/F}X(F) = X(E)$. Using (contravariant) Yoneda's lemma one shows that this functor is in fact *representable*, so that if X is a (group) scheme then $\mathrm{Res}_{E/F}X$ is again a (group) scheme. (For basic ideas about Yoneda's lemma and representable functors, see Vakil's notes, for examples.)

Adjoint functor. The next, perhaps more, abstract definition: $\mathrm{Res}_{E/F}X$ is right-adjoint to the *extension of scalars* functor, of which complexification is the basic example. That is, writing $\mathrm{Mor}_F(X, Y)$ the set of F-morphisms from X to Y an F-scheme, one has a bijection

$$\mathrm{Mor}_E(X \times_F E, X) = \mathrm{Mor}_F(Y, \mathrm{Res}_{E/F}X).$$

Again when $Y =$ Spec F, the E-points of X are the F-points of the restriction.

Construction. The following explicit construction will prove that such a scheme exists. Let K be the Galois closure of E/F, and σ in GalK/F. Let $X^\sigma = X_K \otimes_\sigma K$, regarding X as defined over K and σ acting on the coefficients of the defining equations. I'm not sure what that means, but the rest I think is clear: if we then set $V = \prod_\sigma X^\sigma$ we see it's isomorphic to its Galois conjugates V^σ, so it descends to a scheme over F whose rational points are precisely the Galois fixed points. One has to show that V is compatible with Galois action; apparently for each σ there is an automorphism $\psi^\sigma : V^\sigma \to V$ satisfying the cocycle condition: $\psi^{\sigma\tau} = \psi^\sigma \circ \sigma(\psi^\tau)$.

Application to $\mathbf{GL}_n$ base change. We take $G = GL_n$, and E/F as above, then one asks to lift representations from $G(F)$ to $G(E)$. Using the Weil restriction functor above we have $\mathrm{Res}_{E/F}G(F) \simeq G(E)$, which as an algebraic group splits into d copies of $G(F)$, so that when we pass to L-groups we pick up the $[E : F]$ copies of $GL_n(\mathbb{C})$ for ${}^LG(E)$.

3.3 Aside II: Chevalley and Satake Isomorphism.

In our setting we'll take G to be an algebraic group with Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h} =$ Lie A. Define the Weyl group W as the quotient of the normalizer by the centralizer of G. $N_G(A)/Z_G(A)$.

The **Chevalley isomorphism** is the following:

$$\mathrm{Fun}(\mathfrak{g})^G \simeq \mathrm{Fun}(\mathfrak{h})^W$$

where G acts on its Lie algebra by conjugation, so $\mathrm{Fun}(\mathfrak{g})^G$ denotes the functions on $\mathfrak{g}$ invariant under conjugation by G. We may pass to the group setting:

$$\mathrm{Fun}(G)^G \simeq \mathrm{Fun}(A)^W, \text{ or } \mathbb{C}[G]^G \simeq \mathbb{C}[A]^W$$

considering the group ring as conjugation-invariant complex valued functions on G.

As reminder, the **Satake isomorphism** can be interpreted as:

$$\mathcal{H}(G, K) \simeq \mathbb{C}[A^\vee]^W.$$

Where G is quasisplit. The Satake isomorphism has many incarnations, due to its successive generalizations. It starts life with the Satake transform giving an equivalence from the Hecke algebra of G with that of the maximal torus/Cartan subgroup A, which then can be identified with the Hecke algebra of the maximal *split* torus T. Interpret the W-invariant functions on the tori as coming from the group ring of the character group, hence of the dual $T^\vee$.

Further, there is a grander **geometric Satake equivalence**, which gives an equivalence of certain tensor categories,

$$\{\mathcal{D}\text{-modules on the affine Grassmanian of } G\} \simeq \{\text{Representations of } G^\vee \text{ over } \mathbb{C})$$

the most elegant proof of this equivalence is by Mirkovic-Vilonen, where $\mathbb{C}$ can be replaced by a commutative ring, using the Tannakian formalism.

Finally, our **application to base change** is the following: given a map of L-groups (specifically, an L-homomorphism) ${}^LG(F) \to {}^LG(E)$ the Satake isomorphism induces a map of Hecke algebras at *unramified* places $\mathcal{H}_{E_v} \to \mathcal{H}_{F_v}$, allowing for the transfer of spherical functions.

3.4 Aside III: Conjugacy classes in reductive groups

A **regular element** is one that has centralizer of minimal dimension. Ler's look for the centralizer of an element of GL_n. First, we invoke the **rational canonical form** over a field F: Given an endomorphism of x of $V := F^n$, V decomposes into a direct sum of cyclic submodules $V_1, \cdots, V_d$, where the minimal polynomials $m(x, V_i)$ divides $m(x, V_{i+1})$. Moreover, $m(x, V) = m(x, V_d)$ and the characteristic polynomial $c(x, V) = \prod m(x, V_i)$.

An endomorphism of V commuting with x is a direct sum of homomorphisms between the V_i. Now V_i is a quotient of V_j whenever $i < j$, so the dimension of $\hom_x(V_i, V_j) = \min(\dim V_i, \dim V_j)$. When $i > j$ one maps V_j into the unique cyclic submodule of V_i isomorphic to V_j, so $\hom_x(V_i, V_j) \simeq \hom_x(V_i, V_i)$. Therefore the centralizer of x in $\mathrm{End}(V)$ has dimension

$$\sum_{i,j=1}^{d} \min(\dim V_i, \dim V_j) = \sum_{i=1}^{d} (2d - 2i + 1) \dim(V_i)$$

because of the fact $\#\{(i, j) | 1 \leq i, j \leq d \text{ and } \min(i, j) = k\}$=2d-2k+1. So the dimension of the centralizer is at least the rank n of x, with equality if and only if $m(x, V) = c(x, V)$. For x in $M_n(F)$ this is just $F[x]$. For a semisimple matrix $m(x, V) = c(x, V)$ means all its eigenvalues are distinct; while for a unipotent matrix this means it is similar to a Jordan block with 1 only on the diagonal or subdiagonal.

3.5 Noninvariant base change for GL(n)

Let γ, γ' be in $G(F)$. We call γ and γ' **stably conjugate** if they are conjugate in $G(\overline{F})$. For example,

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

shows two non-conjugate elements in $SL_2(\mathbb{R})$ that are conjugate over $\mathbb{C}$.

Lemma 3.5.1. *Two elements in $G'(F) := Res_{E/F}G \rtimes \theta$ are stably conjugate iff they are conjugate in $G(E)$.*

Let π, π' be an admissible irreducible representations of $G(F), G'(F)$, with character distributions $\Theta_\pi, \Theta_{\pi'}$. We say π' is a **base change** of π if $\Theta_{\pi'}(x) = e(G')\Theta_\pi$, where e is a certain Kottwitz constant. This is also called endoscopic correspondence.

Next, the following is known as endoscopic transfer over a local field: A pair of smooth, compactly-supported functions f and ϕ on $G(F)$ and $G'(E)$ are called **associated** if for any regular γ in $G(F)$,

$$J_G(\gamma, f) = \sum_\delta \Delta_G^{G'}(\gamma,\delta) J_{G'}\delta, \phi).$$

where $\Delta_G^{G'}(\gamma,\delta) = 1$ if γ is stably conjugate to δ^l and 0 otherwise, and δ runs over $G(E)$-conjugacy classes in $G'(F)$. Then Labesse introduces a noninvariant version of association: f and ϕ are called **strongly associated** if for all Levis M and parabolics Q in G (resp. $M', Q' \in G'$), one has

$$J_M^Q(\gamma, f) = J_{M'}^{Q'}(\delta, \phi)$$

if $\gamma \in M(F)$ is stably conjugate to $\delta^l \in M'(F)$, and $J_M^Q(\gamma, f) = 0$ if γ is not a norm.

Proposition 3.5.2. *Let F be a local field, ϕ a smooth function on $G'(F)$ compactly supported on regular elements. Then there exists strongly associated ϕ' and f' such that $J_{G'}(\delta,\phi) = J_{G'}(\delta,\phi')$ for δ regular semisimple. Conversely, given f with regular support, if $J_G(\gamma, f) = 0$ if γ is not stably conjugate to some $\delta^l \in G'(F)$ then there exists strongly associated such that $J_G(\gamma, f) = J_G(\gamma, f')$.*

Then one proves a noninvariant fundamental lemma:

Theorem 3.5.3. *Let F be a nonarchimidean local field, $b_{E/F}$ the base change homomorphism between unramified Hecke algebras. Then given $h \in \mathcal{H}_E$, $h_\theta(x \rtimes \theta) := h(x)$ and $b_{E/F}(h)$ are strongly associated.*

To prove this we require the following base change identity:

Proposition 3.5.4. *If ϕ on $G'(\mathbb{A}_F)$, f on $G(\mathbb{A}_F)$ are strongly associated and regular, then $J^{Q'}(\phi) = J^Q(f)$.*

Recall

$$J^{Q'}(\phi) = \sum_{\chi'} \sum_M |W_0^M||W_0^Q|^{-1} \sum_{L_1,L_2} d_M^Q(L_1,L_2) J_{M',\chi'}^{L_1',Q_2'}(\phi)$$

where

$$J_{M,\chi'}^{L_1,Q_2}(\phi) = \sum_{\pi' \in \Pi_{\mathrm{disc}}(M',\chi')} a_{\mathrm{disc}}^{M'}(\pi') \int_{i\mathfrak{a}_M^*} r_{M'}^{L_1'}(\pi'_{\Lambda_{E/F}}) J_{M'}^{Q_2'}(\pi'_{\Lambda_{E/F}S}, \phi_S) \mathrm{tr}\pi'^S_{\Lambda E/F}(h_{M'})\, d\Lambda$$

Labesse's main technical result is the refined base change identity:

Proposition 3.5.5. *Assume normalizing factors are chosen to be compatible with weak base change. Let S be a finite set of places outside which E/F is unramified. Given a Levi M, a character ψ of the $\mathcal{H}(\mathbb{A}_F^S)$, if (f_S, ϕ_S) are strongly associated regular functions, then*

$$l^{\dim\, \mathfrak{a}_M} \sum_{\pi' \in \Pi_{\mathrm{disc}}(M')} \delta_{M'}^{M'}(\pi',\psi) a_{\mathrm{disc}}^{M'}(\pi') J_{M'}^{Q'}(\pi',\phi_S) = \sum_{\pi \in \Pi_{\mathrm{disc}}(M)} \delta_M^{M'}(\pi,\psi) a_{\mathrm{disc}}^M(\pi) J_M^Q(\pi, f_S)$$

where $\delta_G^{G'}(\pi,\psi)$ is 1 if tr $\pi^S(b_{E/F}(h))$ = tr $\psi(h)$ *and 0 otherwise; resp.* tr $\pi'^S(h)$ = tr $\psi(h)$ *for $\delta_{G'}^{G'}(\pi',\psi)$.*

Starting from 12.4, assume inductively that 12.5 holds for proper parabolic subgroups of Q. Then we have

$$l^{\dim \mathfrak{a}_M} \sum_{\pi' \in \Pi_{\mathrm{disc}}(M')} \delta_{M'}^{M'}(\pi', \psi) a_{\mathrm{disc}}^{M'}(\pi') r_{M'}^{L_1'}(\pi') J_{M'}^{Q_2'}(\pi', \phi) = \sum_{\pi \in \Pi_{\mathrm{disc}}(M)} \delta_M^{M'}(\pi, \psi) a_{\mathrm{disc}}^M(\pi) r_M^{L_1}(\pi') J_M^{Q_2}(\pi, \phi).$$

As in Langlands' proof, we cancel terms and are left with the identity

$$\sum_M |W_o^M||W_0^G|^{-1} \Big(\sum_{\chi'} J_{M',\chi'}^{M',Q'}(\phi_s \otimes h_\theta) - \sum_{\chi} J_{M,\chi}^{M,Q}(f_S \otimes b_{E/F}(h)) \Big) = 0$$

From here one deduces the identity using 'separation of infinitesimal characters via multipliers' and the fundamental lemma above.

The trace identity in place, we also require the following rigidity result by Moeglin and Waldspurger:

Theorem 3.5.6. *Let π_1, π_2 be cuspidal unitary automorphic representations of GL_n, $\tilde{\pi}$ the contragredient of π, and S a finite set containing the archimedean places and outside which the representations are unramified. Then $L^S(s, \pi_1 \times \tilde{\pi}_2)$ is regular nonzero for $Re(s) > 1$, and at $s = 1$ it has a simple pole if $\pi_1 \simeq \pi_2$, otherwise it is regular and nonzero.*

Using this one proves the second crucial ingredient:

Proposition 3.5.7. *Let π_1, π_2 be discrete automorphic representations of $G(\mathbb{A}_F)$ and ψ a character of $\mathcal{H}(G'(\mathbb{A}_F^S))$, then $\delta_G^{G'}(\pi_1, \psi) = \delta_G^{G'}(\pi_2, \psi) = 1$ if and only if there exists a character ξ of $M(\mathbb{A}_F)/N_{E/F}M'(\mathbb{A}_E)$.*

Let π_1', π_2' be discrete automorphic representations of $G(\mathbb{A}_E)$ and ψ a character of $\mathcal{H}(G'(\mathbb{A}_F^S))$, then $\delta_{G'}^{G'}(\pi_1', \psi) = \delta_{G'}^{G'}(\pi_2', \psi) = 1$ if and only if $\pi_1' \simeq \pi_2'$.

The π_i occur discretely, hence are parabolically induced from a tensor of m_i Speh representations $\mathrm{Speh}(\sigma_i)$, where σ_i are unitary cuspidal representations of $M_i = GL(d_i)$, and $n = m_i d_i r_i$. One is led to

$$\prod_\xi L^S(s, \sigma_1 \times \tilde{\pi}_1 \times \xi) = \prod_\xi L^S(s, \sigma_1 \times \tilde{\pi}_2 \times \xi)$$

where ξ runs over Grossencharacters of $F^\times / N_{E/F} E^\times$. The previous theorem shows that the left has a pole, and from this one deduces that $\pi_2 = \pi_1 \otimes \xi$.

Remark: $L^S(s, \pi_1 \times \pi_2)$ is the Rankin-Selberg convolution L-function. Initially defined by Rankin and Selberg as an convolution integral, it was later interpreted as an automorphic L-function. At the nonarchimedean unramified places we use the Satake isomorphism to write $L_v(s, \pi_1 \times \pi_2) = \det(1 - q_v^{-s} \alpha_v \beta_v)$ where α_v, β_v are the semisimple ${}^LG$-conjugacy classes corresponding to π_1, π_2. Then $L^S(s, \pi_1 \times, \pi_2)$ is the product over all $v \notin S$ of the $L_v(s, \pi_1 \times \pi_2)$.

Finally, we have the main theorem: Let $\delta_G^{G'}(\pi, \pi') = 1$ if π_v' is the base change of π_v and 0 otherwise.

Theorem 3.5.8. *Let F be a global field and E a prime cyclic extension of F. (i) Given $\pi' \in \Pi_{\mathrm{disc}}(G')$ there is $\pi \in \Pi_{\mathrm{disc}}(G)$ such that $\delta_G^{G'}(\pi, \pi') = 1$, unique up to twists by characters ξ. (ii) Given $\pi \in \Pi_{\mathrm{disc}}(G)$ there is a unique $\pi' \in \Pi_{\mathrm{disc}}(G')$ such that $\delta_G^{G'}(\pi, \pi') = 1$. (iii) π' is a strict base change of π if $\delta_G^{G'}(\pi, \pi') = 1$.*

Let's prove (i). Propositions 12.6 and 12.8 show that given π' such that $a_{\mathrm{disc}}^{G'}(\pi') \neq 0$, for S large enough and strongly associated (f_S, ϕ_S) we have

$$a_{\mathrm{disc}}^{G'}(\pi') \mathrm{tr}\ \pi_S'(\phi_S) = \sum_{\pi \in \Pi_{\mathrm{disc}}(G)} \delta_G^{G'}(\pi', \pi) a_{\mathrm{disc}}^G(\pi) \mathrm{tr}\ \pi_S(f_S).$$

We want to show a π exists with $\delta_G^{G'}(\pi', pi) = 1$ and $a_{\mathrm{disc}}^G(\pi) \neq 0$. It suffices to produce a strongly associated (f_S, ϕ_S) such that $\mathrm{tr}\ \pi_S'(\phi_S) \neq 0$. Let $\gamma' \in G_v'$ be a regular semisimple element such that the $\Theta_{\pi'} \neq 0$. Then using regular transfer (12.3) one constructs strongly associated (f_v, ϕ_v) with $\mathrm{tr}\ \pi_v'(\phi_v) \neq 0$ by taking ϕ_v with support in a small enough neighborhood of γ' with $J_{G'}(\gamma', \phi_v)$ nonzero at γ'. Note finally that similar results are proved over local F.

3.6 Aside IV: L-group exercise

Let's work through the construction of the L-group of GL_3, over a base field k to start. A torus T is a group isomorphic to copies of the multiplicative group scheme $\mathbb{G}_m$. As is standard, we take the diagonal matrices as our maximal torus

$$T = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$$

where indeed $T \simeq \mathbb{G}_m^3$ over k, so T is a split torus. We have its character group $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ and cocharacter group $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$. Concretely, we have morphisms $\alpha_i : T \to \mathbb{G}_m$ by the projection $\alpha_i(t) = t_i$, and the $\lambda : \mathbb{G}_m \to T$ sending x to a_{ii}, for example:

$$\lambda_2(x) = \begin{pmatrix} 1 & & \\ & x & \\ & & 1 \end{pmatrix}$$

thus $X^*(T) = \bigoplus \mathbb{Z}\alpha_i$ and $X_*(T) = \bigoplus \mathbb{Z}\lambda_i$ for $1 \le i \le 3$. Note that both are isomorphic to $\mathbb{Z}^3$.

The adjoint action of $t \in T$ on the Lie algebra $\mathfrak{g} = \mathfrak{gl}_3$, the group of 3×3 matrices is

$$\mathrm{Ad}(t)(aij) = t(a_{ij})t^{-1} = (t_i t_j^{-1} a_{ij}).$$

The generalized eigenvectors must then be the e_{ij} with $a_{ij} = 1$ and 0 elsewhere. i.e., $\mathrm{Ad}(t)e_{ij} = t_i t_j^{-1} e_{ij}$ for any t in T. This corresponds to the character $(\alpha_i - \alpha_j)(t)e_{ij} = t_i t_j^{-1} e_{ij}$ and we get to decompose

$$\mathfrak{g} = \bigoplus_{i,j} \mathfrak{g}_{\alpha_i - \alpha_j} = \bigoplus_{i,j} \langle e_{ij} \rangle.$$

The set of roots is $\Phi = \{\alpha_i - \alpha_j : i \neq j\}$. Using the pairing $\langle \alpha_i, \lambda_j \rangle = \delta_{ij}$ we define for each root α the coroot $\alpha^\vee$ by condition $\langle \alpha, \alpha^\vee \rangle = 2$. In our case, $(\alpha_i - \alpha_j)^\vee = (\lambda_i - \lambda_j)$. Finally we have the root datum of our reductive group, $\Psi = (X^*(T), \Phi, X_*(T), \Phi^\vee)$. Inverting the datum $\check{\Psi} = (X_*(T), \Phi^\vee, X^*(T), \Phi)$ this defines the dual torus $\check{T}$, and thus a dual reductive group $\check{G}$. Since the roots and coroots of GL_n are isomorphic,

$$G\check{L}_n = GL_n.$$

but more interestingly SL_n and PGL_n are dual, also SO_{2n+1} and Sp_{2n+1}.

We're halfway to the L-group. Next we need a Galois action. From the action of $\Gamma := \mathrm{Gal}(\bar{k}/k)$. on k, we get an action of $\sigma \in \Gamma$ on $\gamma \in \mathrm{Aut}(G(\bar{k}))$ by $\sigma(\gamma) = \sigma\gamma\sigma^{-1}$. Defining the 1-cocycle $c : \Gamma \to \mathrm{Aut}(G_{\bar{k}})$ to be $c(\sigma) = \gamma^{-1}\sigma(\gamma)$, this satisfies the cocycle relation

$$c(\sigma_1\sigma_2) = c(\sigma_1)\sigma_1(c(\sigma_2)),$$

leading to the pointed set $H^1(\Gamma, \mathrm{Aut}(G_{\bar{k}})$. But what we really want is an action of Γ on $\mathrm{Aut}(\Psi_0)$, where we've chosen a basis for Φ. Given the Γ action on $T \subset G$, the character group $X^*(T)$ inherits the action

$$\sigma(\alpha(t)) = \sigma\alpha\sigma^{-1}$$

where the first σ^{-1} acts on T, while the second σ acts on $\mathbb{G}_m$.

Implicitly we have made a choice a maximal torus T, which also fixes our choice of Borel subgroup B (e.g., upper triangular matrices). It turns out that $\sigma(\gamma)$ sends the B to another Borel, which being defined as the minimal parabolic we may conjugate it back to B. In fact, choosing the conjugating element g appropriately, we obtain an automorphism of Ψ_0. So for $\rho \in \mathrm{Aut}(\Psi_0)$ we set $\sigma(\rho) = \mathrm{Inn}(g)\sigma\rho\sigma^{-1}$.

[*Not relevant*: It turns out that Γ preserves inner automorphisms, so the exact sequence holds:

$$0 \to \mathrm{Inn}(G) \to \mathrm{Aut}(G) \to \mathrm{Aut}(\Psi_0) \to 0$$

Considering the group $\mathrm{GL}_2(\mathbb{R})$, its based root datum is $(\mathbb{Z}^2, (1,-1), \mathbb{Z}^2, (1,-1))$. Then we have

$$0 \to PGL_2(\mathbb{C}) \to \mathrm{Aut}(GL_2(\mathbb{C})) \to \{\pm 1\} \to 0.]$$

Also note that if G is split over k, the above discussion shows that Γ must act trivially on G_k. Finally, we define the L-group of G over k to be the semidirect product with the complex group

$${}^L G := \check{G}(\mathbb{C}) \rtimes \mathrm{Gal}(\bar{k}/k).$$

In particular, since GL_3 is split since it contains our split torus T, so ${}^L GL_3 = \check{G}L_n(\mathbb{C}) \times \mathrm{Gal}(\bar{k}/k)$.

Now for the restriction of scalars $\mathrm{Res}_{E/F} G$, where E is some field extension of F. Let's work our way up. From 12.4, we showed $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \simeq \mathbb{C}^\times$, which over $\mathbb{R}$ is $\mathbb{R}[x,y,z]$ with the relation $z(x^2+y^2) = 1$. I think because we can vary the two factors we obtain the maximal torus $T(\mathbb{R}) \simeq (\mathbb{R}^\times)^2$. Then looking at $X^*(T)$ the two roots are identical, inverting the root datum we find $\check{T} \simeq \mathbb{G}_m^2$, which is our dual group already.

$${}^L \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m(\mathbb{R}) = (\mathbb{G}_m(\mathbb{C}) \times \mathbb{G}_m(\mathbb{C})) \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R}).$$

The Galois action on $\mathbb{C}$ exchanges 1 with i, so on $X^*(T)$ it exchanges to two roots, and on $\check{T}$ it permutes the $\mathbb{G}_m$ factors. (In general, given a separable extension E/F, $L \times_F F \simeq F \times \cdots \times F$.)

From this and the exercise on affine varieties it we might believe that over F,

$${}^L GL_3(E) = (GL_3(\mathbb{C}) \times \cdots \times GL_3(\mathbb{C})) \rtimes \mathrm{Gal}(E/F)$$

where the number of copies is $[E:F]$, or the number of embeddings of E into $\bar{F}$, and $\mathrm{Gal}(E/F)$ acts by permuting the factors. This construction supposedly holds for the restriction of any algebraic group. As a reminder, the pertinent application here is that the base change map of L-groups induces a map of Hecke algebras by the Satake isomorphism.

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4 Endoscopy theory

4.1 Warmup on group cohomology

Let Γ be a group acting on a set G, denoted $\sigma(g)$ or ${}^\sigma g$. The zeroth cohomology is the fixed point set

$$H^0(\Gamma, G) := G^\Gamma = \{g \in G : \sigma(g) = g \text{ for all } \sigma \in \Gamma\}.$$

Now let G be a group. A map $c : \Gamma \to G$ is a 1-cocycle if $c(\sigma_1\sigma_2) = c(\sigma_1)\sigma_1(c(\sigma_2))$ holds for any σ_1, σ_2. Call two 1-cocyles c_1, c_2 equivalent if $c_2(\sigma) = g^{-1}c_1(\sigma)\sigma(g)$ holds for some g in G. Then we define

$$H^1(\Gamma, G) := \{\text{1-cocycles } c : \Gamma \to G\}/\sim$$

In the general setting the cohomology groups are defined functorially, which we'll not discuss here.

Given a homomorphism $f : H \to G$ of groups with Γ actions, the map of cocycles $c_b(\sigma) = f(c_a(\sigma))$ induces a map $H^1(\Gamma, H) \to H^1(\Gamma, G)$. If H and G are abelian then it's a group homomorphism, if not then H^1 is not a group in general, but we have a distinguished class $g^{-1}\sigma(g)$, which allows us to define the kernel of the given map.

Now let G be a connected linear algebraic group with a homomorphism $f : G \to H$. We say (G, f) is a *cover* of H if f is onto with finite kernel. We also call G *simply connected* if it is connected and every cover of it is an isomorphism. Third, the *derived subgroup* G_{der} is the largest normal subgroup such that G/G_{der} is abelian.

4.2 Endoscopic character

Let G be a reductive group over a characteristic 0 field k, G_{sc} the simply connected cover of G_{der}. Given γ in G_k, denote I_γ the centralizer of γ (Labesse uses the subgroup in G generated by $Z(G)$ and the image in G of the centralizer of γ in G_{sc}). In particular, if γ is regular semisimple then I_γ is a torus.

Suppose γ, γ' in G_k are regular semisimple and stably conjugate, then for a g in $G_{\bar{k}}$ and any σ in Γ_k,

$$\gamma' = \sigma(\gamma') = \sigma(g\gamma g^{-1}) = \sigma(g)\gamma\sigma(g)^{-1} = \sigma(g)g^{-1}\gamma' g\sigma(g)^{-1} = c(g)^{-1}\gamma' c(g)$$

That is, $c(g) = g\sigma(g)^{-1}$ defines a 1-cocycle in $H^1(k,T)$, trivial in $H^1(k,G)$. Also from the long exact sequence on cohomology induced by $1 \to T \to G \to T\backslash G \to 1$ we have

$$H^0(k, T\backslash G) \to H^1(k,T) \to H^1(k,G).$$

One checks that $\gamma \mapsto c$ defines a bijection from G_k-conjugacy classes in the stable conjugacy class of δ to

$$\mathcal{D}(k,T) := \ker(H^1(k,T) \to H^1(k,G))$$

So we see that $I_\gamma\backslash G(k) = H^0(k, T\backslash G)$ is the stable class of γ by the exact sequence

$$1 \to T(k)\backslash G(k) \to T\backslash G(k) \to \mathcal{D}(k,T) \to 1.$$

Where the left are naturally conjugacy classes of γ, and the right parametrizes conjugacy classes in the stable class. From abelianized cohomology (taken on faith) one also has a long exact sequence

$$\cdots \to H^0_{ab}(k,G) \to H^1(k,G_{sc}) \to H^1(k,G) \to H^1_{ab}(k,G)$$

which gives an injection

$$\mathcal{D}(k,T) \hookrightarrow \mathcal{E}(k,T) := \ker(H^1(k,T) \to H^1_{ab}(k,G)) = \mathrm{im}(H^1(k,T_{sc}) \to H^1(k,T)).$$

When k is nonarchimedean local, $H^1(k,G_{sc}) = 1$ (Kneser) and in fact it is a bijection. Note $\mathcal{D}(k,T)$ is in general only a set, but it turns out that $\mathcal{E}(k,T)$ is abelian, and finite for k local. Then by the composition

$$H^0(k,T\backslash G) \to \mathcal{D}(k,T) \hookrightarrow \mathcal{E}(k,T),$$

a character κ of $\mathcal{E}(k,T)$ defines a character on $H^0(k,T\backslash G)$, called *endoscopic characters*. By Tate-Nakayama duality below this will give us the endoscopic group, and also define the κ-orbital integrals below.

4.3 Rational representative

First we need a black box: Let T be a torus over a local field k and split over a Galois extension k'.

Theorem 4.3.1. *(Tate-Nakayama duality) There is a canonical isomorphism of Tate cohomology groups*

$$\hat{H}^i(k'/k, X_*(T)) \to \hat{H}^{2+i}(k'/k, T_k)$$

This is a result of local class field theory. In particular, we have an isomorphism

$$\hat{H}^{-1}(k'/k, X_*) := \hat{H}_0(\Gamma, X_*) = X_*^{N_\Gamma}/I_\Gamma X_* \to \hat{H}^1(k'/k, T_k) = H^1(k,T)$$

where $X_*^{N_\Gamma}$ is the kernel of $x \mapsto \sum \sigma(x)$ and I_Γ the augmentation ideal $I_\Gamma := \{\sum n_\sigma \sigma : \sum n_\sigma = 0\} \subset \mathbb{Z}[\Gamma]$, all sums over Γ.

Here is a retelling: let G be a reductive group over k and $Z(\check{G})$ the center of its complex dual group. Tate-Nakayama gives a canonical isomorphism from Galois cohomology to the character group

$$H^1(k,G) \to \pi_0(Z(\check{G})^\Gamma)^*.$$

The important point is that an endoscopic character will correspond to an element of $\check{G}$.

Let k be a global field. Combining the above with the long exact sequence of cohomology (via the derived functor etc.) for the exact sequence $1 \to T_{\bar{k}} \to T_{\mathbb{A}_{\bar{k}}} \to T_{\mathbb{A}_{\bar{k}}}/T_{\bar{k}}$ and the isomorphism

$$H^1(k, T_{\mathbb{A}_{\bar{k}}}) \simeq \bigoplus_v H^1(k_v, T)$$

(by Shapiro's lemma etc.), we can measure the obstruction to a strongly regular semisimple $G_{\mathbb{A}}$-conjugacy class γ having a rational representative γ_v for every v by

$$\operatorname{coker}(H^1(k, T \to \bigoplus_v H^1(k_v, T)) \simeq \operatorname{im}(\bigoplus_v \pi_0(\check{T}^{\Gamma_v})^* \to \pi_0(\check{T}^{\Gamma})^*)$$

There is a similar picture for T_{sc} involving $\mathcal{E}(k, T) \subset H^1(k, T)$, and replacing $\pi_0(\check{T}^{\Gamma})$ with the group $\mathcal{K}(k, T) \subset \pi_0((\check{T}/Z(\check{G}))^{\Gamma})$ whose image in $H^1(k_v, Z(\check{G}))$ is trivial for all v.

Langlands' idea was to use Fourier transform on the finite cokernel so that the part of the trace formula corresponding to the stable class γ becomes a sum over characters, where the trivial character defines the stable part of the trace formula, the rest a product of κ-orbital integrals.

4.4 Endoscopic group

Start with $T = I_\gamma$, κ an endoscopic character. Let s be the image of κ in $\check{G}$, and $\check{H} = \check{G}_s$ its connected centralizer in $\check{G}$. Then inverting the roots one obtains the endoscopic group H associated to κ. In general H may not be a subgroup of G, which we will now discuss.

Fixing an embedding of the Weil form of ${}^LT \hookrightarrow {}^LG = \check{G} \rtimes W_k$, we have an L-subgroup $\mathcal{G}_s = {}^LT\check{G}_s$,

$$1 \to \check{G}_s \to \mathcal{G}_s \to W_k \to 1$$

a split extension of W_k, acting on $\check{G}_s$ by outer automorphisms, and let G_s be a quasisplit group over k dual to $\check{G}_s$. It contains a maximal torus $T_s = G_{\delta'}$, with δ, δ' related by an isomorphism of T, T_s. We have

$$(\delta, \kappa) \to (G_s, \mathcal{G}_s, s, \delta'),$$

a correspondence motivating the definition of *endoscopic datum* $(G_s, \mathcal{G}_s, s, \xi)$ for a reductive group G, where G_s is a quasisplit group over k, the *endoscopic group*; $\mathcal{G}_s$ a split extension of W_k by $\check{G}_s$; s semisimple in $\check{G}$; ξ an embedding of $\mathcal{G}_s$ in ${}^LG$ such that $\xi(\check{G}_s)$ is the connected centralizer of s in $\check{G}$ and $\xi(u)s = s\xi(u)a(u)$, for $u \in \mathcal{G}_s$, and a a locally trivial 1-cocycle from W_k to $Z(\check{G})$. This is so that ξ extends to an L-homomorphism.

Example 4.4.1. The centralizer of any semisimple s in GL_n is a product of GL_{n_i}, so its endoscopic datum is represented by a Levi M, and the only datum where $\operatorname{im}(\xi)$ is not contained in a proper Levi is GL_n itself. For other classical groups like $Sp(2n)$, its endoscopic groups are of the form $Sp(2n-2k) \times Sp(2k)$.

Once again, the endoscopic group G_s is *not a subgroup* of G in general. Nevertheless we can still transfer stable conjugacy classes. Assume for simplicity that G_s is split, so the inclusion $\check{G}_s \subset \check{G}$ induces an inclusion of Weyl groups $W_s \subset W$, and there is a canonical map $\mathfrak{t}/W_s \to \mathfrak{t}/W$. If γ_s in $\mathfrak{g}_s(k)$ has characteristic polynomial $a_s \in \mathfrak{t}/W_s(k)$ mapping to the characteristic polynomial a of γ in $G(k)$, we will say the stable class of γ_s transfers to to the stable class of γ.

The *Kostant section* is a section of the characteristic polynomial morphism $\mathfrak{g} \to \mathfrak{t}/W$, distinguishing a conjugacy class by a stable class a in $\mathfrak{t}/W(k)$. The section helps define the transfer factor.

4.5 Stable distributions

Let's first motivate the problem, leaving precise definitions for later. Let k be a local field. Denote $\Delta_{\text{reg}}(G_k)$ the set of strongly regular stable conjugacy classes, and let δ be such a class. Given $f_v \in C_c^\infty(G_v)$, a *stable orbital integral* is

$$\mathbf{SO}_\delta(f_v) = \sum_{\gamma\sim\delta} \mathbf{O}_\gamma(f_v) = \sum_{\gamma\sim\delta} \int_{I_\gamma(k_v)\backslash G_v} f_v(g^{-1}\gamma g)dg,$$

a finite sum over conjugacy classes in the stable class of δ; and aslo the κ-orbital integral

$$\mathbf{O}_\delta^\kappa(f_v) = \sum_{\gamma\sim\delta} \kappa(\gamma)\mathbf{O}_\gamma(f_v).$$

One difficulty here is that $\kappa(\gamma)$ depends on the choice of representative δ; to correct this the Langlands-Shelstad *transfer factor* $\Delta(\gamma,\delta)$ was introduced, depending only on the classes of δ and γ. Second, the global stable class $\delta_{\mathbb{A}}$ is a product of local rational representatives, so we need each ordinary class $\gamma_{\mathbb{A}}$ to also have a rational representative. We will measure the obstruction to this cohomologically.

An invariant distribution on G_v is a *stable distribution* if its value at f_v depends only on the stable orbital integrals $\mathbf{SO}_\delta(f_v)$, i.e., it belongs to the closure of the space spanned by $\mathbf{SO}_\delta(f_v)$. The terms in the invariant trace formula are not stable, and the aim here is to stabilize the trace formula.

Theorem 4.5.1. *Let H be an endoscopic group of a reductive group G. Then*

1. *(Transfer) For every $f \in C_c^\infty(G_k)$ there exists a $f^H \in C_c^\infty(H_k)$ such that*

$$\mathbf{SO}_{\gamma^H}(f^H) = \Delta(\gamma_H,\gamma)\mathbf{O}_\gamma^\kappa(f) \qquad (*)$$

for all γ^H,γ strongly regular semisimple with characteristic polynomial of γ^H mapping to that of γ.

2. *(Fundamental lemma) (*) holds for $f = 1_{\mathfrak{g}(\mathcal{O}_k)}$ and $f^H = 1_{\mathfrak{h}(\mathcal{O}_k)}$.*
3. *(*) holds for any $f \in \mathcal{H}_G$ and $f^H = b(f) \in \mathcal{H}_H$, where b is induced from the map of L-groups.*

Langlands and Shelstad proved this for archimedean k around 1987. The nonarchimedean case waited another thirty years for the work of Laumon, Waldspurger, Ngô, and many others, completed in 2008.

References

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5 Fundamental lemma

The fundamental lemma, initially arising in the theory of endoscopy, is a broader phenomenon. So one may speak of 'fundamental lemmas'. We will not be able to treat the fundamental lemma in any completeness, but I will introduce the main objects and the broad outline of its proof. But first let's review the statement of the lemma.

Let G be a reductive group over a local field F_v. Given an element γ in G, denote $I_\gamma(F_v)$ its centralizer in G. Define the *orbital integral* of a smooth compactly supported function f_v on $G(F_v)$:

$$\mathbf{O}_\gamma(f) := \int_{I_\gamma(F_v)\backslash G(F_v)} f_v(g^{-1}\gamma g)dg.$$

A character κ of I_γ defines a subgroup H of G, called the *endoscopic subgroup*. We say $\gamma_H \in H$ and $\gamma \in G$ *correspond* if their characteristic polynomials are equal. Also, call two elements γ and γ' in G *stably conjugate* if they are conjugate over $\bar{F}_v$. The transfer conjecture states that

Conjecture 5.0.1. *For every $f_v \in C_c^\infty(G(F_v))$ there exists an $f_v^H \in C_c^\infty(H(F_v))$ such that*

$$(f) := \sum_{\gamma^H} \mathbf{O}_{\gamma^H}(f_v^H) = \Delta(\gamma_H, \gamma) \sum_{\gamma} \kappa(\gamma)\mathbf{O}_\gamma(f_v) \qquad (*)$$

for all strongly regular semisimple γ^H, γ with equal characteristic polynomial, and the sums taken over stable conjugacy classes. $\Delta(\gamma^H, \gamma)$ is called the transfer factor (which we won't define).

The left is called a stable orbital integral $\mathbf{SO}_{\gamma^H}$ and the summands on the right are κ-orbital integrals $\mathbf{O}_\gamma^\kappa$. The **fundamental lemma** asserts that the identity (*) holds for f_v, f_v^H chosen to be the unit elements $1_{G(\mathcal{O}_v)}, 1_{H(\mathcal{O}_v)}$ of the respective Hecke algebras.

5.1 Transfer to characteristic p

The series of reductions are as follows:

1. Waldspurger: The fundamental lemma (FL) implies the transfer conjecture.
2. Hales: FL for units at almost all v implies the FL for Hecke algebra at all v.
3. Waldspurger, Hales: Lie algebra FL implies Lie group FL.
4. Waldspurger, Cluckers-Hales-Loeser: FL for local fields of char p implies FL for char 0.

What is finally proven is the following:

Theorem 5.1.1. (Ngô) *Let $k = \mathbb{F}_q$, F_v its field of fractions $k((\pi))$ and $\mathcal{O}_v = k[[\pi]]$. Let G be a reductive group scheme over $\mathcal{O}_F$ where the order of its Weyl group does is not divisible by char(k). Let H be an endoscopic group of G defined by the character κ and L-embedding ξ. Let a and a_H be corresponding stable semisimple conjugacy classes. Then there is an identity of κ-orbital integrals and stable orbital integral:*

$$\Delta_G(a)\mathbf{O}_a^\kappa(1_{\mathfrak{g}(\mathcal{O}_F)}) = \Delta_H(a_H)\mathbf{O}_{a_H}^\kappa(1_{\mathfrak{h}(\mathcal{O}_F)})$$

where Δ_G and Δ_H are transfer factors, and are powers of q.

The proof ingredients were developed as follows:

1. Goresky-Kottwitz-MacPherson: Local method using equivariant cohomology of Khazdan-Lusztig's affine Springer fibers.
2. Ngô: Global method using perverse sheaves in cohomology of Hitchin fibration.
3. Laumon and Ngô: Proved FL for unitary groups, followed by Ngô for all reductive groups.

This is the journey we shall now describe. Let's set the stage: unless otherwise stated, fix X a smooth, geometrically connected, projective curve of genus g over a finite field k, and $\bar{X} = X \times_k \bar{k}$. Denote F the function field of X, F_v the fraction field of the completed local ring at a closed point v in X, $\mathcal{O}_v$ its ring of integers, and k_v the residual field.

5.2 A first example: $\mathbf{SO}_5$ and $\mathbf{Sp}_4$

Let F_v be a nonarchimedean local field with residue field k with $q > 2$ elements. Let $0, \pm t_1, \pm t_2$ be the eigenvalues of an element γ in $\mathfrak{so}(5)$ and assume there is an odd r such that $|\alpha(\gamma)| = q^{-r/2}$ for every root α of $\mathfrak{so}(5)$. Using the eigenvalues Hales constructs the elliptic curve E_a over k given by

$$y^2 = (1 - x^2\tau_1)(1 - x^2\tau_2)$$

where τ_i is the image of t_i^2/ϖ^r in the residue field. By direct calculation he shows that the stable orbital integral of f at γ is equal to the number of points on the elliptic curve up to rational functions A and B,

$$\mathbf{SO}_\gamma(f) = A(q) + B(q)|E_\gamma(k)|$$

Next with $\mathfrak{sp}(4)$, there is an element a' with eigenvalues $\pm t_1, \pm t_2$. (Twisted) endoscopy predicts an f' on $\mathfrak{sp}(4)$ with stable orbital integral equal to that of f. But calculating the integral of f' gives a similar formula with another elliptic curve $E'_{\gamma'}$, having different j-invariant. The desired equality is obtained by producing an isogeny between E_γ and $E'_{\gamma'}$. As we shall see, these elliptic curves are 'spectral curves'.

5.3 Examples of $\mathbf{GL}_n$ Hitchin fibres

Now let $k = \mathbb{F}_q$ and consider $\mathbb{P}^1_k = \mathrm{Spec}k[y] \cap \{0\}$. Fix the divisor $D = 2[0]$, the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1_k}(D)$ is

$$\mathrm{Spec}(\bigoplus_{i=0}^{\infty} \mathcal{O}_{\mathbb{P}^1_k}(-iD)t^i) \to \mathbb{P}^1_k.$$

Define the scheme $\mathcal{A}$ classifying characteristic polynomials $t^2 - a_1(y)t - a_2(y)$ where for $i = 1, 2$,

$$a_i(y) \in H^0(\mathbb{P}^1_k, \mathcal{O}(iD)) = \{a_i(y) \in k[y] : \deg(a_i) \le 2i\}$$

such that the discriminant $a_1^2 - 4a_2$ is non vanishing at $y = 0$. The spectral curve Y_a is the preimage of $t^2 - a_1(0)t - a_2(0)$ in the total space. Assume further that $a_1 = 0$. The following cases hold:

1. a_2 has only distinct roots. Then Y_a is a smooth projective curve of genus one, i.e., an elliptic curve.
2. a_2 has a double root and two distinct roots. Then Y_a is integral and its only singularity is a node.
3. a_2 has a triple root. Then Y_a is integral and its only singularity is a cusp.
4. a_2 is a square of a polynomial with two distinct roots. Then Y_a has two irreducible components which intersect transversally.

The Hitchin fibres, defined below, can be identified with the compactified Jacobians of each Y_a.

5.4 An extended example: $\mathbf{GL}_n$

Though stability is trivial for $G = \mathrm{GL}_n$ since stable conjugacy is the same as ordinary conjugacy, the geometric constructions on GL_n as a prototype will give us intuition for arbitrary reductive groups.

5.4.1. Chevalley characteristic and Kostant section. Consider the Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$. Fix a Cartan subalgebra $\mathfrak{t}$ of diagonal matrices and Weyl group $W = S_n$. By Chevalley, the adjoint action of G on $\mathfrak{g}$ (i.e., conjugation) induces an isomorphism

$$k[\mathfrak{g}]^G \xrightarrow{\sim} k[\mathfrak{t}]^W$$

Let $\mathfrak{c} := \mathrm{Spec}(k[\mathfrak{t}]^W)$ is the affine space of degree n monic polynomials, isomorphic to $\mathbf{A}^n$ by

$$p_A := X^n - a_1X^{n-1} + \cdots + (-1)^n a_n \mapsto (a_1, \ldots, a_n).$$

The *Chevalley characteristic* $\chi : \mathfrak{g} \to \mathfrak{c}$ sends A to its characteristic polynomial $\det(XI - A)$, or equivalently,

$$\chi : A \mapsto a = (\mathrm{tr}(A), \mathrm{tr}(\wedge^2 A), \ldots, \mathrm{tr}(\wedge^n A)).$$

The *Kostant section* ϵ of the map χ constructs a matrix with the given characteristic polynomial. The companion matrix obtained by the endomorphism X of $R := k[X]/(p_A)$

$$\epsilon : (a_1, \ldots, a_n) \mapsto \begin{pmatrix} 0 & \ldots & 0 & (-1)^{n-1}a_n \\ 1 & \ddots & \vdots & (-1)^{n-2}a_{n-1} \\ \vdots & \ddots & 0 & \vdots \\ 0 & \ldots & 1 & a_1 \end{pmatrix}.$$

The Kostant section is a similar but different from the companion matrix which generalizes to reductive groups.

The centralizer of a companion matrix can be identified with the centralizer of X in $GL(R)$, and one checks that

$$J_a \simeq R^\times \subset GL(R) \text{ and } I_\gamma \simeq R \subset \mathfrak{gl}(R).$$

More generally, if two regular elements in $\mathfrak{g}$ have the same characteristic polynomial a, then their centralizers are canonically isomorphic. Also, for every a there is a regular centralizer J_a isomorphic to I_γ for any γ such that $\chi(\gamma) = a$.

5.4.2. Affine Springer fibre. Now for the affine setting. Define the *affine Grassmannian*

$$\mathrm{Gr} := G(F_v)/G(\mathcal{O}_v) = \{\mathcal{O}_v\text{-modules of rank } n, \text{ i.e., lattices in } F_v^n\},$$

where we deduce the latter from $G(F_v)$ acting transitively on lattices with $G(\mathcal{O}_v)$ fixing $\mathcal{O}_v$-lattices.

Given a Kostant section $\epsilon(a) = \gamma$, consider the following subset of $\mathfrak{g} \times G(F_v)/G(\mathcal{O}_v)$,

$$\widetilde{\mathfrak{g}}(F_v) = \{(\gamma, g) : \mathrm{Ad}(g)^{-1}(\gamma) \in \mathfrak{g}(\mathcal{O}_v))\}.$$

The fibres of its projection to the first factor $\pi : \widetilde{\mathfrak{g}}(F_v) \to \mathfrak{g}(F_v)$ are called the *affine Springer fibres*,

$$\mathcal{M}_v(a) := \pi^{-1}(\gamma) = \{g \in G(F_v)/G(\mathcal{O}_v) : \mathrm{Ad}(g)^{-1}(\gamma) \in \mathfrak{g}(\mathcal{O}_v))\}.$$

Now the standard lattice $\Lambda_0 := \mathcal{O}_v^n$ is γ-stable if and only if $\gamma \in \mathfrak{g}(\mathcal{O}_v)$. Given another lattice $\Lambda = g\Lambda_0$, Λ is γ-stable if and only if L_0 is $g^{-1}\gamma g$-stable if and only if $\mathrm{Ad}(g)^{-1}(\gamma) \in \mathfrak{g}(\mathcal{O}_v))$. Thus the affine Springer fibre parametrizes γ-stable lattices. One also writes

$$\mathrm{Gr}_\gamma := \mathcal{M}_v(a) = \{\mathcal{O}_v\text{-modules } \Lambda \text{ of rank } n \text{ in } F_v^n : \gamma(\Lambda) \subset \Lambda\}.$$

5.4.3. Local orbital integrals. Let γ be a regular semisimple element. Its centralizer I_γ is a torus T_{F_v}. We want to count γ-stable lattices in F_v^n modulo the action of T_{F_v}. Any element of T fixes γ, so the action preserves the fiber Gr_γ. Restricting to the subgroup $X_*(T)$, the stabilizer of a lattice is a compact discrete subgroup hence a finite group, which must be trivial since $X_*(T)$ is a free $\mathbb{Z}$-module. So $X_*(T)$ acts freely. Now

$$\mathbf{O}_\gamma(1_{\mathfrak{g}(\mathcal{O}_v)}) = \int_{X_*(T)\backslash G_F} 1_{\mathfrak{g}(\mathcal{O}_v)}(x^{-1}\gamma x)dx = \sum_{X_*(T)\backslash G_F/G_{\mathcal{O}_v}} \int_{G_{\mathcal{O}_v}} 1_{\mathfrak{g}(\mathcal{O}_v)}(k^{-1}x^{-1}\gamma xk)dk$$

setting $\mathrm{vol}(G_{(\mathcal{O}_v)}) = 1$,

$$= \sum_{X_*(T)\backslash G_F/G_{\mathcal{O}_v}} 1_{\mathfrak{g}(\mathcal{O}_v)}(x^{-1}\gamma x) = |X_*(T)\backslash \mathrm{Gr}_\gamma|$$

Thus local orbital integrals are equivalent to counting points on the affine Springer fibre modulo the action of the group

$$\mathcal{P}_v(a, k) = J_a(F_v)/J_a(\mathcal{O}_v) \simeq I_{\epsilon(a)}(F_v)/I_{\epsilon(a)}(\mathcal{O}_v).$$

5.4.4. As stacks. Schemes can be seen, by the functor of points $\mathrm{Hom}(-, M)$, as sheaves of sets on the category of (Sch/S) with the étale topology. A *stack*, by contrast, is a sheaf of groupoids, where a groupoid is a category in which every morphism has an inverse.

The affine Springer fibre represents the functor from schemes to groupoids

$$\mathcal{M}_v(a) : S \mapsto \begin{cases} E, & \text{rank } n \text{ vector bundles on } \mathrm{Spec}(\mathcal{O}_v) \times S \text{ with} \\ \gamma, & \text{a section of } E \text{ over } \mathrm{Spec}(\mathcal{O}_v) \times S \\ \iota, & \text{a trivialization of } E \text{ over } \mathrm{Spec}(F_v) \times S \end{cases}$$

(the affine Grassmannian is similarly defined, without the second datum) while the Picard stack represents

$$\mathcal{P}_v(a) : S \mapsto \begin{cases} F, & J_a\text{-torsors on } \mathrm{Spec}(\mathcal{O}_v) \times S \\ \iota, & \text{a trivialisation of } F \text{ over } \mathrm{Spec}(F_v) \times S \end{cases}$$

In particular, $\mathcal{P}_v(a, k) \simeq J_a(F_v)/J_a(\mathcal{O}_v) \simeq I_{\epsilon(a)}(F_v)/I_{\epsilon(a)}(\mathcal{O}_v)$.

The corresponding global object is the *Hitchin stack* associating to schemes the groupoids:

$$\mathcal{M} : S \mapsto \begin{cases} E, & \text{rank } n \text{ vector bundles on } X \times S \text{ with} \\ \phi, & \text{a twisted endomorphism } E \to E \otimes \mathcal{O}_X(D) \end{cases}$$

where D is an even, effective divisor on X, and (E, ϕ) is called a *Higgs pair*. One also has a Picard stack

$$\mathcal{P} : S \mapsto \Big\{ \ J_a\text{-torsors on } X \times S \text{ for every } a \text{ in } \mathcal{A}$$

where the scheme $\mathcal{A}$ is the Hitchin base over which the Hitchin stack fibers, which we next introduce.

5.4.5. Hitchin fibration. The *Hitchin fibration* is the morphism $f : \mathcal{M} \to \mathcal{A}$ sending a pair

$$(E, \phi) \mapsto (\mathrm{tr}(\phi), \mathrm{tr}(\wedge^2 \phi), \ldots, \mathrm{tr}(\wedge^n \phi))$$

where $\mathrm{tr}(\wedge^i \phi)$ is the trace of the endomorphism $\wedge^i \phi : \wedge^i E \to \wedge^i E \otimes \mathcal{O}_X(iD)$. The fibers $\mathcal{M}_a := f^1(a)$ of the map will be the global analogue of the affine Springer fibres.

The *Hitchin base* $\mathcal{A}$ is the affine space of characteristic polynomials,

$$X^n - \mathrm{tr}(\phi) X^{n-1} + \cdots + (-1)^n \mathrm{tr}(\wedge^n \phi)$$

identified with the space of global sections of $X \times S$ with values in $\mathfrak{c}_D := \mathfrak{c} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$

$$\mathcal{A} = \bigoplus_{1 \leq i \leq n} H^0(X, \mathcal{O}_X(iD))$$

Define ($\mathcal{A}^{\mathrm{ani}}$ and) $\mathcal{A}^{\heartsuit}$to be the subset of $\mathcal{A}(\bar{k})$ corresponding to the (anisotropic) regular semisimple subset of $\mathfrak{c}$. These are open subschemas of $\mathcal{A}$. Define similarly $\mathcal{M}^{\heartsuit}$ and $\mathcal{M}^{\mathrm{ani}}$ by pulling back along f.

5.4.6. Global orbital integrals. Attach to the divisor $D = \sum_{v \in |X|} n_v [v]$ an idèle $\varpi_D := (\varpi_v^{n_v})$. Then the degree morphism deg: $\mathbb{A}_F^{\times} \to \mathbb{Z}$ which is trivial on $F^{\times}$ and $\mathcal{O}^{\times}$ gives $\deg(\varpi_D) = \deg(D)$. Let

$$H_a := \{(g, \gamma) \in G_{\mathbb{A}}/G_{\mathcal{O}} \times \mathfrak{g}_F : \deg(\det(g)) = 0, \chi(\gamma) = a, \text{ and } g^{-1} a g \in \varpi_D^{-1} \mathfrak{g}(\mathcal{O})\}$$

Define a G_F-action by left multiplication and conjugation respectively, forming the quotient stack $[G_F \backslash H_a]$.

Let $(\gamma_v) = \varpi_D \gamma \in \mathfrak{g}_{\mathbb{A}}$. The third (integrality) condition is equivalent to $g_v^{-1}\gamma_v g_v \in \mathfrak{g}(\mathcal{O}_v)$ for all v, and this coset is exactly the affine Springer fibre Gr_{γ_v}. Choosing a to be regular semisimple, the Hitchin fibre is

$$\mathcal{M}_a(k) \simeq [G_F \backslash H_a] = [T_F \backslash H_a] = [T_F \backslash \prod_{v \in |X|}' \mathrm{Gr}_{\gamma_v}](k).$$

If we normalize measures so that $\mathrm{vol}(G_{\mathcal{O}_v}) = \mathrm{vol}(T_{\mathcal{O}_v}) = 1$ and define $T^0_{\mathbb{A}}$ to be the elements of degree 0, then

$$|\mathcal{M}_a|(k) = \mathrm{vol}(T_F \backslash T^0_{\mathbb{A}}) \prod \int_{T_v \backslash G_v} 1_{\mathfrak{g}(\mathcal{O}_v)}(g_v^{-1}\gamma_v g_v) dg_v.$$

The expression is finite if and only if the Hitchin fiber has finitely many k-rational points. Also, γ_v is in $\mathfrak{g}(\mathcal{O}_v)$ for almost all v, so the orbital integral is almost always 1. At other places, the integral is taken over a compact subset (the conjugacy class of a semisimple element is closed) hence finite.

5.4.7. Spectral curve of X. Consider the total space of the line bundle $\mathcal{O}_X(D)$

$$\Sigma_D = \bigoplus_{i=1}^{\infty} \mathcal{O}_X(-iD).$$

Given a $\bar{k}$-point a of $\mathcal{A}$, we have the characteristic polynomial $p_a : \Sigma_D \to \Sigma_D^n$

$$p_a(X) = X^n + a_1(v)X^{n-1} + \cdots + a_n(v) = 0, \qquad a_i \in H^0(X, \mathcal{O}_X(iD))$$

The *spectral curve* Y_a is the preimage under p_a of the zero section of Σ_D^n, tracing out a closed curve in Σ_D defined by $p_a(X) = 0$. It is a n-fold finite cover of X, generically étale Galois of Galois group $W = S_n$.

Related to this is the *cameral cover* $\tilde{X} \to X$ obtained by pulling back a Higgs pair along the fiber product $\tilde{\mathfrak{g}} \to \mathfrak{g}$ of the quotient map $\mathfrak{t} \to \mathfrak{t}/W$ and $\chi : \mathfrak{g} \to \mathfrak{t}/W$. (Not quite sure how this works.)

5.4.8. Picard stack. The Beauville-Narasimhan-Ramanan correspondence states that when a is regular, $\mathcal{M}_a$ is isomorphic to the moduli stack of coherent torsion-free sheaves $\mathcal{F}$ of generic rank 1 on Y_a with a certain trivialization. When X is a curve this is the compactified Jacobian. In general,

$$\mathcal{M}_a \simeq \overline{\mathrm{Pic}^0(Y_a)}.$$

where $\mathrm{Pic}^0(Y_a)$ is the neutral component of the *Picard scheme*: the smooth commutative group scheme of degree 0 line bundles on Y_a. When Y_a is smooth these are just line bundles, and $\mathcal{M}_a$ is just the Jacobian. The group of symmetries $\mathcal{P}_a$ can be identified with the $\mathrm{Pic}(Y_a)$ acting on $\mathcal{M}_a$ by tensoring sheaves.

Now let $a : \bar{X} \to \mathfrak{c}_D$ be a point in $\mathcal{A}^\heartsuit(\bar{k})$. Consider the normalization of the spectral curve $\xi : Y_a^\flat \to Y_a$, and the pullback

$$\xi^* : \mathrm{Pic} Y_a \to \mathrm{Pic} Y_a^\flat$$

The exact sequence

$$0 \to \mathcal{O}_{Y_a}^\times \to \xi_* \mathcal{O}_{Y_a^\flat}^\times \to \xi_* \mathcal{O}_{Y_a^\flat}^\times / \mathcal{O}_{Y_a}^\times \to 0$$

implies ξ^* is surjective on $\bar{k}$-points, whose kernel consists of invertible sheaves on Y_a that pullback to the trivial bundle on $Y_a^\flat$. Define the *Serre invariant* of a normalization $\tilde{A}$ of A, $\delta_A := \mathrm{length}(\tilde{A}/A)$, in this case

$$\delta_a = \sum_{y \in Y_a} \delta_{\hat{\mathcal{O}}_{Y_a,y}} = \dim H^0(Y_a, \xi_* \mathcal{O}_{Y_a^\flat}^\times / \mathcal{O}_{Y_a}^\times) = \dim \ker \xi^*.$$

Also, the connected components $\pi_0(\mathrm{Pic} Y_a^\flat) \simeq \mathbb{Z}^{\pi_0(\mathrm{Pic} Y_a^\flat)}$.

5.4.9. Subschemes of $\mathcal{A}$. Let's describe some important subschemes of $\mathcal{A}$:

1. $\mathcal{A}^{\heartsuit}$: The functor

$$\mathcal{B} : S \mapsto \begin{cases} a : S \to \mathcal{A}^{\heartsuit}, & \text{an } S \text{ point of } \mathcal{A}^{\heartsuit} \\ \xi : Y_a^{\flat} \to Y_a, & \text{a normalization of the spectral curve } Y_a. \end{cases}$$

   is representable by a finite type k-scheme. The forgetful functor $\mathcal{B} \to \mathcal{A}^{\heartsuit}$ is a bijection on the level of $\bar{k}$-points.

2. $\mathcal{A}^{\delta}$: The locally closed subset of a in $\mathcal{A}$ such that $\delta_a = \delta$, for fixed $\delta \in \mathbb{N}$. When $\delta = 0$, we have $\mathcal{A}^0 = \mathcal{A}^{\text{sm}}$, the open dense subset of a in $\mathcal{A}$ such that Y_a is smooth. Denote Ψ the finite set of connected components of $\mathcal{B}$, giving a stratification $\mathcal{A}^{\heartsuit} \times_k \bar{k} = \bigsqcup_{\Psi} \mathcal{A}_{\psi}$. The invariants δ_a and $\pi_0(Y_a^{\flat})$ are constant on each stratum. We may group together strata with equal δ_a,

$$\mathcal{A}^{\delta} = \bigsqcup_{\delta(\psi)=\delta} \mathcal{A}_{\psi}.$$

3. $\mathcal{A}^{\text{good}}$: The open set of all strata $\mathcal{A}^{\delta}$ of codimension $\geq \delta$, and call its complement $\mathcal{A}^{\text{bad}}$. Note that this condition fails outside the anisotropic set, hence $\mathcal{A}^{\text{good}} \subset \mathcal{A}^{\text{ani}}$.

4. $\mathcal{A}^{\infty}$. The open set of elements a in $\mathcal{A}$ such that the cameral cover $\tilde{X}_a \to X$ is étale over ∞, a fixed geometric point of $X(\bar{k})$. Note that $\mathcal{A}^{\infty} \subset \mathcal{A}^{\heartsuit}$.

5. $\tilde{\mathcal{A}}$: The *étale open subset* of $\mathcal{A}$, whose $\bar{k}$-points are pairs $(a, \tilde{\infty})$ where $a \in \mathcal{A}^{\infty}(\bar{k})$ and $\tilde{\infty}$ is a $\bar{k}$-point of $\tilde{X}_a$ over ∞. It is irreducible, smooth and the forgetful morphism to $\mathcal{A}^{\infty}$ is a W-torsor. There is a homomorphism $X_*(T) \to \pi_0(\mathcal{P}_a)\big|_{\tilde{\mathcal{A}}}$ which will give a geometric description of endoscopic characters.

For the fundamental lemma we will work with the cohomology of $\mathcal{A}$, which these various subschemes will give us control over.

5.5 Another example: SL_2

SL_2 is the simplest nontrivial case, GL_2 considered as trivial in terms of endoscopy. Now first the Lie algebra

$$\mathfrak{sl}_2 = \{A \in \mathfrak{gl}_2 : \mathrm{tr}(A) = 0\}$$

implies the characteristic polynomials are precisely $t^2 + c$. The Chevalley morphism is the determinant

$$\chi : \mathfrak{sl}_2 \to \mathbb{C}, \qquad \chi(A) = \det(A).$$

The determinant c generates $k[\mathfrak{g}]^G$, and the Kostant section maps

$$\epsilon : \mathbb{C} \to \mathfrak{sl}_2, \qquad \epsilon(c) = \begin{pmatrix} 0 & -c \\ 1 & 0 \end{pmatrix}.$$

Consider the family of elements

$$\xi(\epsilon) = \begin{pmatrix} 0 & \epsilon t^2 + t^3 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}_2(k((t))) \qquad \epsilon \in k$$

The affine Springer fibre is

$$\mathrm{Gr}_{\xi(\epsilon)} = \begin{cases} \mathbb{P}^1 \text{ if } \epsilon = 0 \\ \text{an infinite string of } \mathbb{P}^1 \text{ connected at the end points if } \epsilon \neq 0 \end{cases}$$

The spectral curve $Y_{\xi(\epsilon)}$ is a family of elliptic curves $y^2 = \epsilon t^2 + t^3$. If $\epsilon = 0$ it is a cuspidal elliptic curve, otherwise it is a nodal elliptic curve.

The Hitchin stack is the moduli space of pairs

$$\mathcal{M}_{SL_2} : S \mapsto \begin{cases} E, \text{ rank 2 vector bundles on } X \times S \text{ with trivialized determinant } \wedge^2 E \simeq \mathcal{O}_X \\ \phi, \text{ a trace 0 twisted endomorphism } E \to E \otimes_{\mathcal{O}_X} \mathcal{O}_X(D) \end{cases}$$

and the Hitchin base is the space of determinants

$$\mathcal{A}_{SL_2} = \Gamma(X, \mathcal{O}_X(D)^{\otimes 2}).$$

The parameter a in $\mathcal{A}_{SL_2}$ can be seen as an element of $\mathcal{A}_{GL_2}$, hence the spectral cover is a two-fold cover of X given by the equation $t^2 + a$ on the total space of $\mathcal{O}_X(D)$

$$\pi : Y_a \to X.$$

The Hitchin fibre, when a is not identically zero and Y_a is reduced, is isomorphic to the moduli

$$\mathcal{M}_{SL_2}(a) \simeq \{F \in \overline{Pic}(Y_a) : \det(\pi_* F) \simeq \mathcal{O}_X\}.$$

The symmetry group $\mathcal{P}_a$ for SL_2 is the Prym variety

$$\mathcal{P}_a \simeq \{\mathrm{Norm} : Pic(Y_a) \to Pic(X)\}$$

acting again on the fibres by tensor product. The Hitchin base $\mathcal{A}_{SL_2} \backslash \{0\}$ can be cut up into three natural pieces indexed by endoscopic groups:

1. $\mathcal{A}^{\mathrm{st}}$: Y_a is irreducible, $\tilde{Y_a} \to Y_a$ is ramified. Then $\pi_0(\mathcal{P}_a)$ is trivial.
2. $\mathcal{A}^{U1}$: Y_a is irreducible, $\tilde{Y_a} \to Y_a$ is unramified. Then $\pi_0(\mathcal{P}_a) \simeq \mathbb{Z}/2\mathbb{Z}$.
3. $\mathcal{A}^{GL_1}$: Y_a is reducible. Then $\pi_0(\mathcal{P}_a) \simeq \mathbb{Z}$.

The anisotropic locus $\mathcal{A}^{\mathrm{ani}}$ is the union of (1) and (2).

5.6 From GL_n to reductive groups

Now let's generalize the GL_n picture to a reductive group G. Still we have that local orbital integrals are mass formulas of affine Springer fibres, whose global analogue is the Hitchin fibration, carrying an action of the Picard stack. We list the key changes below:

1. The Kostant section is defined generally as follows: let x_+ to be the sum of one nonzero vector x_α in each root space $\mathfrak{g}_\alpha$, and similarly x_- for $x_{-\alpha}$ in $\mathfrak{g}_{-\alpha}$. The restriction to $x_- + C_{\mathfrak{g}}(x_+)$ is an isomorphism, whose inverse is the desired section.
2. The group scheme of centralizers is $I = \{(g, x) \in G \times \mathfrak{g} : \mathrm{ad}(g)x = x\}$, smooth and commutative over the regular set $\mathfrak{g}^{\mathrm{reg}}$, containing $\epsilon(\mathfrak{c})$, as is $J := \epsilon^* I$. Note that over the regular semisimple set $\mathfrak{c}^{\mathrm{rs}}$, J is a torus. There is a canonical G-equivariant isomorphism $\chi^* J \to I$ over $\mathfrak{g}^{\mathrm{reg}}$. That is, $J_a \simeq I_A$ for every regular A such that $\chi(A) = a$.

   Fix a point a in $\mathfrak{c}(\mathcal{O}_F) \cap \mathfrak{c}^{\mathrm{rs}}(F)$. a defines a regular semisimple stable conjugacy class $\chi^{-1}(a)$ of $\mathfrak{g}(F)$, also a representative $\gamma_0 = \epsilon(a) \in \mathfrak{g}(F)$ of this class. The centralizers of $\chi^{-1}(a)$ are tori, canonically isomorphic to I_{γ_0}.
3. The affine Springer fibre no longer classifies γ-stable lattices, but we still use the affine Grassmannian.
4. For the Higgs pair, replace the datum of a vector bundle with a principle G-bundle.

5. A Higgs field is now a section of $(E \times^G \mathfrak{g}) \otimes \mathcal{O}_X(D)$.. Given a principle G-bundle E we form the associated bundle of $E \times^G \mathfrak{g}$, the quotient of $E \times \mathfrak{g}$ by the G-action $g : (e, v) \mapsto (eg, g^{-1}v)$.

   As a stack, a Higgs pair is the same as giving a map $X \times S \to [\mathfrak{g}/(G \times \mathbb{G})]$, while the composition into the classifying stack of principle G-bundles $BG = [\cdot/G]$ and $B\mathbb{G}_m = [\cdot/\mathbb{G}_m]$ defines a principle G-bundle and a line bundle $\mathcal{O}_X(D)$ respectively.

6. The Hitchin base, then, can be seen as a map $X \times S \to [\mathfrak{c}/\mathbb{G}_m]$. It is still the space of global sections of $\mathfrak{c} \otimes \mathcal{O}_X(D)$. The fibration is induced by the Chevalley characteristic $\chi : [\mathfrak{g}/(G \times \mathbb{G})] \to [\mathfrak{c}/\mathbb{G}_m]$.

7. The Hitchin fibres are related to the spectral curve, not necessarily through the compactified Picard scheme.

8. The group of symmetries is still the classifying stack of J_a-torsors. By G-equivariance J descends to a smooth group scheme over $[\mathfrak{c}/\mathbb{G}_m]$. Let $a : X \times S \to [\mathfrak{c}/\mathbb{G}_m]$ be an S-point of $\mathcal{A}$, pulling back J we get a commutative group scheme J_a. Given (E, ϕ) in $\mathcal{M}_a$, we have a map

$$\chi^* J_a \to \mathrm{Aut}(E, \phi),$$

   giving an action of $\mathcal{P}_a$ on $\mathcal{M}_a$. Varying a, we obtain the stack $\mathcal{P}$ on $\mathcal{A}$.

9. Now let a be anisotropic. When the fibres of J_a over X are not connected, Ngô introduced the following correction: Fix a smooth commutative group scheme J'_a with connected fibres over X with a homormophism to J_a isomorphic over U_a, and let $\mathcal{P}'_a$ be the classifying stack of J'_a torsors on X. Locally, we have a smooth commutative group scheme with connected fibres over $\mathcal{O}_v$ isomorphic to $J_{a,v}$ over the generic fiber, and consider the stack $\mathcal{P}_v(J'_a)$ of J'_a-torsors which trivialize over F_v. The isomorphism $H^1(F_v, J_{a,v}) \simeq H^1(k, \mathcal{P}_v(J'_a))$ leads to

$$|[\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)](k)_\kappa| = \mathrm{vol}(J'_a(\mathcal{O}_v))\mathbf{O}^\kappa_a(1_{\mathfrak{g}(\mathcal{O}_v)}).$$

   and the corresponding product formula

$$[\mathcal{M}_a/\mathcal{P}'_a] = \prod_{v \in \bar{X} \setminus U_a} [\mathcal{M}_v(a)/\mathcal{P}_v(J'_a)]$$

   gives

$$|[\mathcal{M}_a/\mathcal{P}'_a)](k)_\kappa| = \prod_{v \in \bar{X} \setminus U_a} \mathrm{vol}(J'_a(\mathcal{O}_v))\mathbf{O}^\kappa_a(1_{\mathfrak{g}(\mathcal{O}_v)}).$$

10. The connected components $\pi_0(\mathcal{P})$ of $\mathcal{P}^\heartsuit$ is a sheaf of abelian groups over $\mathcal{A}^\heartsuit$, where the fibre over each geometric point a is $\pi_0(\mathcal{P}_a)$. There is a canonical homomorphism

$$X_*(T) \times^W \mathrm{Irr}(\tilde{X}_a) \to \pi_0(\mathcal{P}_a)$$

11. The invariant δ_a now requires we invoke the *Néron model* of J_a, denoted J^{ab}_a, is the universal smooth abelian group scheme over $\bar{X}$ with a map $J_a \to J^{\mathrm{ab}}_a$ isomorphic on U_a, the open set of regular points defined by a. The induced map on $\mathcal{P}_a$ leads to a Chevalley dévisage

$$0 \to \mathcal{P}^{\mathrm{aff}}_a \to \mathcal{P}_a \to \mathcal{P}^{\mathrm{ab}}_a \to 0$$

   where $\mathcal{P}^{\mathrm{aff}}_a$ is an affine algebraic group, and the connected component of $\mathcal{P}^{\mathrm{ab}}_a$ is an abelian stack. The invariant

$$\delta_a := \dim \mathcal{P}^{\mathrm{aff}}_a$$

   measures the affine part of $\mathcal{P}_a$.

5.7 Aside I: some geometric representation theory

5.7.1. Flag variety. Let G be a connected reductive group. A Borel subgroup is a maximal closed connected solvable subgroup. Equivalently, it is minimal among parabolic subgroups P, which are defined to be such that G/P is complete. All Borel subgroups are conjugate and normal in G. Define the *flag variety* $\mathcal{B}$ to be the set of Borel subgroups of G.

Fix a Borel B, then $\mathcal{B}$ can be identified with G/B as follows: G acts transitively on $\mathcal{B}$ by conjugation, and by normality the stabilizer of B is itself. So we obtain the desired bijection. Moreover, a Borel subalgebra is defined as the Lie algebra of a Borel subgroup, so by abuse of notation we may also identity $\mathcal{B}$ with the set of Borel subalgebras of $\mathfrak{g}$.

Example. The upper triangular matrices in GL_n form a Borel B. Let V be a finite dimensional vector space, and let $\{V_i\}$ and $\{U_i\}$ be complete flags in V., i.e, a chain of vector spaces $U_1 \subset \cdots \subset U_{\dim V}$. Define g in $GL(V)$ to be the map of basis vectors $v_i \mapsto u_i$, so $GL(V)$ acts transitively on the set of complete flags. Identifying $GL(V)$ with GL_n by the basis of $\{V_i\}$, we see that B stabilizes the flag since multiplying (v_i) with an element of B does not change the basis.

Therefore we can identify each B with a flag in V and $GL(V)/B$ with the set of complete flags, whence the name. Similarly, parabolic subgroups stabilize some flag, so G/P is called the *generalized flag variety.*

5.7.2. Springer resolution. Let G be a reductive group with Borel subgroup B and unipotent radical U. The *Springer resolution* is the resolution of singularities for the unipotent variety $\mathcal{U}$ of G by projecting

$$\tilde{\mathcal{U}} = \{(x, gB) : g^{-1}xg \in U\} \subset G \times G/B$$

onto the first factor. (Every unipotent element of G has a conjugate in U.) For Lie algebras, define also

$$\tilde{\mathcal{N}} = \{(x, gB) : \mathrm{Ad}(g)^{-1}(x) \in \mathfrak{u}\} \subset \mathfrak{g} \times G/B.$$

The *generalized* or *Grothendieck-Springer resolution* is similarly defined, replacing U by G and $\mathfrak{u}$ by $\mathfrak{g}$. The fibres of the projection are then called the (generalized) Springer fibres.

5.7.3. Grassmannian. The *Grassmanian* $\mathrm{Gr}(r, V)$ is the set of r-dimensional linear subspaces of V. It is a projective variety, given by the Plücker embedding into $\mathbb{P}(\bigwedge^r(V))$ by sending a subspace U with basis $\{u_i\}$ to $u_1 \wedge \cdots \wedge u_r$. Second, $GL(V)$ acts transitively on $\mathrm{Gr}(r, V)$, so we may write $\mathrm{Gr}(r, V) = GL(V)/\{\text{stabilizer}\}$. Third, there is a description of the Grassmannian as the scheme of a representable functor from S-schemes to quasicoherent sheaves. This is the interpretation we will see in the affine setting. Note that the flag variety embeds into a product of Grassmannians $\mathrm{Gr}(1, V) \times \cdots \times \mathrm{Gr}(n, V)$.

5.7.4. Principle bundle. We'll want to know about principle bundles over schemes when we define our stacks. Let X be a k-scheme with an action of an affine algebraic group G. A *G-fibration* over X is a scheme E with a G-action $E \times G \to E$ and a G-equivariant morphism $E \to X$. A morphism between fibrations is a morphism commuting with the morphisms to X. A G-fibration is then called trivial if it is isomorphic to the fibration $G \times X$, the G-action given by multiplication on the second factor. Now, a *principle G-bundle* is a locally trivial G-fibration.

Example. Let V be a k-vector space of dimension n, and E a principle GL_n-bundle. We can form the *associated bundle* $E \times^G V$ given as the quotient of $E \times V$ by the G-action $g : (e, v) \mapsto (eg, g^{-1}v)$. This is a rank n vector bundle. Conversely, given a rank n vector bundle, its associated frame bundle is a principle GL_n-bundle.

The following moduli problem is standard in motivating the definition of stacks: let X be a smooth projective curve of genus g over S. The contravariant functor, or presheaf of sets, associating to an S-scheme M the isomorphism classes of rank n vector bundles on $X \times M$ is not a representable functor. Rather than rigidify the moduli problem, one may replace the representing scheme with a stack.

5.7.5. Stack. We will only define stacks informally here. Schemes can be seen, by the functor of points $\mathrm{Hom}(-, M)$, as sheaves of sets on the category of (Sch/S) with the étale topology. A sheaf of sets F is representable by M if it is isomorphic to $\mathrm{Hom}(-, M)$. A stack, by contrast, is a sheaf of groupoids as a 2-category. A groupoid is a category in which every morphism has an inverse. There is an equivalent definition of stack as a category over (Sch/S) fibered in groupoids, which is more common but we will not use here.

Example. Let X be a projective scheme over S. The functor associating to an S-scheme M the category whose objects are vector bundles V on $X \times M$ of rank r with fixed Chern classes is a stack. The key notion is that unlike schemes, points of a stack may have nontrivial automorphisms, e.g., multiplying a vector bundle by a scalar. For this reason orbifolds, which can be seen as topological stacks, generalize manifolds.

Example. The main example of a stack we will use is the quotient stack. Let X be an S-scheme with an action of a group G. Define the contravariant functor $[X/G]$ from S-schemes to groupoids, associating to an S-scheme M the category of principle G-bundles over M with a G-equivariant morphism to X. In the special case where X is a point $\mathrm{Spec}(k)$, the stack $[\mathrm{pt}/G]$ is the moduli stack of all principle G-bundles, also known as the classifying stack BG. Further, if $G = \mathbb{G}_m$ then $B\mathbb{G}_m$ classifies line bundles.

5.8 Cohomology of the Hitchin base

5.8.1. A story of perverse sheaves. Start with a topological space X over a field k, with a sheaf K of k-vector spaces over it. We can choose an injective resolution

$$0 \longrightarrow K \longrightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \dots$$

inducing a complex of abelian groups

$$0 \longrightarrow \Gamma(X, I^0) \xrightarrow{d^0} \Gamma(X, I^1) \xrightarrow{d^1} \dots$$

and then take cohomology to get $H^i(X, K)$.

In general, a *complex of sheaves* K is a diagram

$$\dots \longrightarrow K^{i-1} \xrightarrow{d^{i-1}} K^i \xrightarrow{d^i} K^{i+1} \xrightarrow{d^{i+1}} \dots$$

with $d^{i+1} \circ d^i = 0$. Call it a *bounded* complex if $K^i = 0$ for $|i|$ large enough. Complexes of sheaves form an abelian category, so we may form the *cohomology sheaves* $H^i(K) = \ker(d^i)/\mathrm{I'm}(d^{i-1})$ whose stalks at a point x is the cohomology of the complex of stalks at x.

Identifying morphisms that are homotopic and formally adding inverses to quasi-isomorphisms (morphisms inducing isomorphism in cohomology), we obtain the *(bounded) derived category* $D^b(X)$ of sheaves on X. Its objects are still complexes of sheaves.

Let I be a bounded complex of injective sheaves. Given a continuous map $f : X \to Y$, we may push forward the complex I by the *direct image* $f_* I$, the complex of sheaves on Y satisfying

$$H^i(U, f_* I) = H^i(f^{-1}(U), I).$$

More generally, given a bounded complex K, replace it with an injective resolution I and then push forward to get its direct image $Rf_* K$ in $D^b(Y)$. Its cohomology sheaves on Y are denoted $R^i f_* K$. When Y is a point, $Rf_* K = R\Gamma(X, K) = \Gamma(X, I)$, the complex of abelian groups

$$\dots \longrightarrow \Gamma(X, I^{i-1}) \longrightarrow \Gamma(X, I^i) \longrightarrow \Gamma(X, I^{i+1}) \longrightarrow \dots.$$

A bounded complex K is quasi-isomorphic to some I, unique up to unique isomorphism, This is an injective resolution of K. There are related functors $f_!, f_*, f^!, f^*$ which will be important.

A *constructible* subset is one obtained from a finite sequence of unions, intersections and complements in X. A *local system* is a locally constant sheaf with finite dimensional stalks. A *constructible complex* is one whose cohomology sheaves are local systems on each stratum, for some finite stratification of X by constructible sets. Denote the constructible bounded derived category $D^b_c(X)$.

Finally, a *perverse sheaf* is a constructible complex with certain restrictions on the dimension of the support of its stalk cohomology and of its stalk cohomology with compact supports. The *intersection complex* $IC_X(L)$ of a local system L on an open $U \subset X$ is a complex of sheaves on X uniquely extending the shifted complex $L[\dim X]$ on U. Perverse sheaves are built up from these.

To each complex in $D^b_c(X)$ we can canonically associate the collection of perverse cohomology sheaves ${}^pH^n(K)$, which are themselves perverse sheaves. This is similar to identifying a constructible sheaf with a constructible complex trivial outside of $i = 0$.

5.8.2. Decomposition. We invoke the purity theorem of Beilinson-Bernstein-Deligne:

Theorem 5.8.1. *Let K be a pure complex of sheaves of weight w. The perverse sheaves ${}^pH^i(K)$ are pure of weight $w+i$, and*

$$K \simeq \bigoplus_{n\in\mathbb{Z}} {}^pH^n(K)[-n].$$

which leads to their decomposition theorem:

Theorem 5.8.2. *Let $f : X \to Y$ be a proper morphism. There is an isomorphism in $D^b_c(Y)$*

$$Rf_*IC_X \simeq \bigoplus_{n\in\mathbb{Z}} {}^pH^n(Rf_*IC_X)[-n]$$

where the perverse sheaves ${}^pH^n(Rf_*IC_X)$ are semisimple. There is a decomposition of $Y = \coprod Y_i$ into locally closed subsets and a canonical decomposition into intersection complexes of semisimple local systems

$$ {}^pH^n(Rf_*IC_X) \simeq \bigoplus_i IC_{\overline{Y}_i}(L_i)$$

We concentrate now on the anisotropic locus of the Hitchin fibration, $\tilde{f}^{\text{ani}} : \tilde{\mathcal{M}}^{\text{ani}} \to \tilde{\mathcal{A}}^{\text{ani}}$, where the tilde indicates we are working over the étale open subset. By the purity theorem, the direct image decomposes into perverse sheaves:

$$\tilde{f}^{\text{ani}}_*\bar{\mathbb{Q}}_l = \bigoplus {}^pH^n(\tilde{f}^{\text{ani}}_*\bar{\mathbb{Q}}_l)[-n]$$

Now the action of $\mathcal{P}$ on $\mathcal{M}$ gives an action of the sheaf of abelian groups $\pi_0(\tilde{\mathcal{P}}^{\text{ani}})$, a quotient of the constant sheaf, on the summands. Thus the cocharacter group X_* acts through π_0, so that a character of the finite abelian group descends to an endoscopic character:

$$X_*(T) \to \pi_0(\tilde{\mathcal{P}}^{\text{ani}}) \xrightarrow{\kappa} \overline{\mathbb{Q}}_l^\times.$$

Consequently the perverse cohomology breaks into κ-isotypic pieces:

$$ {}^pH^n(\tilde{f}^{\text{ani}}_*\bar{\mathbb{Q}}_l) = \bigoplus_{\kappa\in\check{T}} {}^pH^n(\tilde{f}^{\text{ani}}_*\bar{\mathbb{Q}}_l)_\kappa$$

We also need the κ-Grothendieck-Lefschetz trace formula adapted to stacks:

Proposition 5.8.3. *Let P^0 be the connected component of P, a commutative k-group scheme acting on a k-scheme M. Let Λ be a torsion-free subgroup of P such that P/Λ and M/Λ are finite type. The following formula holds:*

$$|P^0(k)||M/P(k)|_\kappa = \sum_n (-1)^n \text{tr}(\text{Frob}, H^n_c(M/\Lambda, \mathbb{Q}_l)_\kappa)$$

If Λ' is a σ-invariant finite index subgroup of Λ, there is a canonical isomorphism $H^n_c(M/\Lambda')_\kappa \simeq H^n_c(M/\Lambda)_\kappa$.

5.8.3. Endoscopic transfer. Recall that the endoscopic character κ defines an endoscopic group H of G. Each piece ${}^pH^n(\tilde{f}_*^{\mathrm{ani}}\bar{\mathbb{Q}}_l)_\kappa$ is supported on the locus $\tilde{\mathcal{A}}_\kappa^{\mathrm{ani}}$ consisting of elements a where the κ action factors through $\pi_0(\tilde{\mathcal{P}}_a)$. This locus is not connected; its connected components are classified by a homomorphism $\rho : \pi_1(X, \infty) \to \pi_0(\check{H})$. One checks that $\tilde{\mathcal{A}}_\kappa$ is precisely the Hitchin base associated to the group H, $\tilde{\mathcal{A}}_H$.

Now a morphism $\mathfrak{c}_H \to \mathfrak{c}_G$ induces a closed immersion $\iota_{\kappa,\rho} : \mathcal{A}_H \to \mathcal{A}$.

Theorem 5.8.4. *Let G be a quasisplit reductive group and $r = \mathrm{codim}(\tilde{\mathcal{A}}_H)$. Then there is an isomorphism*

$$\bigoplus_n {}^pH^n(\tilde{f}_*^{ani}\bar{\mathbb{Q}}_l)_\kappa[2r](r) \simeq \bigoplus_\rho (\iota_{\kappa,\rho})_* \bigoplus_n {}^pH^n(\tilde{f}_{H,*}^{ani}\bar{\mathbb{Q}}_l)_\kappa$$

The main ingredient in the proof is Ngô's support theorem:

Theorem 5.8.5. *Let Z be the support of a geometrically simple factor of ${}^pH^n(\tilde{f}_*^{any}\bar{\mathbb{Q}}_\ell)_\kappa$. If Z meets $\nu(\tilde{\mathcal{A}}_H^{good})$ for some endoscopic group H then $Z = \nu(\tilde{\mathcal{A}}_H^{good})$. In fact, there is a unique such H.*

Alternatively, let $f : M \to A, g : P \to A$ be a δ-regular abelian fibration of relative dimension d. Assume A is connected and M is smooth. Let F be an intersection cohomology sheaf that is a summand in $Rf_*\mathbb{C}_M$ and let $Z \subset A$ be the support of F. Then there exists an open $U \subset A$ intersecting Z with a nonzero local system L on $U \cap Z$ such that the tautological extension by zero of L to U is a summed of $R^{2d}f_*\mathbb{C}_M|_U$. In particular, if the fibres of f are irreducible (e.g., over the anisotropic locus) then $Z = A$.

Using the isomorphism of perverse sheaves Ngô proves

Theorem 5.8.6. *(Stabilization) For every k point a_H of $\tilde{\mathcal{A}}_H^{\mathrm{ani}}(k)$ with image a in $\tilde{\mathcal{A}}^{\mathrm{ani}}(k)$, the global equality holds:*

$$|[\mathcal{M}_a/\mathcal{P}'_a](k)_\kappa| = q^r|[\mathcal{M}_{H,a_H}/\mathcal{P}'_a](k)|,$$

(Fundamental lemma) For all k-points a_H in $\mathfrak{c}_H^{\mathrm{rs}}(F_v) \cap \mathfrak{c}_H(\mathcal{O}_v)$ with image a in $\mathfrak{c}^{\mathrm{rs}}(F_v) \cap \mathfrak{c}(\mathcal{O}_v)$ the local equality holds:

$$|[\mathcal{M}_{v,a}/\mathcal{P}_v(J'_a)](k)_\kappa| = q^{\deg(v)r_v(a)}|[\mathcal{M}_{H,v,a_H}/\mathcal{P}_v(J'_a)](k)|,$$

where $r = \sum_v \deg(v)r_v(a)$.

5.9 Aside II: Cohomological definitions

The cohomological background is substantial, as it invokes Deligne's purity theorem relating to his proof of the Weil conjectures. In particular, we need the technology of derived categories, perverse sheaves and ètale cohomology of stacks rather than schemes. A nice exposition of this is by De Cataldo and Migliorini, cited below.

5.9.1. Derived categories and derived functors. Let's blithely sketch some categorical definitions:

1. A *monodical category* is a category A equipped with a bifunctor $\otimes : A \times A \to A$ associative up to natural isomorphism and an object I which is both a left and right identity for $\otimes$.

   An *enriched category* is a category whose hom-sets are replaced with objects from a monoidal category.

   A *pre-additive category* is a category whose hom-sets are abelian groups and composition of morphisms is bilinear; i.e., it is a category enriched over the monoidal category of abelian groups.

   An *additive category* is a preadditive category admitting all finitary coproducts

   A *pre-abelian category* is an additive category in which every morphism has a kernel and a cokernel.

   An *abelian category* is a pre-abelian category such that every monomorphism and epimorphism is normal.

2. A *(cochain) complex* in an additive category A is a system $X = (X^n, d^n)$ where $d^n : X^n \to X^{n+1}$ are morphisms of objects in $\mathcal{A}$ such that $d^{n+1} \circ d^n = 0$ for any $n \in \mathbb{Z}$. A morphism $f : X \to Y$ is a system of morphisms $f^n : X^n \to Y^n$ such that $d^n_Y f^n = f^{n+1} d^n_X$ for all n.

   The *shift functor* is the functor $[k]$ sending a complex X to $X[k] = (x^{n+k}, (-1)^k d_X^{n+k})$, and a morphism f to $f[k] = f^{n+k} : X[k] \to Y[k]$.

   The *cone* $C(f)$ of a morphism f is the complex $X[1] \oplus Y$ with differential $\begin{pmatrix} -d_X & 0 \\ f & d_Y \end{pmatrix}$. There is a canonical morphism of complexes $X \xrightarrow{f} Y \to C(f) \to X[1]$.

   A *homotopy category* HA has objects same as CA, the category of complexes of A, modulo the homotopy relation $f \sim g$ if there is a system of morphisms s such that $f - g = sd + ds$.

   The *cohomology groups* $H^n(X)$ are $\ker d^n / \mathrm{Im} d^{n-1}$, where X is a complex in an abelian category A..

3. A *triangle* (X, Y, Z) in an additive category A with an automorphism $T : A \to A$ is a system of morphisms $X \to Y \to Z \to TZ$. A morphism of triangles (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) is a system of morphisms (f_1, f_2, f_3) such that the expected diagram commutes.

   A *distinguished triangle* in HA is a triangle isomorphic to the triangle $(X, Y, C(f), X[1])$.

   A *triangulated category* D is an additive category with an automorphism $T : X \mapsto X[1]$ and a family of distinguished triangles satisfying certain axioms. A triangulated subcategory is a full subcategory D' closed under shifts and taking triangles (if (X, Y, Z) is a triangle with $X, Y \in D'$ then $Z \in D'$).

4. A *multiplicative systems of morphisms* S in a category A is a family of morphisms satisfying conditions similar to those of a multiplicative set in a commutative ring.

   The *localization of a category* A with multiplicative system S is the universal category A_S such that any functor $F : A \to B$ mapping morphisms in A to invertible morphisms in B factors through A_S.

   An *acyclic* complex is a complex X such that $H^n(X) = 0$ for all n.

   A *quasi-isomorphism* is a morphism of complexes inducing an isomorphism of cohomology groups.

   The *derived category* of an abelian category A is the category $DA := HA/N = HA_{S(N)}$ where $N \subset HA$ is the subcategory of acyclic objects, and $S(N)$ is the family of quasi-isomorphisms in HA. The functor $H^n : HA \to A$ factors uniquely through DA.

5. A *injective* in an abelian category A is an object I such that for any monomorphism $X \to Y$ the induced map $\mathrm{Hom}(Y, I) \to \mathrm{Hom}(X, I)$ is a monomorphism.

   An abelian category has *enough injectives* if for every object X there exists a monomorphism $X \to I$ where I is an injective. The category of sheaves over a topological space has enough injectives.

   Let A be an additive category. Define $C^+A = \{X \in CA : X^n = 0, n \ll 0\}$. If A is an abelian category with enough injectives, and I the subcategory of injectives, then the natural functor $H^+I \to D^+A$ is an equivalence of categories. (Similarly we define C^+, and $C^b = C^+ \cap C^-$.)

   The *right derived functor* of an additive functor $F : A_1 \to A_2$ of abelian categories with enough injectives is the induced functor $RF : D^+A_1 \to D^+A_2$. The n-th right derived functors are $R^nF = H^n \circ RF : D^+A_1 \to A_2$.

   If $F : A_1 \to A_2$ is a left exact functor then $R^0F(X) = F(X)$ and for any sequence $0 \to X \to Y \to Z \to 0$ in A there is a long exact sequence in A_2

$$0 \to FX \to FY \to FZ \to R^1FX \to R^1FY \to R^1FZ \to \dots$$

6. A *t-structure* on a triangulated category D is a pair $(D^{\leq 0}, D^{\geq 0})$ of full subcategories of D such that

   (a) $\mathrm{Hom}(D^{\leq 0}, D^{\geq 1}) = 0$, where $D^{\leq n} = D^{\leq 0}[-n]$ and $D^{\geq n} = D^{\geq 0}[-n]$,

   (b) $D^{\leq 0} \subset D^{\leq 1}$, and $D^{\geq 1} \subset D^{\geq 0}$,

   (c) for all X in D there is a triangle (A, X, B) where $A \in D^{\leq 0}$ and $B \in D^{\geq 1}$.

   There is a canonical t-strucutre on $D = DA$ for any abelian category A given by

$$D^{\leq 0} = \{X \in D : H^nX = 0, n > 0\}, \qquad D^{\geq 0} = \{X \in D : H^nX = 0, n < 0\}$$

   with functors $\tau_{\leq n} : D \to D^{\leq n}$ and $\tau_{\geq n} : D \to D^{\geq n}$ given by

$$\tau_{\leq n}(X) = \dots X^{n-2} \to X^{n-1} \to \ker(d^n) \to 0 \to 0 \to \dots$$
$$\tau_{\geq n}(X) = \dots 0 \to 0 \to \mathrm{coker}(d^{n-1}) \to X^{n+1} \to X^{n+2} \to \dots$$

7. The *core* of a t-structure is the abelian category $\mathrm{Core}(D) = D^{\leq 0} \cap D^{\geq 0}$. Define $H^nX = H^0(X[n])$, where $H^0 := \tau_{[0,0]} : D \to \mathrm{Core}(D)$, with $\tau_{[n,m]} = \tau_{\geq m}\tau_{\leq n}$.

   A *recollement* of a triangulated category D consists of a pair of exact functors $D_1 \xrightarrow{i_*} D \xrightarrow{j^*} D_2$ where $i : Z \hookrightarrow X$ is a closed embedding and $j : U \hookrightarrow X$ its open complement satisfying certain axioms.

$$D^{\leq 0} = \{X \in D : i^*X \in D_1^{\leq 0}, j^*X \in D_2^{\leq 0}\}, \quad D^{\geq 0} = \{X \in D : i^!X \in D_1^{\geq 0}, j^!X \in D_2^{\geq 0}\}$$

   define a t-structure, where D, D_1, D_2 are triangulated categories with t-structures.

5.9.2. Perverse sheaves. In this section we fix a topological space X over a field k. Denote $\mathrm{Sh}(X)$ the category of sheaves F of k-vectors spaces over X, and $D(X) = D(\mathrm{Sh}(X, k))$ its derived category. Let $f : X \to Y$ be a continuous map of (locally compact, Hausdorff) topological spaces.

1. Let F be a sheaf on X. The *direct image sheaf* f_*F of F is defined by $\Gamma(U, f_*F) = \Gamma(f^{-1}(U), F)$.

   Let F be a sheaf on Y. The *inverse image sheaf* f^*F of F on X is the sheaf associated to the presheaf $V \mapsto \varinjlim \Gamma(U, F)$, taken over open neighbourhoods $U \supset f(V)$, for any open V in X.

   An *adjoint pair* is a pair of functors $F : C \to D$ and $G : D \to C$ such that $\mathrm{Hom}_C(FY, X) \simeq \mathrm{Hom}_D(Y, GX)$ for all $X \in C, Y \in D$. In particular, (f^*, f_*) is an adjoint pair.

2. A *proper* morphism is a morphism f such that the preimage of any compact in Y is compact in X.

   Let F be a sheaf on X. The *direct image sheaf with proper support* is the subsheaf $f_!F \subset f_*F$ given by
   $$\Gamma(U, f_!F) = \{s \in \Gamma(f^{-1}U, F) : \text{supp } s \text{ is proper}\}.$$
   Let F be a sheaf on Y and $a_X : X \to \text{pt.}$. The *sections with compact support* $a_{X!}F$ are defined by
   $$\Gamma_c(X, F) = \{s \in \Gamma(X, F) : \text{supp } s \text{ is compact.}\}$$
   Now $f_!$ does not have a right adjoint in general. It does in the case of an embedding $i : Z \hookrightarrow X$ of a locally closed subset ($Z = Y \cap V$, where Y is closed and V is open). Define the $\Gamma_Z : \text{Sh}(X) \to \text{Sh}(X)$
   $$\Gamma_Z(U, F) := \{s \in \Gamma(U, F) : \text{supp } s \subset Z\}$$
   and the functor $i^! = i^* \circ \Gamma_Z : \text{Sh}(X) \to \text{Sh}(Z)$. Then $(i_!, i^!)$ is an adjoint pair.

3. The *sheaf cohomology groups* are $H^n(X, F) = R^n\Gamma(X, F)$ given by the right derived functor of $\Gamma(X, -) : \text{Sh}(X) \to \text{Mod}(k)$. Similarly we have $H^n_c(X, F)$.

   The *local cohomology sheaves with supports in Z* are $H^i_Z F = R^i\Gamma_Z F$.

   The *cohomological dimension* of a left exact functor $F : A \to B$ of abelian categories with enough injectives is defined as $\text{hd}(F) = \sup\{n \geq 0 : R^n F X \neq 0 \text{ for some } X \text{ in } A\}$

   The *cohomological dimension* of a locally compact topological space is $\text{hd}(X) = \text{hd}(a_{X!})$. From now on assume all $\text{hd}(f^!)$ and $\text{hd}(X)$ are finite.

   Verdier duality: given $f_!$ with finite cohomological dimension, its right derived functor $Rf_! : D^+(X) \to D^+(Y)$ has a right adjoint $f^! = Rf^!$, thus $(Rf_!, f^!)$ is an adjoint pair.

4. An (algebraic) *partition* of X is a finite set S of locally (Zariski) closed subsets of X, called S-*stratum*, whose disjoint union is X. A partition S' is a partition of S if any S-stratum is a union of S'-strata.

   A *stratification* is a partition S such that the Zariski closure of any stratum is a union of strata. The pair (X, S) is called a stratified space. Any partition has a refinement which is a stratification.

   A *local system* or *smooth sheaf* on X is a locally constant sheaf of finite dimensional k-vector spaces on X.

   An S-*constructible sheaf* F on X is a sheaf such that $F|_s =: i^*F$ is a local system for every stratum $i : s \hookrightarrow F$ in S. An *constructible sheaf* is a sheaf that is S-constructible for some stratification S.

   A bounded complex is *constructible* if its cohomology sheaves are constructible. Denote $D^b_c(X)$ (resp. $D^b_S(X)$) the triangulated subcategory of bounded constructible (resp. S-constructible) complexes.

5. An S-*perversity function* is a function $p : S \to \mathbb{Z}$ on a stratified space (X, S).

   A *perverse t-structure* on the category $D = D^+(X)$ or $D^b(X)$ is defined as
   $${}^pD^{\leq 0} = \{X \in D : i^*_S X \in D^{\leq p(s)}, s \in S\}, \quad {}^pD^{\geq 0} = \{X \in D : i^!_S X \in D^{\geq p(s)}, s \in S\}$$
   Define the category of *p-perverse sheaves* on D by $\text{Perv}(X, p) = \text{Core}(D) = {}^pD^{\leq 0} \cap {}^pD^{\geq 0}$.

   A *regular stratification* is a stratification S of X with smooth strata and for any stratum $i : s \hookrightarrow F$ and any local system F on s the sheaves $R^n i_* F$ are S-consructible. The category $D^b_S(X)$ can be given a t-structure as above when X has a regular stratification.

   A *perversity function* is a function $p : \mathbb{N} \to \mathbb{Z}$ such that for $m \leq n$ one has $p(n) \leq p(m)$ and $2m + p(m) \leq 2n + p(n)$. Given a perversity we can defined the S-perversity $p(S) = p(-\dim S)$.

   Define the category of *perverse sheaves* on $D = D^b_c(X)$ as $\text{Perv}(X, p) = \text{Core}(D)$ with t-structure
   $${}^pD^{\leq 0} = \{X \in D : \text{there exists a regular stratification } S \text{ such that } X \in {}^pD^{\leq 0}_S(X)\} (\text{resp. } {}^pD^{\geq 0})$$
   There is a cohomology functor ${}^pH^n : D^b_c(X) \to \text{Perv}(X, p)$ for all $n \in \mathbb{Z}$ analogous to H^n above.

6. A *quasi-finite* map is a map $f : X \to Y$ such that $f^{-1}(y)$ is finite for all y in Y.

   The *intermediate extension* $f_{!*}$ of a quasi-finite morphism f is defined as follows: given a sheaf F in $\mathrm{Perv}(X,p)$ we have $f_!F \in D^{\leq 0}(X)$ and $f_*F \in D^{\geq 0}(X)$, hence a canonical map $f_!F \to f_*F$. By

   $$\mathrm{Hom}(f_!F, f_*F) \simeq \mathrm{Hom}(\tau_{\geq 0} f_!F, \tau_{\leq 0} f_*F) := \mathrm{Hom}({}^p f_!F, {}^p f_*F)$$

   we define the intermediate extension of F to be the image of the canonical map ${}^p f_!F \to {}^p f_*F$. This defines a functor $f_{!*} : \mathrm{Perv}(X,p) \to \mathrm{Perv}(X,p)$.

   The *middle perversity* is defined as $p_{\mathrm{mid}}(n) = -n$.

   A *simple perverse sheaf* is of the form $j_{!*}(L[\dim U])$ where $j : U \hookrightarrow X$ is smooth, locally closed and L is any irreducible local system. These are the simple objects in $\mathrm{Perv}(X, p_{\mathrm{mid}})$.

   The *intersection cohomology sheaf* of X is $IC_X = j_{!*}(\mathbb{C}_X[\dim X])$, where $j : U \hookrightarrow X$ is a smooth open subspace.

   The *twisted intersection cohomology sheaf* is $IC_X(L) = j_{!*}(L[\dim X])$, where L is any local system on U.

   The *kith intersection cohomology group* is $IH^k(X) = H^k(X, IC_X[-\dim X])$.

7. A $\bar{\mathbb{Q}}_l$ sheaf F on a variety X is *punctually pure of weight* w if for every $n \geq 1$ and x in $X(\mathbb{F}_{q^n})$, the eigenvalues of the action of Frob^n on F_x are algebraic numbers such that all their complex conjugates have absolute values $q^{nw/2}$.

   A $\bar{\mathbb{Q}}_l$ sheaf F is *mixed* if it admits a finite filtration with punctually pure quotients. The weights of F are the weights of the nonzero quotients.

   A *mixed complex* is a complex whose cohomology sheaves are mixed. The category of mixed complexes is a full subcategory of $D^b_c(X, \bar{\mathbb{Q}}_l)$

   A mixed complex is *pure of weight* w if its cohomology sheaves $H^i(F)$ are punctually pure of weights at most $w + i$.

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6 Beyond Endoscopy

6.1 Introduction (Altuğ)

Langlands functoriality: let G, H be split reductive groups with an L-homomorphism

$$^L H \xrightarrow{\rho} {}^L G \xrightarrow{\sigma} GL(V) \qquad (*)$$

Conjecture: there should be a corresponding transfer of automorphic representations

$$\pi_H \mapsto \pi_G,$$

such that the following equality holds:

$$L(s, \pi_H, \rho \circ \sigma) = L(\sigma, \pi_G, \sigma).$$

In terms of the conjectural Langlands group $L_{\mathbb{Q}}$, there should be maps

$$\psi_{\pi_H} : L_{\mathbb{Q}} \to {}^L H \text{ and } \psi_{\pi_G} : L_{\mathbb{Q}} \to {}^L G$$

commuting with ρ.

For endoscopic H, ρ is 'simple', leads to stable trace formula. For arbitrary ρ, how to even start? We would like to use the trace formula to isolate representations coming from transfers π_H in the trace formula of G. The idea is to use automorphic L-functions: it is believed that poles of $L(s, \pi, \sigma)$ identifies such functorial transfers.

Digression: the spectral decomposition of G is the following

$$L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}}) = L^2_{\text{cont}} \oplus L^2_{\text{disc}} = L^2_{\text{cont}} \oplus L^2_{\text{cusp}} \oplus L^2_{\text{res}}$$

For π_G in L^2_{cusp}, $L(s, \pi_G)$ is holomorphic at $\text{Re}(s) = 1$ when $G = GL_n$, expected to hold for general G. The associated Dirichlet series is

$$L(s, \pi_G) = \sum_{n=1}^{\infty} \frac{a_{\pi_G}(n)}{n^s}.$$

By analogy with $L(s, \chi)$ which has a pole at $s = 1$ if χ is trivial, and holomorphic otherwise, we separate the spectrum into representations that are transfers and those which are not:

$$L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}}) = L^2_{\text{trans}} \oplus L^2_{\text{not trans}},$$

and we expect π_G to be in L^2_{trans} if $L(s, \pi_G, \sigma)$ has a pole at $\text{Re}(s) = 1$ for some $\sigma : {}^L G \to GL(V)$, while the $L(s, \pi_G, \sigma)$ for π_G in $L^2_{\text{not trans}}$ should be holomorphic at 1.

Example. Let $G = GL_2$, π a Maass form or holomorphic cusp form. Then $L(s, \pi)$ is holomorphic at 1. On the other hand, $L(s, \pi, \text{Sym}^2$ has a pole at $s = 1$ if and only if it π is a transfer from a GL_1 form (dihedral representation); it is holomorphic otherwise.

Now fix S a finite set of places containing all ramified primes, and choose f_v^G in $C_c^\infty(G_v)$ for all v in S. Form the partial L-function

$$\sum_{\pi_G \in L^2_{\text{disc}}} L^S(s, \pi_G, \sigma) \prod_{v \in S} \text{tr}(\pi_{G_v}(f_v^G)) \qquad (1)$$

and compare with

$$\sum_{\pi_H \in L^2_{\text{cusp}}} L^S(s, \pi_H, \sigma) \prod_{v \in S} \text{tr}(\pi_{H_v}(f_v^H)). \qquad (2)$$

One would like, for given G, H, and σ, to get workable expressions, and show that

$$\text{Res}_{s=1}(1) = \oplus\{\text{transfers for which } L(s,\pi,\rho\circ\sigma) \text{ has a pole}\}$$

and by decomposing $L^2(G) = \bigoplus\{H,\sigma\}$, compare (1) and (2) to establish functoriality.

Stable transfers of $f_v^G \mapsto f_v^H$ and $\pi_G \mapsto \pi_H$ should lead to stable traces

$$\text{tr}_{\text{st}}(\pi_G(f_v^G)) = \text{tr}_{\text{st}}(\pi_H(f_v^H)).$$

We are only interested in the poles of

$$L^S(s,\pi,\sigma) = \sum_{(n,S)=1} \frac{a_{\pi_\sigma}(n)}{n^s}$$

Fact: there are functions f_r in $\prod_{v|r} G_v$, $v \notin S$ such that $\text{tr}(\pi_r(f_{r,\sigma})) = a_{\pi,\sigma}(r)$. This leads to a sum of the form

$$\sum_\pi L^S(s,\pi,\sigma)\{\dots\} = \sum_\pi\{\dots\}\sum_n \frac{\text{tr}(\pi(f))}{n^s} = \sum_n \frac{1}{n^s}\sum_\pi\{\dots\}\text{tr}(\pi(f))$$

which by the stable trace formula should be equal to

$$\sum_n \frac{1}{n^s}(\text{geometric side} - (\text{residual} + \text{continuous spectra})).$$

We want to get an expression for this.

Example. $G = GL_2$, $S = S_\infty$, $\rho = \text{std}$. Suppose we wanted only holomorphic forms of weight k, then we have

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{|m|\leq 2\sqrt{n}} \theta_\infty(\frac{m}{2\sqrt{n}})L(1,m^2-4n)$$

where $\theta_\infty(x)$ is in $C_c(\mathbb{R})$ and

$$L(s,m^2-4n) = \sum_{f^2|m^2-4n} f^{1-2s} L(s, \Big(\frac{(m^2-4n)/f^2}{\cdot}\Big)$$

where the $(\frac{-}{\cdot})$ is the Kronecker symbol.

1. Where does this come from? The geometric side is a sum of orbital integrals weighted by volume factors. Roughly, the volume factor is

$$L(1,m^2-4n) = \prod_v \text{Orb}(-)\text{vol}(-)$$

, which are class numbers of quadratic fields, whereas the orbital integral is

$$\theta_\infty(\frac{m}{2\sqrt{n}} = \text{Orb}(f_\infty)$$

involving elements of trace m and determinant n.

2. Difficulty: $L(1,m^2-4n)$ requires analysis of class numbers; while not smooth, $\theta_\infty(x) = (\text{smooth})\sqrt{|x^2-1|}$ so we understand the singularity.
3. Idea: Express as something like a Dirichlet series

$$L(1,m^2-4n) = \sum_f f \sum_l \frac{1}{l}\Big(\frac{-}{l}\Big),$$

then switch the order of summation, apply Poisson summation, collect the main terms. (The main term is 0 if there are no discrete series!)

In general, the contribution of the non tempered spectrum is $\zeta(s+\frac{1}{2})\zeta(s-\frac{1}{2})$, giving a pole at $s=\frac{3}{2}$. The second term is p-adically oscillating fast.

Results. This is done for GL_2 with $\sigma = \mathrm{Sym}^1, \mathrm{Sym}^2$; for the relative trace formula we have for GL_2 with $\sigma = \mathrm{Sym}^1, \mathrm{Sym}^2, \mathrm{Asai}$

6.2 Endoscopy and Beyond

This section describes highlights from Langlands' 2004 lecture introducing the method. The principle of functoriality can be roughly stated as

1. If H and G are reductive groups over a global field F and G is quasi-split, then to each homomorphism
$$\varphi : {}^L H \to {}^L G$$
there is an associated transfer of automorphic representations of H to those of G.
2. To an automorphic representation π of G, there is an associated algebraic subgroup ${}^\lambda H_\pi$ of G. If ρ is a representation of ${}^L G$, that is,
$${}^\lambda H_\pi \hookrightarrow {}^L G \xrightarrow{\rho} GL(V)$$
then the multiplicity $m_H(\rho)$ of the trivial representation of ${}^\lambda H_\pi$ in the restriction of ρ to ${}^\lambda H_\pi$ is the order $m_\pi(\rho)$ of the L-function $L(s,\pi,\rho)$ at $s=1$.

Langlands lists the following touchstones:

a. $H = GL(2), G = GL(m+1), \phi = \mathrm{Sym}^n$ the m-th symmetric power representation. Functoriality in this case leads to the Ramanujan-Petersson conjecture and the Selberg conjecture.

b. $H = \{1\}, G = GL(2), \phi = \mathrm{Std}$ the standard representation. Then ${}^L H$ is the Galois group of F, and the problem (1) is of associating to an automorphic form two-dimensional Galois representation. Functoriality in this case leads to the Artin conjecture.

c. $G = GL(2)$, and π to be an automorphic representation such that at every infinite place v, π_v is associated two-dimensional representation of the Galois group (rather than the Weil group). Show that H_π is finite.

6.2.1. The group ${}^\lambda H_\pi$. How to define this group? Difficulties arise in trying to define it by (2):

- First, following Arthur's classification of automorphic representations of representations into those which are of Ramanujan type and those that are not. That is, representations which are expected to satisfy the Ramanujan conjecture. The contribution to the trace formula of representations not of Ramanujan type will be expressible in terms of traces of groups of lower dimension.
- Second, we do not know that for π of Ramanujan type, $L(s,\pi,\rho)$ is analytic on $\mathrm{Re}(s) \geq 1$ except for a finite number of poles on $\mathrm{Re}(s) = 1$. Moreover, why should there exist an ${}^\lambda H$ such that $m_H(\rho) = m_\pi(\rho)$ for all ρ. Even then, its conjugacy class under $\hat{G}$ may not be unique and there may be a finite number of groups ${}^\lambda H_\pi$. (But if ${}^L G = GL_n(\mathbb{C})$ then ${}^\lambda H_\pi$ is uniquely determined by $m_H(\rho)$.)

Langlands idea is to study ${}^\lambda H_\pi$ using the trace formula, given the condition $m_\pi(\rho) = \mathrm{tr}\pi(f^\rho)$ where f^ρ is some generalized function on $G(\mathbb{A}_F)$. Recalling the automorphic L-function

$$L^S(s,\pi,\rho) = \prod_{p\notin S} \det(1-\rho(A(\pi_p)p^{-s})^{-1} = \sum_{n,(n,S)=1} a_{\pi,\rho}(n)n^{-s}$$

he suggests an alternative definition of $m_\pi(\rho)$:

For $c > 0$ large enough and $X > 0$, one has from the Euler product

$$-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{L'}{L}(s,\pi,\rho)X^s\frac{ds}{s} = \frac{1}{2\pi i}\sum_p\sum_k \ln(p)\int_{c-i\infty}^{c+i\infty}\frac{\mathrm{tr}\rho(A(\pi_p)^k)}{p^{ks}}X^s\frac{ds}{s}$$

the terms for which $X < p^k$ are 0 by moving the contour to the right; the finite number of terms for which $X > p^k$ are calculated as a residue at $s = 0$. So the LHS is now

$$\sum_{p^k<X}\ln(p)\mathrm{tr}\rho(A(\pi_p)^k)$$

Now if the L-function can be continued to Re(s)≥ 1 except for a possible finite number j of poles on $\mathrm{Re}(s) = 1$ then the RHS is morally equal to

$$\sum_j \frac{m1+it_j}{1+it_j}X^{1+it_j} + o(X)$$

Then we can define

$$m_\pi(\rho) = \lim_{M\to\infty}\lim_{X\to\infty}\frac{1}{M}\int_X^{X+M}\sum_{p^k<Y}\ln(p)\mathrm{tr}\rho(A(\pi_p)^k)\frac{dY}{Y}$$

where initially $m_\pi(\rho)$ is defined as the residue of the L-function at $s = 1$. In the simple case where there is only a pole at $s = 1$ and π is of Ramanujan type, this reduces to

$$\lim_{X\to\infty}\frac{1}{X}\sum_{p<X}\ln(p)\mathrm{tr}(\rho(A(\pi_p))) \tag{*}$$

6.2.2. The role of the trace formula. For a comparison of trace formulas, one might hope to calculate

$$\sum_{{}^\lambda H_\pi}\sum_{{}^\lambda H_\pi \prec {}^\lambda H}\mu_\pi m_\pi(\rho)\prod_{v\in S}\mathrm{tr}(\pi_v(f_v)) \tag{**}$$

where ρ is an arbitrary complex representation of G, S a finite set of places of F including archimedean and places where G is quasi-split and split over an unramified extension, f_v a suitable function on G_v, μ_π the multiplicity, the outer sum over Ramanujan type representations unramified outide S, and the inner sum by a partial ordering $H' \prec H$ if $m_{H'}(\rho) \leq m_H(\rho)$ for all ρ.

For the test function, let $f_q, q \notin S$ be the unit element of the Hecke algebra at q and $f_p, p \notin S$ in the Hecke algebra such that

$$\mathrm{tr}(\pi_p(f_p)) = \mathrm{tr}(\rho(A(\pi_p)))$$

if π_p is unramified. Finally, for $v \in S$ we allow f_v to be general, and set $f^p(g) = \prod_v f_v(g_v)$. One has

$$\mathrm{tr}(\pi(f^p)) = \begin{cases}\mathrm{tr}(\pi_S(f_S))\mathrm{tr}(\rho(A(\pi_p))) & \text{if } \pi_v \text{ is unramified outside } S\\ 0 & \text{otherwise}\end{cases}$$

To compare trace formulas, one would want a transfer of test functions $f \to f^H$ from $G(\mathbb{A}_F)$ to $H(\mathbb{A}_F)$ and compare it with the analogous formula for H. For this, Langlands notes that for endoscopy the transfer is defined by a correspondence of conjugacy classes; whereas in general it will not be so simple:

- the transfer will require much more knowledge of local harmonic analysis, especially of irreducible characters,
- there will be analytic problems to overcome in taking the limit of the trace formula, and

- on the groups ${}^{\lambda}H$ which occur are now essentially arbitrary subgroups of G.

Now combining (*) and (**), one has roughly

$$\lim_{X\to\infty} \frac{1}{X} \sum_{p<X} \ln(p)\mathrm{tr}(R_\bullet(f^p)) \tag{6.1}$$

where $R_\bullet$ denotes the regular representation on $L^2(G(\mathbb{A}_F)$, and the subscript indicates that our first approximation is to take only the discrete spectrum, and eventually one wants to isolate the representations of Ramanujan type. Moreover, as p tends to infinity, the support of f^p is not compact, hence one would like a trace formula that is valid for non-compactly supported test functions. This is explored by Müller and his collaborators.

With this last expression, the trace formula once again enters into the analysis.

6.3 The test case: $\mathbf{GL}_2$

Now let's walk through the case Langlands considers: take $F = \mathbb{Q}$, $G = GL(2)$, and automorphic representations whose central character is trivial on $\mathbb{R}^+$. By multiplicity one, $\mu_\pi = 1$ for all π. For $GL(2)$, the representations in $L^2_{\mathrm{disc}}(G)$ not of Ramanujan type are the one-dimensional representations. Their traces will be subtracted from the elliptic term of the trace formula. For the terms in the trace formula we refer the reader to §2.2.

Fix $S = \{\infty\}$, which is to say we only allow ramification at infinity. The representations being unramified at finite places, the central character is then trivial; moreover the the (μ, ν) in the spectral sums reduce to the single term $\mu = \nu$. A word of caution: we remind the reader that the test function

$$f_\infty f_p^m \prod_{q\neq p} f_q$$

is not compactly supported, the latter being the usual input into the trace formula. If $\rho = \mathrm{Sym}^m$, choose $f_p = T_p^m/p^{m/2}$ where T_p^m is the characteristic function on 2 by 2 matrices over $\mathbb{Z}_p$ with $|\det X| = p^{-m}\}$, for $m \geq 0$.

Our expectation will be for Sym^n with $n = 1$, which is to say the standard representation, the relevant L-function will have no pole and hence the limit will be zero. We examine the terms:

- The first term is the identity term,

  $$\sum_{Z_\mathbb{Q}} \frac{1}{X}\mu(Z_+G_\mathbb{Q}\backslash G_\mathbb{A}) \sum_p \ln(p) f^m(z)$$

  but since $f^m(z) = f_\infty(z)p^{-m/2}$ if $z = \pm p^{m/2}$ and 0 otherwise, so its limit in (6.1) is zero.
- The orbital integrals $\omega(\gamma, f_v)$ and $\omega_1(\gamma, f_v)$ are zero unless there is an element n and k such that $k^{-1}n^{-1}\gamma nk$ lies in the support of f_v. Fixing f_∞, the eigenvalues of γ away from p must be units.
- We next consider the spectral term (vi). We consider the regular representation ξ_s on the unitarily induced representation

  $$\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \mapsto \left|\frac{a}{b}\right|^{\frac{s}{2}}$$

  and we also note that $M(0) = -I$. Thus its contribution is

  $$\frac{1}{4}\frac{1}{X}\sum_p \ln(p)\mathrm{tr}(\xi_0(f))\mathrm{tr}(\xi_0(f_p^m))$$

  Since $\mathrm{tr}(\xi_0(f_p^m)) = m+1$. Apart from the elliptic term , this will be the only nonzero contribution to the limit.

- For the spectral term (vii), the scalar factor of the intertwining operator is

$$\frac{m'}{m}(s) = -\frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{1-s}{2}) - \frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{1+s}{2}) - \frac{\zeta'}{\zeta}(1-s) - \frac{\zeta'}{\zeta}(1+s)$$

  multiplied by $\mathrm{tr}(\xi_s(f))\mathrm{tr}(\xi_s(f_p^m))$. The first factor is the Fourier transform of a smooth function of compact support; the second is equal to $p^{ims/2} + p^{i(m-2)s/2} + \cdots + p^{i(2-m)s/2} + p^{-ims/2}$. Since $m'(s)m^{-1}(s)\mathrm{tr}(\xi_s(f))$ is L^1 on the imaginary axis, the Riemann-Lebsegue lemma ensures that if m is odd then the limit tends to zero as p tends to infinity.

- The spectral term (viii) is zero at finite places because our f_q is spherical for all q. We are left to consider

$$\frac{1}{4\pi}\int_{-i\infty}^{i\infty} \mathrm{tr}(R^{-1}(s)R'(s)\xi_s(f_\infty)) \sum_{k=0}^{m} p^{i(m-2k)s} d|s|$$

  Then by an estimate of Arthur,

$$|\mathrm{tr}(R^{-1}(s)R'(s)\xi_s(f_\infty))| = O(s^{-2})$$

  as s tends to infinty, then the Riemann-Lebesgue lemma applies again so that when m is odd, the contribution is zero.

- For even m, the formulas are somewhat complicated. We outline the ideas briefly. One must consider

$$\frac{1}{4\pi}\int_{-i\infty}^{i\infty} \frac{m'}{m}(s)\mathrm{tr}(\xi_s(f_\infty)) + \mathrm{tr}(R^{-1}(s)R'(s)\xi_s(f_\infty))\ d|s|$$

  For the scalar factor, one splits off the terms involving $\zeta(s)$ and integrates the logarithmic derivatives term by term. From the pole at $s = 0$ one also picks up a contribution $-\mathrm{tr}(\xi_s(f_\infty))/2$. The total contribution here is a sum of atomic measures that ought to cancel with the elliptic part.

  The remaining logarithmic derivatives are to be combined with (iv) and (v) to obtain invariant distributions.

- For the geometric term (iv), one shows that only the terms for $v = \infty, p, 2$ are nonzero. The ∞ term is combined with the logarithmic derivative of the intertwining operator to form an invariant distribution. The other two are already invariant, and unbounded in p. Its appearance is apparently unexpected and ought to cancel with an elliptic term.

- For the geometric term (v), we also summarize: one shows again that the terms for $v = \infty, p$ are nonzero; the first is combined with the logarithmic derivative again, while the second cancels with part of the unexpected term from (iv).

- We are left with the elliptic term, (ii) the most difficult part of the analysis. We now turn to this in earnest.

6.4 The elliptic terms

We examine the elliptic terms more closely; it is a sum over global regular elliptic elements, i.e., whose characteristic polynomial is irreducible over $\mathbb{Q}$. Take an element γ of $GL_2(\mathbb{Q})$, denote by r its trace and $\frac{N}{4}$ its determinant. Its characteristic polynomial is $X^2 - rX + N/4$, so only γ such that r is integral and, by the choice of test function f^p, such that $N = \pm 4p^m$ appear. The usual quadratic polynomial gives the eigenvalues of γ:

$$\frac{r}{2} \pm \frac{\sqrt{r^2 - N}}{2}$$

Their difference is $\pm\sqrt{r^2 - N}$, thus γ will be elliptic if and only if $r^2 - N$ is square. Given an integral binary qudratic form, we may write $r^2 - N = s^2 D$ where s is a positive integer and D is a fundamental discriminant of $\mathbb{Q}(\sqrt{D})$, i.e., $D \equiv 0, 1 \pmod 4$. If $r^2 = N$, then take $D = 0$; if it is square then take $D = 1$.

Furthermore, at a real completion we may factor $\gamma = z\delta u$ where z is central with entries $|\det \delta|^{1/2}$, δ is an element of $SL_2(\mathbb{R})$, and $u = \mathrm{diag}(\mathrm{sgn}(\det \gamma), 1)$.

The elliptic orbital integral begins as

$$\sum_{\gamma \text{ ell}} \mathrm{vol}(\gamma) \prod_v \mathbf{O}_\gamma(f_v)$$

where for short we have written vol(γ) for the measure of $Z_+ G_{\mathbb{Q}} \backslash G_{\mathbb{A}}$. First we recall Langlands' preliminary setup: Warning: in the following there is a typo–you may want to assume $k = m$ where it seems right!

6.4.1. Preliminaries: Langlands' analysis Let $E = \mathbb{Q}(\sqrt{D})$ be a imaginary quadratic field such that $D \neq -2, -3$. Then any element of $G_\gamma(\mathbb{R})$ has eigenvalues $\sigma e^{\pm i\theta}$. Then the volume of $Z_+ G_\gamma(\mathbb{Q}) \backslash G_\gamma(\mathbb{A}) = Z_+ E^\times \backslash \mathbb{A}_E^\times$ is equal to $2\pi h_\gamma / \omega_\gamma$ where h_γ is the class number of E and ω_γ the number of roots of unity of E. If E is real quadratic, then the measure is $2h_\gamma R_\gamma$ where R_γ is the regulator. Then by the class number formula we have the more elegant formula

$$\mathrm{vol}(\gamma) = \sqrt{|D|} L(1, \left(\frac{D}{\cdot}\right))$$

We sketch the following proof of Langlands

Lemma 6.4.1. *The integral*

$$\int_{G_\gamma(\mathbb{Q}_p) \backslash G(\mathbb{Q}_p)} T_p^m(g^{-1}\gamma g) dg = \begin{cases} p^k = |\gamma_1 - \gamma_2|^{-1} & \gamma \text{ is split} \\ \frac{p^k(p+1)}{p-1} - \frac{2}{p-1} & \gamma \text{ is not split, } E_\gamma \text{ is unramified} \\ \frac{p^{k+1}}{p-1} - \frac{1}{p-1} & \gamma \text{ is not split, } E - \gamma \text{ is ramified.} \end{cases}$$

is 0 unless the eigenvalues γ_1, γ_2 *of* γ *are integral and* $|\gamma_1 \gamma_2| = p^{-m}$.

Proof. We prove this by lattice counting. The characteristic function $T_p^m(g^{-1}\gamma g)$ is 1 if and only if $g^{-1}\gamma g$ takes the lattice $L_0 = \mathbb{Z}_p \oplus \mathbb{Z}_p$ onto itself (hence γ stabilizes the lattice gL_0), and has determinant with absolute value p^{-m}. Knowing L is equivalent to knowing g modulo $G(\mathbb{Z}_p)$ on the right; multiplying g by an element of $G_\gamma(\mathbb{Q}_p) = E_\gamma \otimes \mathbb{Q}_p^\times$ is equivalent to multiplying L by the same.

If E_p is split, then we can normalize L up to multiplication by requiring

$$L \cap \{(0, z) | z \in \mathbb{Q}_p\} = \{(0, z) | z \in \mathbb{Q}_p\}$$

and that its projection on to the first factor is $\mathbb{Z}_p$. Then the x such that $(1, x)$ lie in L are determined modulo $\mathbb{Z}_p$ by L. Multiplying by diag(a, b) where $|a| = |b| = 1$, we may replace $(1, x)$ with $(1, bx/a)$ so we can consider only the absolute value $|x|$.

The measure in $G_\gamma(\mathbb{Q}_p) \backslash G(\mathbb{Q}_p)$ of the set of g defining L is equal to the index in $G_\gamma(\mathbb{Z}_p)$ of the stabilizer of L. This is the number of y mod $\mathbb{Z}_p$ such that $|y| = |x|$ or if $|y| \leq 1$ if $|x| \leq 1$, thus the number of lattices that can be obtained from a given one by multiplying with an element of $G_\gamma(\mathbb{Z}_p)$. The condition that L be fixed by γ asks that γ_1, γ_2 be integral and that

$$(\gamma_1, \gamma_2 x) = \gamma_1(1, x) + (0, (\gamma_2 - \gamma_1)x)$$

lie in L, hence that $(\gamma_1, \gamma_2 x)$ be integral.

The arguments for the rest are similar, considering the quotient $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ with lattices in E_p. □

Now, we want to show that

Lemma 6.4.2. *For $i = m$ at p, and $i = 1$ for all $q \neq p$, one has*

$$\prod_q \int_{G_\gamma(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} T_p^i(g^{-1}\gamma g)dg = \sum_{f|s} f \prod_{q|f} (1 - \frac{1}{q}\Big(\frac{D}{q}\Big))$$

where $(\frac{D}{q})$ is the Kronecker symbol

Recall that for a number $n = up_1^{e_1} \dots p_k^{e_k}$, the Kronecker symbol is defined as

$$\Big(\frac{D}{q}\Big) = \Big(\frac{D}{u}\Big) \prod_{i=1}^k \Big(\frac{D}{p_i}\Big)^{e_i}$$

when p_i is odd, it is the Legendre symbol which gives $1, -1, 0$ depending on whether D is a nonzero quadratic residue mod p, a quadratic non-residue mod p, or 0 mod p. If p_i is even, define it to be 0, 1, -1, if D is even, ± 1 mod 8, or ± 3 mod 8. For the unit u, we define it to be $1, -1$ if D is nonnegative or negative.

Proof. By multiplicativity it is enough to consider for each prime q,

$$1 + \sum_{j=1}^k q^j (1 - \frac{1}{q}\Big(\frac{D}{q}\Big))$$

If $(\frac{D}{q}) = 1$, this is $1 + q^k - 1$.

If $(\frac{D}{q}) = -1$, this is

$$1 + (q^{k-1})\frac{q+1}{q-1} = q^k \frac{q+1}{q-1} - \frac{2}{pq-1}$$

If $(\frac{D}{q}) = 0$, this is

$$\sum_{j=1}^k q^j = \frac{q^{k+1}}{q-1} - \frac{1}{q-1}$$

then the result follows from the preceding lemma. □

Then we have for $\det \gamma = \pm p^k$, the expression

$$\mathrm{vol}(\gamma)\mathbf{O}_\gamma(f^p) = \frac{1}{p^{k/2}}\mathbf{O}_\gamma(f_\infty)\sqrt{|D|}L(1, \Big(\frac{D}{\cdot}\Big))\Big\{\sum_{f|s} f \prod_{q|f}(1 - \frac{1}{q}\Big(\frac{D}{q}\Big))\Big\}$$

Finally, changing variables $f \mapsto s/f$ and rearranging, we arrive at

$$\frac{1}{p^{k/2}}\mathbf{O}_\gamma(f_\infty)\sqrt{|D|}\sum_{f|s}\frac{1}{f}L(1, \Big(\frac{D/f^2}{\cdot}\Big))$$

where here again $D = r^2 - N$. We now examine the archimedean orbital integral in greater detail.

6.4.2. Singularities of archimedean orbital integrals. Assume for the moment γ is regular semisimple, and f_∞ in $C_c^\infty(Z_+\backslash G(\mathbb{R})$ as before. Then

$$\mathbf{O}_\gamma(f_\infty) = g_1(\gamma) + g_2(\gamma)\frac{|\gamma_1\gamma_2|^{\frac{1}{2}}}{|\gamma_1 - \gamma_2|}$$

where g_1, g_2 are smooth class functions depending on f and z, compactly supported on a Weyl group invariant neighbourhood of z. In particular, g_1 is supported on the elliptic torus. We see that $(\gamma_1 - \gamma_2)^2/\gamma_1\gamma_2 = 4(r^2/N - 1)$ by definition, so we may write

$$g_1(r, N) + g_2(r, N)\frac{1}{2}\left|\frac{r^2}{N} - 1\right|^{-\frac{1}{2}}$$

Moreover, since f_∞ is assumed Z_+ invariant, we see that $g_i(ar, a^2N) = g_i(r, N)$ for any $a > 0$. Thus taking $a = \sqrt{|N|}$ we see that the g_i depend only on the ratio $r/\sqrt{|N|}$, and the sign of N. We thus write

$$g_1^\pm(\frac{r}{\sqrt{|N|}}) + g_2^\pm(\frac{r}{\sqrt{|N|}})\frac{1}{2}\left|\frac{r^2}{N} - 1\right|^{-\frac{1}{2}}$$

where $g_i^\pm(x) = g_i(\pm 1, x)$. Note that if N is negative, the torus G_γ is split and hence g_1 vanishes.

Recall that only γ for which $N = 4\det\gamma = \pm 4p^k$ give a nonzero contribution to the elliptic terms, therefore $\sqrt{|N|}/2 = p^{k/2}$, and

$$\frac{1}{p^{k/2}}\mathbf{O}_\gamma(f_\infty)\sqrt{D} = 2|\frac{r^2}{N} - 1|^{\frac{1}{2}}g_1^\pm(\frac{r}{\sqrt{|N|}}) + g_2^\pm(\frac{r}{\sqrt{|N|}})\frac{1}{2} := \theta_\infty(\frac{r}{\sqrt{|N|}})$$

Moreover, since we require also that $r^2 - N$ be nonsquare, we finally have the elliptic

$$\sum_{\gamma \text{ ell}} \text{vol}(\gamma)\prod_v \mathbf{O}_\gamma(f_v) = \sum_\pm\sum_r \theta_\infty^\pm(\frac{r}{2p^{k/2}})\sum_f \frac{1}{f}\, L(1, \Big(\frac{(r^2 \pm 4p^k)/f^2}{\cdot}\Big)) \tag{6.2}$$

where the sum r is taken over $r \in \mathbb{Z}$ such that $r^2 \pm 4p^k$ is not square, and the sum over f is over square divisors $f^2|m^2 \pm 4p^k$ such that $(m^2 \pm 4p^k)/f^2 \equiv 0, 1 \bmod 4$, i.e., is a discriminant. Finally, we are in a position analyze the elliptic terms in the trace formula, which follows the work of Altuğ.

6.4.3. Approximate functional equation: Altuğ's analysis We treat the inner sum over f. Define

$$L(s, D) := \sum_{f^2|D}{}' f^{1-2s}L(s, \Big(\frac{Df^{-2}}{\cdot}\Big)), \quad \Lambda(s, D) = (\frac{|D|}{\pi})^{\frac{s}{2}}\Gamma(\frac{s + \iota_D}{2})L(s, D)$$

where the $'$ indicates summation over f such that $D/f^2 \equiv 0, 1 \bmod 4$, and ι_D is one if $D > 0$ and zero if $D < 0$, we have the functional equation $\Lambda(s, D) = \Lambda(1 - s, D)$. Now define

$$F(x) = \frac{1}{2K_0(2)}\int_x^\infty e^{-y-\frac{1}{y}}\frac{dy}{y}$$

where $K_s(z)$ denotes the s-th modified Bessel function of the second kind. Then one has the approximate functional equation

$$L(s, D) = \sum_{f^2|D}{}' f^{1-2s}\sum_{l=1}^\infty l^{-s}\Big(\frac{D/f^2}{\cdot}\Big)F(\frac{lf^2}{A}) + (\frac{|D|}{\pi})^{\frac{1}{2}-s}\sum_{f^2|D}{}' f^{2s-1}\sum_{l=1}^\infty l^{s-1}\Big(\frac{D/f^2}{\cdot}\Big)H_{D,s}(\frac{lf^2A}{|D|})$$

where

$$H_{D,s}(y) = \frac{1}{2\pi i}\int_{1-i\infty}^{1-i\infty} \pi^{s-1/2}\frac{\Gamma(\frac{1+u-s+\iota_D}{2})}{\Gamma(\frac{s-u+\iota_D}{2})}(\pi y)^{-u}\tilde{F}(u)\, du.$$

for some $A > 0$, and $\tilde{F}(u)$ is the Mellin transform of $F(x)$. The idea is to consider the integral

$$\frac{1}{2\pi i}\int_{(\sigma)} L(s + u, D)\tilde{F}(u)A^u du$$

for σ large enough so that the integral and sum are absolutely convergent. Interchange the sum and integral, and apply Mellin inversion to obtain the first term. Then shifting the contour to $\sigma < 0$ we pick up the pole at u giving $L(s, D)$, then using the functional equation on the integral we obtain the second term.

We will be interested in

$$L(1,D) = \sum_{f^2|D}' f^{1-2s} \sum_{l=1}^{\infty} l^{-s} \Big(\frac{D/f^2}{\cdot}\Big) \Big[F(\frac{lf^2}{A}) + \frac{lf^2}{|D|^{1/2}} H_{D,1}(\frac{lf^2 A}{|D|})\Big]$$

which we substitute into the final form (6.2) of the last section.

6.4.4. Poisson summation Now for the outer sum. The issues with applying Poisson directly are the following: (1) the m sum does not run over all $\mathbb{Z}$, and adding in the missing values gives poles in the $L(1,-)$ term, (2) the sum over $L(1,-)$ does not converge absolutely, so interchanging sums is not immediate, and (3) the functions $\theta_\infty^\pm$ are singular, not smooth.

The approximate functional equation resolves (1) and (2), giving an absolutely convergent sum. To address (3), a judicious choice of A will smooth $\theta_\infty^\pm$. Let $a > 0$ and $\Phi(x)$ be a Schwartz function on $\mathbb{R}$. Then the functions

$$\theta_\infty^\pm \Phi(|1-x^2|^{-a}), \qquad \frac{1}{\sqrt{|1-x^2|}} \theta_\infty^\pm \Phi(|1-x^2|^{-a})$$

are smooth. By definition of $\theta_\infty^\pm$ we see that the only thing to check is $x = 1, -1$. Then one shows that the left and right derivatives at these points are 0, and also all higher derivatives.

Now then, choosing $A = |m^2 \pm 4p^k|^a$ where $0 < a < 1$, one sees that the expression in brackets above is smooth, hence Poisson summation can be applied. Adding the square terms to (6.2) we have by Poisson summation

$$\frac{p^{k/2}}{2} \sum_\pm \sum_{f=1}^{\infty} f^{-3} \sum_{l=1}^{\infty} l^{-2} \sum_{\xi \in \mathbb{Z}} \mathrm{Kl}_{l,f}(x, \pm p^k) \int \theta^\pm(x) [F(\frac{lf^2(4p^k)^{-a}}{|x^2 \mp 1|^a}) + \frac{lfp^{-k/2}}{2\sqrt{|x^2 \mp 1|}} H(\frac{lf^2(4p^k)^{a-1}}{|x^2 \mp 1|^{1-a}})] e(\frac{-x\xi p^{k/2}}{2lf^2}) dx$$

where H is H_0 or H_1 depending if $x^2 \mp 1$ is positive or negative, and Kl is the Kloosterman sum

$$\mathrm{Kl}_{l,f}(\xi, \pm p^k) = \sum \Big(\frac{(a^2 \mp 4p^k)/f^2}{l}\Big) e(\frac{a\xi}{4lf^2})$$

with the sum running a mod $4lf^2$ such that $a^2 \mp 4p^k \equiv 0$ mod f^2 and $(a^2 \mp 4p^k)/f^2 \equiv 0, 1$ mod 4. To get this, one first interchanges the l and f summation; since the assumptions imply that the sum over m is periodic mod $4lf^2$, one breaks the sum into arithmetic progressions mod $4lf^2$, then apply Poisson summation to obtain the expression.

With this in mind, define the auxiliary Dirichlet series

$$D(s,n) = \sum_{f=1}^{\infty} f^{-2s-1} \sum_{l=1}^{\infty} l^{-s-1} \mathrm{Kl}_{l,f}(0,n) = 4\frac{\zeta(2s)}{\zeta(1+s)} \prod_{p|n} \frac{1 - p^{-s(v_p(n)+1)}}{1 - p^{-s}},$$

where $v_p(n)$ is the p-adic valuation of n. A lot of computation proves the last equality.

6.4.5. Contribution of special representations We want to isolate the contribution of the trivial representation and the contribution of the residue of Eisenstein series to the dominant term $\xi = 0$ in the Poisson sum. Altug shows that this term is equal to the sum of

$$2p^{\frac{k}{2}} \frac{1 - p^{-k-1}}{1 - p^{-1}} \sum_\pm \int_{-\infty}^{\infty} \theta_\infty^\pm(x) dx - (k+1) \sum_\pm \int_{x^2 \mp 1 > 0} \frac{\theta_\infty^\pm(x)}{\sqrt{|x^2 \mp 1|}} dx + p^{\frac{k}{2}} \sum_\pm \int_{-\infty}^{\infty} \theta_\infty^\pm(x) \frac{1}{2\pi i} \Big[\cdots\Big] dx$$

where the expression in brackets is the sum of

$$2\int_{(-1)}\tilde{F}(u)(\frac{|x^2\mp 1|^a}{(4p^k)^{-a}})^u\frac{\zeta(2u+2)}{\zeta(u+2)}\frac{1-p^{-(u+1)(k+1)}}{1-p^{-u-1}}du$$

and

$$\Big(\frac{\pi p^k}{|x^2\mp 1|}\Big)^{\frac{1}{2}}\int_C\tilde{F}(u)\frac{\Gamma(\frac{\iota_{x^2\mp 1}+u}{2})}{\Gamma(\frac{\iota_{x^2\mp 1}+1-u}{2})}(\frac{|x^2\mp 1|^{1-a}}{\pi(4p^k)^{a-1}})^u\frac{\zeta(2u)}{\zeta(u+1)}\frac{1-p^{-u(k+1)}}{1-p^{-u}}du$$

where C runs along the imaginary axis, tracing a small semicircle to the left of 0 within the zero-free region. This expression is obtained by considering the cases where $x^2\mp 1$ is positive or negative, then interchanging sums and shifting contours.

Finally, Altuğ shows that

$$\operatorname{tr}(\mathbf{1}(f_p^k)) = 2p^{\frac{k}{2}}\frac{1-p^{-k-1}}{1-p^{-1}}\sum_{\pm}\int_{-\infty}^{\infty}\theta_\infty^\pm(x)dx,$$

which is the first term in the dominant term, and

$$\operatorname{tr}(\xi_0(f_p^k)) = \frac{1}{2}(k+1)\sum_{\pm}\int_{x^2\mp 1>0}\frac{\theta_\infty^\pm(x)}{\sqrt{|x^2\mp 1|}}dx$$

which is $-\frac{1}{2}$ times the second term.

6.5 Geometrization: Frenkel-Langlands-Ngô

In this section we summarize the paper of Frenkel-Langlands-Ngô, which takes a geometric approach to the Poisson summation formula as utilized above.

6.5.1. Stability. Arthur proposed a classification of automorphic forms, contingent upon functoriality. To an automorphic representation π_G one can associate a homomorphism $\phi_\pi : SL_2 \to {}^LG$, together with the conjugacy class $\{A_G(\pi_v)\}$ in the connected centralizer ${}^\lambda G_{\phi,v} = \operatorname{Cent}_{{}^LG_v}(\phi(SL_2))$ at almost every place v, and a homomorphism of the local Weil group $W_{F_v} \to {}^\lambda G_{\phi,v}$. One also constructs a homomorphism of the automorphic Galois group $\xi : L_F \to {}^\lambda G_\phi$, into the global centralizer generated by the local classes.

More useful are the groups ${}^\lambda H_\pi$ defined as the Zariski closure of the image of ξ. One looks for a reductive group H with a surjective homomophism $\psi : {}^LH \to {}^\lambda H \subset {}^\lambda G_\phi$ and central kernel, and a representation π_H such that $\{A_G(\pi_v) = \{\psi(A_H(\pi_{H,v})\}$. We will call π_H Ramanujan if $\{A_H(\pi_{H,v})\}$ are unitary. For such classes, the eigenvalues of $\rho_H\{A(\pi_{H_v})\}$ have absolute value 1 for any representation ρ_H of H, and $L(s,\pi_H,\rho_H)$ is holomorphic for $\operatorname{Re}(s)>1$, and has finitely many possible poles and no zeroes on $\operatorname{Re}(s)=1$.

The representations ρ_H and $\phi\times\psi$ define a representation of the product $SL_2\times {}^LH$, decomposing into a direct sum $\bigoplus \sigma_j\oplus\rho_H^j$ where each σ_j is and irreducible m_j+1-dimensional representation of SL_2. Then one has

$$L_S(s,\pi,\rho) = \prod_j\prod_i L_S(s+i,\pi_H,\rho_j).$$

For ϕ_π trivial, one expects the related π_H to be Ramanujan.

Consider the sum

$$\sum_\pi\prod_{v\in S}\operatorname{tr}(\pi_v(f_v))L_S(s,\pi_G,\rho_G)$$

where the functions f_v are smooth, compactly supported, and π is unramified outside S. This should allow us to isolate (stable) classes of π_G. There remains the possibility having several H associate to π which are isomorphic but not conjugate, seemingly related to the failure of multiplicity one.

We want to discard the different H in two steps: first, subtract the contribution of the pairs $\phi \times \psi$ for which ϕ is nontrivial, which should leave behind the representations of Ramanujan type; second, isolate the remaining representations for which $L(s, \pi, \rho)$ has a pole at $s = 1$. The previous expression (**) is modified in terms of stable L-packets π^{st} of Ramanujan type,

$$\sum_{\pi^{\mathrm{st}}} \sum_{{}^\lambda H_\pi \prec {}^\lambda H} \mu(\pi^{\mathrm{st}}) m(\pi^{\mathrm{st}}) \prod_{v \in S} \mathrm{tr}(\pi_v^{\mathrm{st}}(f_v))$$

where $m(\pi^{\mathrm{st}})$ is the stable multiplicity, which has been defined only in certain cases, and is possibly fractional. Recall that $L_S(s, \pi, \rho)$ is independent of π in the L-packet.

If ρ is irreducible and nontrivial, the multiplicity is nonzero only for π with ${}^\lambda H_\pi \neq {}^L G$, corresponding to H of lower dimension than G. Denoting by $\mu(\rho_H)$ the multiplicity of the trivial representation in the restriction ρ_H, the expression becomes

$$\sum_H \mu(\rho_H) \sum_{\pi_H^{\mathrm{st}}} m(\pi_H^{\mathrm{st}}) \prod_{v \in S} \mathrm{tr}(\pi_v^{\mathrm{st}}(f_v))$$

the first sum taken over groups H attached to ${}^\lambda H$.

Now, here is the strategy suggested by FLN. Define the Hecke operator $K_v^{\rho,(n)}$ such that

$$\mathrm{tr}(\pi_v(K_v^{\rho,(n)})) = \mathrm{tr}(\rho^{(n)}(A(\pi_v)))$$

where $\rho^{(n)}$ denotes the n-th symmetric power of ρ. Set

$$\mathbf{L}_v(s, \rho) = \sum_{n=0}^{\infty} q_v^{-ns} K_v^{\rho,(n)}$$

so that $\mathrm{tr}(\pi_v(\mathbf{L_v}(s, \rho))) = L_v(s, \pi_v, \rho)$ when π_v is unramified, and zero elsewhere. An irreducible representation π_v is called unramified if the group G_v is quasiplit over F_v and split over an unramified extension, and if it contains a nonzero vector fixed by a hyperspecial subgroup. Call a packet π_v^{st} unramified if every element is unramified for a given hyperspecial subgroup. The subgroups $G(\mathfrak{o}_v)$ are hyperspecial for almost all v. Choose S such that this is so outside of S, making the above two equations valid outside of S, but for a single element in the packet. This element and the Hecke algebra are fixed by $G(\mathfrak{o}_v)$, while for the other elements, $\mathrm{tr}(\pi_v(K_v^{\rho,(n)})) = 0$. Then the factorization

$$L_S(s, \pi, \rho) \prod_{v \in S} \mathrm{tr}(\pi_v^{\mathrm{st}}(f_v)) = \prod_{v \notin S} \mathrm{tr}(\pi_v(\mathbf{L}_v(s, \rho))) \prod_{v \in S} \mathrm{tr}(\pi_v^{\mathrm{st}}(f_v))$$

implies that the sum

$$\sum_{\pi^{\mathrm{st}}} m(\pi^{\mathrm{st}}) L_S(s, \pi, \rho) \prod_{v \in S} \mathrm{tr}(\pi_v^{\mathrm{st}}(f_v))$$

is given by the stable trace formula for the function $f = (\otimes_S \mathbf{L}(s, \rho)) \bigotimes (\otimes^S f_v)$.

6.5.2. The Steinberg-Hitchin base. Let G be a split, simply connected semisimple group over F. Recall from the Fundamental Lemma the Chevalley isomorphism $F[\mathfrak{g}]^G \to F[\mathfrak{t}]^W$ and the characteristic map $\mathfrak{g} \to \mathfrak{c} = \mathrm{Spec}(F[\mathfrak{t}]^W)$. We have the corresponding *Steinberg map* on groups

$$G \to T/W \simeq \mathbb{A}^n$$

where $n = \mathrm{rank}\, G$. Since G is semisimple, its derived group G^{der} is quasisplit, and assume further that G^{der} is simply connected. Let T be a maximal split torus in G, and define $T^{\mathrm{der}} = T \cap G^{\mathrm{der}}$ and W the Weyl group of G relative to T . Define the *Steinberg quotient* $A^{\mathrm{der}} = T^{\mathrm{der}}/W$ for G^{der}, isomorphic to $\mathbf{A}^{n-1}$.

Next, let Z be the neutral component of the centre of G. One has

$$1 \to A := G_{\text{der}} \cap Z \to G_{\text{der}} \times Z \to G \to 1.$$

The principal part of the trace formula is the sum over (stable) regular elliptic conjugacy classes. Certain cohomological complications aside, they can be described as products of elements of Z_F and classes in G_{der}.

For $G = GL_2$, then $G^{\text{der}} = SL_2$, $G/G^{\text{der}} = \mathbb{G}_m$, then the Steinberg-Hitchin base $\mathfrak{A}$ is two dimensional, and equal to $\mathbf{A} \times \mathbb{G}_m$; the map $\mathfrak{c}$ to it from G is given by $t \mapsto (a, b)$ where the characteristic polynomial of t is $x^2 - bx + a$. The parameter η is a choice of a modulo the set of powers x^2 in $F^\times$.

Now let G be split, semisimple, and simply connected, with a torus T. Let $\alpha_1, \ldots, \alpha_r$ be its simple roots with fundamental weights $\mu_1, \ldots, \mu_r$ defined by $\mu_i(\hat{\alpha}_j) = \delta_{ij}$. Let ρ_i be representations with highest weight μ_i, and call $b_i = \text{tr}(\rho_i)$. The b_i are algebraically independent over F, and $T/W \simeq \text{Spec}F[b_1, \ldots, b_r]$. When $G = G_{\text{der}}$, the base $\mathfrak{A}$ is equal to its linear part, $\mathfrak{B}$.

A simply connected, semisimple quasisplit group G_φ is defined by a $\text{Gal}(\bar{F}/F) := \Gamma_F$ cocycle φ with values in the automorphism group of the Dynkin diagram of G. Thus one as a right Γ_F action on the fundamental representations $\sigma(\rho_i(g)) = \rho_i(\varphi(\sigma)(g))$. Moreover, there is a Galois action on $G_{\bar{F}}$ by $\sigma_\varphi(g) = \sigma(\varphi(\sigma)(g))$, then if h_σ is a cocycle with values in the centre of $G_{\bar{F}}$ then $\rho_i(h_\sigma)$ is a scalar in $\bar{F}$. Consequently, we are able to twist the Steinberg-Hitchin base by a cocycle.

Now we pass to the general case where G_{der} is simply connected. Assume the characteristic fo F is prime to $|A|$. There is the a decomposition $G_{\bar{F}} = G_{\text{der}}(\bar{F})Z_{\bar{F}}$ such that $\sigma(g_1)^{-1}g_1 = \sigma(g_2)^{-1}g_2$ in $A_{\bar{F}}$, and the image $\eta(g)$ of the cocycle $\sigma g_1{}^{-1}g_1$ in $H^1(F, A)$ is well defined. Let $\mathfrak{h}$ be the subset of $H^1(F, A)$ obtained in this way.

We want to describe the F-points of $\mathfrak{A}$ of the Steinberg-Hitchin base G as a sum over η in $\mathfrak{h}$ (or $H^1(F, A)$) of rational points of

$$(\mathfrak{B}_\eta(F) \times Z_\eta(F)))/A(F)$$

where $(\mathfrak{B}_\eta(F)$ is a linear space and twisted form of a linear space of the base of G_{der}, of which A_F is in the centre of G_{der}; and Z_η is a Z-torsor.

6.5.3. The regular elliptic terms. First, note that the centralizer of a regular elliptic element is a connected torus. Assume for simplicity that $G = G_{\text{der}}$. Then the elliptic sum is

$$\sum_{\gamma \text{ ell}} \sum_{z \in \mathbb{Z}^s G} \int_{G^+_{\gamma,F} \backslash G_{\mathbb{A}}} f(g^{-1}z\gamma g)dg = \sum_{\gamma} \sum_{z} \text{meas}(T^+_{\gamma,F} \backslash T_{\gamma,\mathbb{A}}) \int_{T_{\mathbb{A}} \backslash G_{\mathbb{A}}} f(g^{-1}\gamma g)dg$$

There is an open dense subset $\mathfrak{A}^{\text{rs}}$, such its preimages in G have tori as centralizers. Now, our objective is to express

$$\int_{G(F_v)} f_v(g_v)dg_v = \int_{G^{\text{rs}}(F_v)} f_v(g_v)dg_v$$

as an integral along the fibers of $\mathbf{c}$. Namely, one has the compactly supported function

$$\theta_v(a, s) = \int_{\mathbf{c}^{-1}(a)} f_v(g)|\omega_a|_v = L_v(s, \sigma_{T/G})|\Delta(t_v)|\mathbf{O}(t_{\text{st}}, f), \quad s \geq 1, \mathbf{c}(t) = a.$$

where $\Delta(t) = \pm t^{-\rho} \prod_{\xi > 0}(\xi(t) - 1)$, and $|\Delta(t)| = 1$ for regular semisimple classes. The function is bounded, but not necessarily smooth, while its Fourier transform does not have compact support, so that the product over all v is not convergent. Then we have

$$\sum_{\gamma} L(1, \sigma_{T/G} \prod_{v} \mathbf{O}(\gamma_{\text{st}}, f_v) = \lim_{s \to 1} \sum_{\gamma} \theta(\mathbf{c}(\gamma_{\text{st}}), s)$$

Observe that $\theta_v(a, s)$ is a variant of the usual stable orbital integral, with an added L-factor, so the global function $\theta(a, s)$ is valid for $\text{Re}(s)$ large and all a in $\mathfrak{A}_{\mathbb{A}}$. Moreover, when $s = 1$ it is equal to the stable orbital integral, if it converges.

The issue is that $\theta(a,s)$ is not smooth, and one applies a truncation technique of Getz. For a particular choice of places S, one can find

$$\theta(\gamma_{\mathrm{st}},s) = L_S(s,\sigma_{T/G})) \prod_{v\notin S} q_v^{-\dim G_{\mathrm{der}}} |G_{\mathrm{der}}|(\kappa_v) \prod_{v\in S} \theta_v(\gamma_{\mathrm{st}},s)$$

where κ_v is the residue field, and apply Poisson summation to the truncated sum

$$\sum_{\mathfrak{B}_S} \prod_{v\notin S} (\dots) \prod_{v\in S} \theta_v(\gamma_{\mathrm{st}},s).$$

We conclude by examining the dominant term

$$\hat{\theta}(0) = \lim_{S\to\infty} \lim_{s\to 1} \hat{\theta}_S(0,s)$$

in the case where $G = G_{\mathrm{der}}$, quasiplit at all places. In this case the only one dimensional representation is the trivial one. The dominant term for a representation ρ of ${}^LG$ for our chosen test function is

$$\mathrm{tr}(\mathbf{1}(f)) = \prod_i \zeta_S(s+i) \prod_S \mathrm{tr}(\pi_v(f_v)) = \int_{G_{\mathbb{A}}} f(g)dg = \prod_v \int_{\mathfrak{A}_{F_v}} \theta_v(a_v,1)da_v$$

where i is given by the embedding of SL_2 in ${}^LG$ attached to a principal unipotent element. More generally, the last term can be expressed as

$$\int_{G_{F_v}} \frac{L_v(s,\sigma_{T/G})}{L_v(1,\sigma_{T/G})} f_v(g_v)dg_v = \int_{\mathfrak{A}_{F_v}} \theta_v(a_v,s)da_v = \hat{\theta}(0,s)$$

Lemma 6.5.1. *Let v be a place where G has reductive reduction. The morphism $\mathbf{c}: G \to \mathfrak{A}$ extends to $\mathfrak{o}_v$. Suppose the volume forms $\omega_G, \omega_{\mathfrak{A}}$ have nonzero extension to the $\mathfrak{o}_v$ model, and f_v the characteristic function on $G_{\mathfrak{o}_v}$. Then for all a_v in $\mathfrak{A}^{rs}_{\mathfrak{o}_v}$, we have*

$$\theta_v(a_v,1) = q_v^{-\dim G + \dim T} \frac{|G(k_v)|}{|T(k_v)|} = L_v(1,\sigma_{T/G}) \frac{|G_{der}(k_v)|}{q^{\dim G_{der}}}$$

where T is the centralizer of a section of g_v in $G^{rs}_{\mathfrak{o}_v}$ over b_v.

Using this, one proves the equality

$$\hat{\theta}(0) = \lim_{S\to\infty} \lim_{s\to 1} \hat{\theta}_S(0,s) = \int_{G_{\mathbb{A}}} f(g)dg$$

The infinite product $\prod \int f_v(g_v)dg_v$ converges. For each v,

$$\lim_{s\to 1} \hat{\theta}_v(0,s) = \int_{G_{F_v}} f_v(g_v)dg_v$$

and it remains to show that for a finite set S',

$$\lim_{S\to\infty} \lim_{s\to 1} \{ \prod_{S\setminus S'} \hat{\theta}_v(0,s) - \prod_{S\setminus S'} \int_{G_{F_v}} f_v(g_v)dg_v \} = 0$$

In particular, one chooses S' such that

$$\prod_{S\setminus S'} \int_{G_{F_v}} f_v(g_v)dg_v = \prod_{S\setminus S'} \frac{|G_{\mathrm{der}}(k_v)|}{q^{\dim G_{\mathrm{der}}}} = \prod_{S\setminus S'} (1 + O(q_v^{-2}))$$

which converges absolutely, and lastly one uses geometric considerations to show that $\hat{\theta}(0,s) = 1 + O(q_v^{-3/2})$.

6.6 Aside: Poisson summation

First recall the classical Poisson summation formula: let f be a Schwartz function on $\mathbb{R}$. Then one has the following

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m)$$

where the second sum is taken over the Pontrjagin dual $\mathbb{Z} \simeq (\mathbb{R}/\mathbb{Z})^*$.

More generally, let G be a locally compact abelian group with a discrete, countable, cocompact subgroup Γ and relatively compact fundamental domain D. Assume further that there exists a pairing $(\cdot,\cdot) : G \times G \to \mathbb{R}/\mathbb{Z}$ inducing an isomorphism $G \simeq G^*$ from the map $x \mapsto e^{2\pi i(x,\cdot)}$. Then given any integrable function f such that

1. $\sum_\Gamma f(x+\gamma)$ is uniformly convergent for all x in D, and
2. $\sum_{\Gamma^*} |\hat{f}(\gamma)|$ is convergent

One has

$$\sum_{\gamma\in\Gamma} f(\gamma) = \frac{1}{\text{meas}(D)} \sum_{\gamma\in\Gamma^*} \hat{f}(\gamma).$$

Let us specialize this to $G = \mathbb{A}_F$ the adeles of a number field F, with F embedded as a discrete cocompact group. For each place v of F, one defines a pairing $(x,y)_v = |\text{Tr}_{F_v|\mathbb{Q}_p}(xy)|_p$ where v lies over p. The dual group of $F^* \simeq F$ leads to $\mathbb{A}_F \simeq \mathbb{A}_F^*$ by the pairing

$$((x_v),(y_v)) = \sum_v (x_v, y_v)_v.$$

We do not define the fundamental domain here, which can be found in Tate's thesis, which shows in this case that $\text{meas}(D) = 1$. Then

$$\sum_{x\in F} f(x) = \sum_{x\in F} \hat{f}(x).$$

More generally, the work of Godement and Jacquet prove an analogous formula for Schwartz functions f on $M_n(\mathbb{A}_F)$, namely, after normalizing measures,

$$\sum_{x\in M_n(F)} f(x) = \sum_{x\in M_n(F)} \hat{f}(x).$$

In FLN, the Poisson sum for $f(a) = \theta(a,s)$ is taken over the $\mathfrak{B}(F_S)$, where $\mathfrak{B}$ is part of the Steinberg-Hitchin base, and $F_S = F \cap \mathbb{A}_F^S$, where $\mathbb{A}_F^S$ consists of points x such that x_v is integral outside of S, the finite set of places related to Getz's truncation.

6.6.1. Connection to L-functions The principle is that given an automorphic L-function, one writes down an integral representation for it, then inside the integrand performs Poisson summation formula to prove the associated functional equation.

In the case of $\zeta(s)$, the Poisson sum was over $\mathbb{Z}$, and we might think of $\zeta(s)$ as an integral over $\mathbb{G}_m$ which sits in $\mathbb{G}_a$. In the case of Hecke L-functions, using Tate's perspective we view the L-function as an integral over GL_1 and the Poisson sum over M_1, similarly the Godement-Jacquet case. In each of these cases the unramified local L-factors could be obtained by $L(s,\pi_v,\rho) = \text{tr}(\pi_s(\mathbf{1}_{M_n(\mathfrak{o}_v)}))$.

Yet, as was already shown by Satake, this last relation does not work in the case of $GSp_4 \hookrightarrow MSp_4$, where one has the similitude character and spinor L-function rather than the determinant and standard L-function. Then as in FLN, one seeks *basic functions* $f_{\rho,s}$ such that $L(s,\pi,\rho) = \text{tr}(\pi_s(f_\rho))$.

Roughly speaking, Ngo's idea is to use Vinberg's theory of monoids: there is an open embedding of a reductive group $G \hookrightarrow M$ into a monoid, a normal affine scheme such that the action of G on itself extends to M. One stratifies M into an infinite union of M_n, and hopes to find $f_\rho = \sum f_{\rho,n}$ from perverse sheaves on M_n by the sheaf-function dictionary. Moreover, Ngo considers M as an open subset of the loop space LM.

6.7 The story so far (Altuğ)

Given a quasiplit group G over $\mathbb{Q}$,

6.7.1. Problem (-1) Write a trace formula.

$$\sum_{\pi \in L^2(G_Q \backslash G_{\mathbb{A}})} \mathrm{tr}(\pi(f)) = \sum_{\gamma \in G_{\mathbb{Q}}^{\#}} \mathrm{vol}(\gamma) \mathbf{O}_\gamma(f)$$

Left hand side is mysterious; right hand side is 'more' approachable. In doing this we encounter issues of convergence of $K_f(x, y) = \sum f(x^{-1}\gamma x)$. Truncation considered by Selberg for SL_2, then Arthur for G, but when you truncate you lose invariance.

6.7.2. Problem 0 Write an invariant trace formula (Arthur)

$$\sum_\pi I_\pi(f) = \sum_\gamma I_\gamma(f)$$

Can prove things like the Jacquet-Langlands correspondence with this.

Goal is to prove functoriality: given groups G, H with G quasiplit with a map of L-groups $\psi : {}^L H \to {}^L G$, want a transfer of $\{\pi_H\}$ to $\{\pi_G\}$, such that $L(s, \pi_H, \rho \circ \psi) = L(s, \pi_G, \rho)$ for all finite dimensional complex representations $\rho : {}^L G \to GL(V)$. Want to do this by comparing the geometric sides of the trace formulas of G and H.

This has been carried out for very trivial ψ, e.g., Arthur recently proved this for H a classical group, $G = GL_n$, and $\phi = \mathrm{Id}$. (If L^H is larger than ${}^L G$, then there should not be many maps ψ.)

6.7.3. Problem 1 Write a stable trace formula. A $\mathbb{Q}$-conjugacy class may not be $\bar{\mathbb{Q}}$-conjugacy class; this problem is referred to as endoscopy.

$$\mathrm{TF}_G = \mathrm{TF}_G^{\mathrm{stab}} - \sum_{{}^L H \hookrightarrow {}^L G} \mathrm{TF}_H^{\mathrm{stab}}$$

where the summands are referred to as the endoscopic terms.

6.7.4. Problem 2 Go beyond trivial ψ. This is beyond endoscopy. Two observations:

1. By Chevalley, given ψ such that ${}^L H$ can be defined as the fixed points of a given representation, one can always find a $\rho : {}^L G \to GL(V)$ such that $\rho|_{{}^L H}$ is trivial, i.e., $\ker(\rho) = {}^L H$
2. Given $\tau : {}^L H \to GL(V)$, one expects that $L(s, \pi_H, \tau)$ has a pole at $s = 1$ or order $m_\pi(\tau)$ equal to the multiplicity of $1_{{}^L H}$ in τ, for π Ramanujan.

So that $\ker(\rho) = {}^L H$ implies $\rho|_{{}^L H} = 1_{{}^L H}$, so that

$$L(s, \pi_H, 1_{{}^L H}) = L(s, \pi_H, \rho \circ \psi) = L(s, \pi_G, \rho)$$

has a pole at $s = 1$. (Think of the Artin conjecture.)

This leads to for $f \in C_c^\infty(G_{\mathbb{A}})$

$$\mathrm{TF}_G^{(\rho)}(f) = \sum_{\pi_G} (\text{sth to do with poles of } L^S(s, \pi_G, \rho)) \mathrm{tr}(\pi_G(f_S))$$

where the 'something to do with poles' should be zero if there is no pole at $s = 1$ and nonzero otherwise, e.g., the order or residue of the pole.

For simplicity, assume poles of $L^S(s,\pi,\rho)$ are simple, where

$$L^S(s,\pi,\rho) = \sum_{n=1}^{\infty} \frac{a_{n,\pi}(\rho)}{n^s}$$

where $a_{n,\pi}(\rho) \in \mathbb{C}$. Then

$$\mathrm{Res}_{s=1} = \lim_{X\to\infty} \frac{1}{X} \sum_{n<X} a_{n,\pi}(\rho)$$

if $L^S(s,\pi,\rho)$ is analytic in a neighborhood of $s=1$. By partial summation, this is implied by

$$\sum_{n<X} a_{n,\pi}(\rho) = O(X^{1-\epsilon}), \qquad \epsilon > 0$$

i.e., there is enough cancellation in the sum. So then we have

$$\mathrm{TF}_G^{(\rho)}(f) = \sum_{\pi_G} \Big(\sum_{n<X} a_{n,\pi}(\rho) \Big) \mathrm{tr}(\pi(f_S)) = (\text{`Something manageable'})$$

Something manageable like 0. If we get to interchange the sum, then

$$\sum_{n<X} \sum_{\pi_G} a_{n,\pi}(\rho)\mathrm{tr}(\pi(f_S)) = \sum_{n<X} \mathrm{TF}^{\mathrm{stable}}(f^{(n,\rho)} f_S) = \sum_{n<X} \mathrm{Geom}(n,\rho)$$

Problem: If there are poles to the right of 1, say at s_0, then the RHS $= X^{1+s_0}$. A solution: if f is smooth, Poisson summation gives

$$\sum_n f(\frac{n}{X}) = X \sum_{\xi} \hat{f}(\xi X) = X\hat{f}(0) + O(X^{-m})$$

for all $m \geq 0$. Observation of FLN: the geometric side

$$\mathrm{Geom}(n,\rho) = \sum_{G^{\#}_{\mathbb{Q}} \text{ stable}} \mathrm{vol}(\gamma)\mathbf{O}_{f_{S,n}}(\gamma) = \sum_{b\in \mathrm{SH\ base}_{\mathbb{Q}}} |\mathrm{Jac(b)}| \text{`vol'}(b)\mathbf{O}_{f_{S,n}}(b_n)$$

Where SH is the affine Steinberg-Hitchin base, $c : \gamma \to \mathrm{tr}(\gamma)$ for γ at SL_2, giving an affine space over which one can do Poisson summation formula. FLN are able to apply poisson summation and isolate the main term, the rest of the terms are a mystery; don't know if this method works for those.

6.8 The r-trace formula

Let G be a connected reductive algebraic group over F, with a representation of its L-group $r : {}^L G \to GL_n(\mathbb{C})$. One would like to construct a stable generalization $S^r_{\mathrm{cusp}}(f)$ of the usual stable trace formula $S^1_{\mathrm{cusp}}(f)$ where the subscript 'cusp' indicates that one is able to and has subtracted the nontempered and continuous spectrum from trace formula. The stable multiplicities of π occurring in $S^1_{\mathrm{cusp}}(f)$ are weighted by the order of the pole of $L(s,\pi,r)$ at $s=1$, but for the trivial representation $L(s,\pi,1) = \zeta_F(s)$, the Dedekind zeta function which has a simple pole at one, so that the weight factor is simply 1.

There should be a decomposition

$$S^r_{\mathrm{cusp}}(f) = \sum_{G'} m_{G'}(r)\iota(G,G')P^{\tilde{G}'}_{\mathrm{cusp}}(f')$$

into *primitive* stable distributions on groups $\tilde{G}'_{\mathbb{A}}$, indexed by isomorphism classes of elliptic 'beyond endoscopic' data $(G', \mathcal{G}', \xi')$ where G' is a quasisplit group over G, $\mathcal{G}'$ a split extension

$$1 \to \hat{G}' \to \mathcal{G}' \to W_F \to 1$$

and an L-embedding $\xi' : \mathcal{G}' \to {}^L G$ such that $\mathrm{Cent}(\mathcal{G}', \hat{G})/Z(\hat{G})^\Gamma$ is finite; also the auxilliary datum $(\tilde{G}', \tilde{\xi}')$ which takes care of the possibility when the image of ξ' is not an L-group. The coefficient $m_{G'}(r)$ is the dimension datum of G at r, which is the multiplicity of the trivial representation of $\mathcal{G}'$ occurring on $r \circ \xi'$. The function f' is the stable transfer of f from $\mathcal{H}(G_S)$ to $S(\tilde{G}'_S, \tilde{\eta}'_S)$ (for this one assumes the local Langlands correspondence for the groups G'_v and $\tilde{G}'_v$).

To define the primitive linear form, suppose first that $\tilde{G}' = G' = G$. One would like to write the primitive trace formula as $P^G_{\mathrm{cusp}}(f) = S^G_{\mathrm{prim}}(f)$ involving contribution of the tempered, cuspidal automorphic representations that are not functorial transfers from a smaller group. But without functoriality, we require an alternative, inductive definition

$$P^G_{\mathrm{cusp}}(f) = S^1_{\mathrm{cusp}}(f) - \sum_{G' \neq G} m_{G'}(1)\iota(G, G') P^{\tilde{G}'}_{\mathrm{cusp}}(f')$$

Then writing down the trace formulas for $S^r_{\mathrm{cusp}}(f)$ and $P^G_{\mathrm{cusp}}(f)$ would be called the r-trace formula and primitive trace formula, respectively.

Following the Beyond Endoscopy method over $\mathbb{Q}$, one wants to establish the formula

$$S^r_{\mathrm{cusp}}(f) = \lim_{X \to \infty} \frac{1}{X} \sum_{p<X} \log(p) S^1_{\mathrm{cusp}}(f^r_p),$$

where f^r_p is a test function depending on r and p, by analyzing the geometric side of the trace formula. The first issue that arises is that $S^1_{\mathrm{cusp}}(f)$ is given by the geometric side minus some spectral terms. If we are able to obtain cancellation of the spectral terms from the geometric expansion, then the limit would only involve geometric terms. This leads us to the nontempered stable characters in $S_{\mathrm{disc}}(f)$, which should be attached to global packets Π_ψ whose parameter

$$\psi : L_F \times SL_2(\mathbb{C}) \to {}^L G$$

is nontrivial on the SL_2 factor.

From FLN, we have the Steinberg-Hitchin base, which for G simply connected and split over F, is an F-vector space A_F of dimension $\mathrm{rk}(G)$, its coordinates parametrized by the irreducible finite dimensional representations ρ_i of G attached to the fundamental dominant weights. Recall also the map from stable semisimple conjugacy classes $\delta \mapsto \oplus \mathrm{tr}(\rho_i(\delta))$ in G_F to A_F. So we have the geometric expansion

$$S^G(f) = \sum_{a \in A_F} \theta_f(a)$$

where $\theta_f(a)$ is the sum of geometric terms in $S^G(f)$ that belong to the fibre of δ over a. As we have seen, FLN are able to apply Poisson summation formula to this sum, and cancel the dominant term with the contribution of the trivial representation term, which is the most nontempered. It corresponds otto the parameter ψ that is trivial on L_F, and as a homomorphism from $SL_2(\mathbb{C})$ to $\hat{G}(\mathbb{C})$, it corresponds to the principal unipotent conjugacy class in $\hat{G}$.

We conclude with some problems suggested by Arthur:

1. Define $\iota(G, G')$. It should differ from endoscopy, but the summation over G' is not determined by $m_{G'}(r)$.
2. The poles of $L(s, \pi, r)$ at $s = 1$ can be captured either by the order or the residue. Analysis using the latter is easier, as Altuğ uses, but is different from the framework presented above, which uses the former.
3. Generalizing Altuğ's work to more general real groups, and then over p-adic fields seems challenging; likely there will be an incarnation of the Fundamental Lemma.
4. The trivial representation contributes only to the dominant term $\theta_f(0)$; is there a natural characterization for other nontempered characters? A possibility lies in contribution of the spectral term $\mathrm{tr}(M(w)I_P(0, f))$ to $\theta_f(0)$ that Altuğ obtains, given an injective map of Kazhdan-Lusztig from unipotent $\hat{G}$ conjugacy classes to W_G conjugacy classes.

6.9 Singularities and transfer

This section is very sketchy! Read at own risk.

6.9.1. Transfer Let F_v be a local field, π an admissible irreducible representation of $SL_2(F_v)$, χ_π its character which is an invariant distribution on $SL_2(F_v)$ and a smooth function on $SL_2^{\mathrm{reg}}(F_v)$. The distribution is a product of this function and the Haar measure. Attach to its L-packet π^{st} the character $\chi_{\pi^{\mathrm{st}}}$ which is a linear combination

$$\chi_\pi^{\mathrm{st}} = \sum_{\pi \in \pi^{\mathrm{st}}} a_\pi \chi_\pi.$$

For $G = SL_2$, the coefficients are at most 1, except possibly when the residue characteristic is even. A map of L-groups $\phi : {}^L H \to {}^L G$ should induce a transfer of stable L-packets $\pi_H^{\mathrm{st}} \mapsto \pi_G^{\mathrm{st}}$, though in fact we ought to work with Arthur parameters $\phi = \sigma \times \psi$. On the other hand, we consider only H that are tori, so that the L-packets contain at most a single element, and the a_π are 1. One expects the formula

$$\chi_{\pi_G^{\mathrm{st}}}(a_G) = \int_{A_H} \chi_{\pi^{\mathrm{st}}}(a_H)\Theta(a_H, a_G) da_G$$

where the function $\Theta(a_H, a_G)$ defines the transfer, and note that it is over the base $A_H \times A_G$ rather than stable conjugacy classes, which was the case in endoscopy. This leads to the transfer of orbital integrals

$$\theta_v^H(a_H) = \int_{A_G} \theta_v^G(a_G, 1)\Theta(a_H, a_G) da_G$$

Example 6.9.1. In the simplest case of $H = \{1\}$, so that ${}^L H = \mathrm{Gal}(\bar{F}/F)$ and $\phi : \mathrm{Gal}(\bar{F}/F) \to {}^L G$, the function f^H is a constant equal to

$$\int_{A_G} f^G(g)\chi_{\pi_G^{\mathrm{st}}}(g) dg = \int_{A_G} \theta_v(a_G, 1)\Theta(1, a_G) dg$$

where $\theta_v(a_G, 1)$ is given in FLN.

Question 6.9.2. *Given f^G smooth compactly supported on G_v^{rs}, can one find an f^H smooth compactly supported on H_v such that*

$$\mathbf{O}^{st}(a_H, f^H) = \frac{1}{|\Delta(a_H)\lambda(a_H)|} \int |\Delta(a_G)\lambda(a_G)\mathbf{O}^{st}(a_G, f^G)\Theta(a_H, a_G) da_G$$

for a_H in A_H^{reg}? Given this, can one give an f^H for a f^G on G_v?

Example 6.9.3. Let H be the multiplicative group of elements of norm one in a quadratic extension E over F. We consider the split case $H \simeq GL_1$ or a split torus of SL_2. We may treat this case uniformly over $F = \mathbb{C}$, $\mathbb{R}$ and $\mathbb{Q}_p$. Let π_H be a character ξ of $F^\times$. Then ${}^L H = GL_1 \rtimes \mathrm{Gal}(E/F)$, and the L-homomorphism is

$$\phi : x \rtimes \sigma \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \rtimes \sigma$$

The representation π_H^{st} is attached to a homomorphism of the Weil group $W_F = F^\times$, namely ξ. The stable character of π_G^{st} is a distribution, given by a stably invariant function on the regular semisimple elements. It is trivial on elliptic elements, while on split classes $\gamma = \mathrm{diag}(t, t^{-1})$ it is

$$\chi_G^{\mathrm{st}}(\gamma) = \frac{\xi(t) + \xi(t)^{-1}}{|\Delta_G(\gamma)|}$$

where $\Delta_G(\gamma) = \pm\sqrt{(1-t^2)(1-t^{-2})}$ as in FLN. On the other hand, $\Delta_H = 1$, so the character of π_H is simply ξ. In terms of the Steinberg-Hitchin base, $a_H = t_H$ and $a_G = c(t) = t_G + t_G^{-1}$, and one has $\lambda(a_H) = 1$ and $\lambda(a_G) = L(1, \sigma_{T_H})$ where T_H is a torus in G isomorphic to H. Hence

$$da_H = L(1, \sigma_H) d^\times t, \qquad da_G = |1 - t^{-2}| = \frac{|\Delta_G(a_G)|}{L(1, \sigma_H)} da_H$$

and

$$\Theta(a_H, a_G) = \frac{\delta_{t_G}(t_H) + \delta_{t_G^{-1}}(t_H^{-1})}{|\Delta_G(\gamma)|}$$

Then we have the following formulas for $\Theta(a_H, a_G) da_H da_G$:

$$\int_{A_G(F)} h(a_H, a_G) \Theta(a_H, a_G) da_H da_G = \int_{A_G} \sum_{a_H \mapsto a_G} \frac{h(a_G, a_H)}{|\Delta(a+G)|} da_G = \int_{A_H} \frac{h(a_H, a_G)}{L(1, \sigma_H)} da_H$$

where $h(a_H, a_G)$ is a function satisfying the above. Furthermore, one has

$$f^H(t_H) = \theta^H(a_H) = \frac{\theta^G(a_G)}{\lambda(a_G)}$$

6.9.2. Singularities

6.9.3. Poisson summation Langlands makes certain simplifying assumptions which introduces some error terms, namely that $L_S(s, \sigma_{T/G} = 1$, and considering the limit as s tends to 1 of $\theta(a, s)$. One has

$$\sum_{b \in B_S}{}' \prod_S \theta_v(b, 1)$$

where the prime indicates summation over b such that the torus defined by b is regular and contains no split subtori. Also,

$$\prod_S L_v(1, \sigma_{T/G}) = \prod_S \frac{1}{1 - q_v^{-1}}$$

In our setting, $B_S = F_S$. If b is split, then $b = \lambda + \lambda^{-1}$ for λ in $F^\times$. Then we have the Poisson summation formula

$$\sum_{b \in B_S} \theta(b) = \sum_{b \in B_S} \hat{\theta}(b)$$

The difference $\theta(b) - \varphi(b) = 0$ if $b \in F^\times$ as S is chosen such that $|\Delta(b)|_v = 1$ if $\theta_S(b) \neq 0$ and for split b, $\varphi_v(b) = |\Delta(b)|_v \theta_v(b)$. We want to interpret the difference

$$\sum_{b \in F} (\theta(b) - \varphi(b)) = \sum_{b \text{ ell}} (\theta(b) - \varphi(b)).$$

By the trace formula one has

$$\sum_{H \text{ ell}} \sum_{\theta_H \neq 1} \operatorname{tr}(\theta_G^{\text{st}}(f)) = \sum_{H \text{ ell}} \sum_{\theta_H \neq 1} \operatorname{tr}(\theta_H(f^H))$$

where the θ_H are made out of nontrivial characters of $H_F \backslash H_{\mathbb{A}}$, and θ_G its functorial image. For elliptic H, the quotient is compact, so the trace formula gives readily

$$\sum_{\theta_H} \operatorname{tr}(\theta_H(f^H)) = \operatorname{vol}(H_F \backslash H_{\mathbb{A}}) \sum_{\gamma \in H_F} f^H(\gamma) = \sum_{\gamma \in H_F} L(1, \sigma_{T/G}) |\Delta(\gamma)| f^H(\gamma)$$

up to an error that vanishes with large S, we may replace this by

$$\sum_{\gamma \in H_F} \varphi(\gamma)$$

where γ maps to b in $B = F$.

7 Galois Representations

7.1 What are they

Let's recall some basic facts about the absolute Galois group. Fix an algebraic closure $\overline{\mathbb{Q}}$. The Galois group is an inverse limit of finite extensions K of $\mathbb{Q}$

$$G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) = \lim_{\leftarrow} \mathrm{Gal}(K/\mathbb{Q}).$$

The norm given for every prime p as

$$|\alpha|_p = |p^r \frac{a}{b}| = p^{-r}$$

defines the p-adic completion $\mathbb{Q}_p$ when p is a finite prime. The inclusion $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ induces the inclusion $G_{\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}}$. Similarly, the norm at the 'infinite' prime is defined to be the Euclidean norm, giving $\overline{\mathbb{Q}} \hookrightarrow \mathbb{R}$ and $G_{\mathbb{R}} \hookrightarrow G_{\mathbb{Q}}$.

It is an important fact the topology on $\mathbb{Q}_p$ is quite different from $\mathbb{R}$. The ring of integers $\mathbb{Z}_p$ is a local ring with maximal ideal $\mathbb{Z}_p/p\mathbb{Z}_p \simeq \mathbb{F}_p$. Note that $\mathbb{Z}_p$ consists of elements $|x|_p \leq 1$ whereas $p\mathbb{Z}_p$ consists of elements $|x|_p < 1$.

A basic relation is the following: fix a prime $p < \infty$,

$$1 \to I_{\mathbb{Q}_p} \to G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p} \to 1$$

where $\mathbb{F}_p$ is identified with the residue field of $\mathbb{Q}_p$. Furthermore $G_{\mathbb{F}_p} \simeq \hat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$. Then the kernel $I_{\mathbb{Q}_p}$ is called the inertia group. One has more:

$$1 \to \bigcup_{n>0} I^n_{\mathbb{Q}_p} \to I_{\mathbb{Q}_p} \to \prod_{\ell \neq p} \mathbb{Z}_\ell \to 1$$

This kernel is referred to as the wild inertia group, denoted $I^{\mathrm{wild}}_{\mathbb{Q}_p}$. Thirdly, we have

$$1 \to I_{\mathbb{Q}_p} \to W_{\mathbb{Q}_p} \to \mathbb{Z} \to 1$$

by considering the dense subset $\mathbb{Z}$ in $\hat{\mathbb{Z}}$, and the preimage $W_{\mathbb{Q}_p}$ of $\mathbb{Z}$ in $G_{\mathbb{Q}_p}$ is known as the Weil group. More generally, it can be defined as a group with dense image in $G_{\mathbb{Q}_p}$ satisfying certain properties related to class field theory.

Now, we are interested in Galois representations, i.e., representations of the Galois group. Here are some basic ones:

1. Artin representations. These are homomorphisms with open kernel,

$$G_{\mathbb{Q}} \to GL_n(\mathbb{C})$$

   Note that the profinite topology and the complex topology on the respective groups force the $G_{\mathbb{Q}}$ to act through a finite quotient.

2. ℓ-adic representations

$$G_{\mathbb{Q}} \to GL_n(\overline{\mathbb{Q}}_\ell)$$

   (a) Fixing a noncanonical ismorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$, we get back Artin representations.

   (b) The cyclotomic characters $\chi_\ell : G_{\mathbb{Q}} \to \mathbb{Z}_\ell^\times$ defined by $\sigma\zeta = \zeta^{\chi_\ell(\sigma)}$ where ζ is a root of unity.

   (c) Let X be a smooth projective variety over $\mathbb{C}$. There is an action of $G_{\mathbb{Q}}$ on its ℓ-adic cohomology $H^i(X(\mathbb{C}), \overline{\mathbb{Q}}_\ell) \simeq H^i_{\text{ét}}(X \times \overline{\mathbb{Q}})$. For example, when X is an elliptic curve E, the Galois action is evident by

$$H^1(E \times \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell) \simeq \mathrm{Hom}(\lim_{\leftarrow} E[\ell^r](\overline{\mathbb{Q}}), \overline{\mathbb{Q}}_\ell) \simeq \overline{\mathbb{Q}}_\ell \times \overline{\mathbb{Q}}_\ell$$

   This is parallel to H^1 equal to $\mathbb{Z}^2$ in singular cohomology.

(d) Weil-Deligne representations extend Galois representations. They consist of the data

$$r : W_{\mathbb{Q}_p} \to GL(V)$$

where V is a finite dimensional vector space over a finite extension of $\mathbb{Q}_\ell$, and a nilpotent endomorphism N of V.

There is some care to be a taken whether $\ell \neq p$ or not. For example, Deligne shows concerning Ramanujan's tau function

$$\Delta(z) = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n$$

that there exists a unique representation $\rho_{\Delta,\ell} : G_\mathbb{Q} \to GL_2(\mathbb{Q}_\ell)$ such that for all $p \neq \ell$, the trace $\mathrm{tr}\rho_{\Delta,\ell}(\mathrm{Frob}_p) = \tau(p)$. But what happens $p = \ell$? There there are many more representations, unlike at $p \neq \ell$, where Deligne has shown that the category of Weil-Deligne representations is equivalent to the category of ℓ-adic Galois representations.

A rich source of Galois representations has been through the cohomology of algebraic varieties. In fact, all the examples above can be cast in such a setting. This motivates the following conjecture:

Conjecture 7.1.1. *(Fontaine-Mazur) Let $R : G_\mathbb{Q} \to GL(V)$ be an irreducible unramified ℓ-adic representation that is de Rham at almost every G_{Q_ℓ}. Then there exists a smooth projective variety X over $\mathbb{Q}$ such that for some $i \geq 0$ and j, V is a subquotient of $H^i(X(\mathbb{C}), \overline{\mathbb{Q}}_\ell) \otimes \chi_\ell^j$.*

Related to this is another conjecture,

Conjecture 7.1.2. *(Tate) Let X be a smooth projective variety over $\mathbb{Q}$. Then there is a decomposition*

$$H^i(X(\mathbb{C}), \overline{\mathbb{Q}}) = \bigoplus_j M_j$$

such that for each prime l and embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_l$, $M_j \otimes \overline{\mathbb{Q}}_l$ is an irreducible subrepresentation of $H^i(X(\mathbb{C}), \overline{\mathbb{Q}})$, and for all p and j,

$$WD_p(M_j) \otimes \overline{\mathbb{Q}}_l \simeq WD_p(M_j \otimes \overline{\mathbb{Q}}_l)$$

These M_j are the so-called 'motives' of Grothendieck which we will see shortly.

7.2 L-functions for Galois representations

1. Consider the elliptic curve $E : y^2 = x^3 + ax + b$ over $\mathbb{Q}$ with $4a^3 + 27b^2 \neq 0$. Then the **elliptic curve L-function** associated to it

$$L(E, s) = \prod L_p(E, p^{-s}) \qquad \mathrm{Re}(s) > \frac{3}{2}$$

where for almost all p, $L_p(E, X)^{-1} = 1 - a_p(E) + pX^2$. It is well known that $p - a_p(E) = |E(\mathbb{F}_p)|$, that is, the points of E over the finite field $\mathbb{F}_p$. The celebrated modularity theorem, completed by Breuil-Conrad-Diamond-Taylor involves the proof that $L(E, s) = L(f, s)$ for some modular form f, hence has analytic continuation and functional equation

$$(2\pi)^{-s}\Gamma(s)L(E, s) = N(E)^{1-s}(2\pi)^{s-2}\Gamma(2 - s)L(E, 2 - s).$$

As a result this proves a special case of the Birch-Swinnerton-Dyer conjecture:

Conjecture 7.2.1. *(BSD) E has infinitely many rational points over $\mathbb{Q}$ if and only if $L(E, s) = 0$ at $s = 1$.*

Note the crucial fact that $L(E,s)$ is not even defined at $s=1$ without analytic continuation! As a result of the work of BCDT, Gross-Zagier, Kolyvagin and others, the conjecture is known to be true when $L(E,s)$ has at most a simple zero at $s=1$.

2. Given a Weil-Deligne representation WD_p, its L-function generalizing the **Artin L-function**, is

$$L(WD_p,s) = \det(1-p^{-s}\mathrm{Frob}_p)|^{-1}_{V^{I_{\mathbb{Q}_p},N=0}}$$

where Frobenius is restricted to the subspace of V on which inertia $I_{\mathbb{Q}_p}$ acts trivially and the endomorphism N is vanishes. For a de Rham ℓ-adic representation R of pure weight w, we have

$$L(\iota R,s) = \prod_p L(\iota WD_p(R),p^{-s}) \qquad \mathrm{Re}(s) > 1+\frac{w}{2}.$$

This specializes to classical cases, for example: $L(1,s)=\zeta(s)$ and $L(\iota H^1(E(\mathbb{C}),\overline{\mathbb{Q}}_l),s) = L(E,s)$. One also expects such $L(\iota R,s)$ to be entire except for a simple pole at $R=\chi_\ell^{-\frac{w}{2}}$, have functional equation and boundedness in vertical strips.

3. Let X be a smooth projective variety over $\mathbb{Q}$, with a model over $\mathbb{Z}[\frac{1}{N}]$ and good reduction over $\mathbb{F}_p$, for $p \not| N$. Then the **Hasse-Weil zeta function** of X is

$$Z(X,t) = \exp(\sum_{n\geq 1}\frac{|X(\mathbb{F}_p)|}{n}t^n) = \prod_{x\in|X(\overline{\mathbb{F}}_p)|}\frac{1}{1-t^{\deg x}}$$

is an element of $Z[[t]]$. The finite field Riemann hypothesis was proved for $Z(X,t)$. More generally, one can define

$$\zeta_N(X,s) = \prod_{p \not| N} Z(X,p^{-s}) = \prod_{i=0}^{2\dim X} L(\iota H^i(X(\mathbb{C}),\overline{\mathbb{Q}}_l),s)^{(-1)^i}$$

where the last equality is a result of Grothendieck. Again this specializes $\zeta(\mathrm{pt}) = \zeta(s)$ and

$$\zeta(E,s) = \frac{\zeta(s)\zeta(s-1)}{L(E,s)}.$$

7.3 Automorphy

First let's briefly review the notion of an automorphic form. Denote the adèle ring of $\mathbb{Q}$ by

$$\mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \hat{\mathbb{Z}} \otimes \mathbb{Q} = \prod_p{}' \mathbb{Q}_p$$

where the ' indicates the restricted direct product, taking $\mathbb{Z}_p$ at almost every place p. Class field theory is our prototye: locally, $\mathbb{Q}_p^\times \simeq W_{\mathbb{Q}_p}^{\mathrm{ab}}$, while globally

$$GL_1(\mathbb{Q})\backslash GL_1(\mathbb{A}) = \mathbb{Q}^\times\backslash\mathbb{A}^\times \xrightarrow{\sim} W_{\mathbb{Q}}^{\mathrm{ab}} \twoheadrightarrow G_{\mathbb{Q}}^{ab}.$$

So functions on the Galois group are functions on the adelic quotient. Roughly, an **automorphic form** on a reductive group G is a function $f : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ such that

1. (K-finite) The translates of f under a maximal compact subgroup of $G(\mathbb{A})$ spans a finite-dimensional vector space. Classically, this descends the function to the upper half plane $SL_2(\mathbb{R})/SO_2(\mathbb{R})$.
2. (Growth) f is a smooth function with a 'slowly increasing' condition. Classically this relates to the holomorphy of modular forms.
3. (Infinitesimal character) If z is an element of the Lie algebra $\mathfrak{z}$ of the center of G then z acts by a character $z(f(g)) = \chi(z)f(g)$. Classically, this is the Laplacian eigenvalue.
4. (Cuspidality) If $\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} f(ng)\, dn = 0$ for any unipotent subgroup N, then furthermore f is a **cusp form**. Classically, these have constant fourier coefficient zero.

A less enlightening characterization is that an automorphic representation is an irreducible constituent of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. The Fontaine-Mazur conjecture fits in the following way:

Conjecture 7.3.1. *Let $R : G_{\mathbb{Q}} \to GL(V)$ be a representation satisfying the conditions of the Fontaine-Mazur conjecture. Then there is a cusp form π of $GL_n(\mathbb{A})$ associated to ιR.*

Going the other way, there are conjectures regarding Galois representations constructed from autormorphic forms, but we will not state them here. Instead we will discuss some results which describe the passage between the two. In the following fix $G = G_n = GL_n$

Base change. Let K/F be a cyclic Galois extension of number fields, R an irreducible de Rham l-adic representation of G_K ramified at finite many primes, and Π a cus form on of $G(\mathbb{A}_K)$ such that $\Pi \simeq \Pi \circ \sigma$ where σ is the generator of $G_{K/F}$. Then we expect a cusp form on $G(\mathbb{A}_F)$ such that for all places v of K,

$$\mathrm{rec}_v(\Pi_v) = \mathrm{rec}_v|_F(\pi_v|_F)|_{W_{K_v}}$$

For $n = 1$ this is true. By setting $\Pi = \pi \circ N_{K/F}$ we get a transfer of π to Π. The transfer of Π to π is proved by class field theory. For $n > 1$ this is the trace formula for base change.

Automorphic induction. Let K/F extension of number fields, R as above. Then $\mathrm{Ind}_{G_K}^{G_F} R$ is a semi-simple de Rham l-adic representation. We expect that if Π is a cusp form on $G_n(\mathbb{A}_K)$ then there is a partition $n[K : F] = n_1 + \cdots + n_r$ and cusp forms on $G_{n_i}(\mathbb{A}_F)$ such that

$$L(\Pi, s) = \prod_{i=1}^{r} L(\pi_i, s)$$

The existence of the π_i are known for $n = 1$ and $[K : F] \leq 3$ by the converse theorems of Jacquet-Langlands and Jacquet-PS-Shalika. If K/F is Galois and soluble this is again Arthur-Clozel. Here is a (strong) form of Artin's conjecture

Conjecture 7.3.2. *Let $R : G_F \to G(\overline{\mathbb{Q}}_l)$ be an irreducible l-adic representation with finite image, and fix $\iota : \overline{\mathbb{Q}}_l \simeq \mathbb{C}$. Then there is a cusp form π on $G(\mathbb{A}_F)$ such that $L(\pi, s) = L(\iota R, s)$. In particular, this implies that $L(\iota R, s)$ is entire except for a possible simple pole at if $n = 1$.*

By a conjecture of Cogdell and PS, such automorphic induction would allow us to write $R = \sum n_i \mathrm{Ind}_{G_{K_i}}^{G_F} \chi_i$ where χ_i are characters of G_{K_i} which do not extend to G_F.

Galois descent. Let K be a totally real Galois extension of $\mathbb{Q}$, Π a cusp form on $G(\mathbb{A}_K)$ with some conditions at the archimedean places and at a finite place, and R an l-adic representation of $G_\mathbb{Q}$ such that $R|_{G_K}$ is irreducible and $R \sim R^* \otimes \psi$ for some character ψ of $G_\mathbb{Q}$. Then if

$$\iota L(WD_v(R|G_K), s) = L(\Pi_v, s)$$

then we expect a cusp form π on $G(\mathbb{A}_\mathbb{Q})$ associated to R in the same sense. This lets us check the automorphy of an l-adic representation of $G_\mathbb{Q}$ by base changing to K.

Lifting theorems. Let $R : G_\mathbb{Q} \to G(\overline{\mathbb{Q}}_l)$ be continuous. Then after conjugation we may assume the image of R is contained in $G(\mathfrak{o}_{\bar{\mathbb{Q}}_l})$, giving the residual representation

$$\bar{R} : G_\mathbb{Q} \to G(\overline{F}_l)$$

Generally, a lifting theorem is of the form: given R, R' with R automorphic and $\bar{R} \simeq \bar{R}'$, to show that R' is also automorphic. For example, Taylor and Wiles show that $R \mod l^r$ is automorphic for all r, using the fact that $GL_2(\mathbb{Z}_3)$ is a pro-soluble group and there is a homomorphism from $GL_2(\mathbb{F}_3) \to GL_2(\mathbb{Z}_3)$ splitting the reduction map. Then they use the Langlands-Tunnel theorem that the Artin representation

$$G_\mathbb{Q} \to GL(H^1(E(\mathbb{C}), \mathbb{F}_3)) \to GL_2(\mathbb{Z}_3)$$

is automorphic.

Reference

Taylor, R., *Galois representations* (2004)

7.4 An infinite fern

This section summarizes the paper of Mazur titled, *An infinite fern in the universal deformation space of Galois representations.*

7.4.1. Deformation theory of Galois representations,

Example 7.4.1. Let $K/\mathbb{Q}$ be the splitting field of $x^3 + ax + 1$ where $27 + 4a^3 = p$ a prime. Then $K/\mathbb{Q}$ is unramified outside of p with Galois group isomorphic to S_3. Up to equivalence there is only one embedding of S_3 in $GL(\mathbb{F}_p)$, so there is only one (continuous) representation

$$\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_p)$$

We want to classify lifts of $\bar{\rho}$ to

$$\rho : G_{\mathbb{Q}} \to GL_2(A)$$

where A is any commutative complete noetherian local ring with residue field $\mathbb{F}_p$ (the 'coefficient ring'). Indeed, there is a **universal deformation ring** $R = R(\bar{\rho})$ isomorphic to $\mathbb{Z}_p[[t_1, t_2, t_3]]$. In particular, we get induced representations

$$\rho_x : G_{\mathbb{Q}} \to GL_2(\mathbb{Z}_p)$$

for every point x in the algebraic homomorphisms

$$X := \mathrm{Hom}(R, \mathbb{Z}_p) \simeq \mathrm{Hom}(\mathbb{Z}_p[[t_1, t_2, t_3]], \mathbb{Z}_p) \simeq pZ_p \times pZ_p \times pZ_p$$

the 'cube' with sides pZ_p. The ring R is universal in the sense that any lift of $\bar{\rho}$ to A is induced by a homomorphism $R \to A$, unique up to unique isomorphism.

Example 7.4.2. Denote $\Lambda = R(\det \bar{\rho})$ the universal deformation ring of the determinant representation

$$G_{\mathbb{Q}} \xrightarrow{\bar{\rho}} GL_2(\mathbb{F}_p) \xrightarrow{\det} \mathbb{F}_p^\times$$

induces a homomorphism of complete local rings $\Lambda \to R(\bar{\rho})$. The operation of twisting by one-dimensional characters gives a natural action on $R(\bar{\rho})$ by a one-parameter formal multiplicative group.

Now the space $X = X(\bar{\rho})$ may be viewed as the $\mathbb{Q}_p$ points of a p-adic analytic variety underlying $\mathrm{Spec}(R(\bar{\rho}))$. The structural homomorphism $\Lambda \to R(\bar{\rho})$ induces a natural mapping of p-adic analytic spaces

$$X(\bar{\rho}) = \mathrm{Hom}(R(\bar{\rho}), \mathbb{Z}_p) \to \Delta = \mathrm{Hom}(\Lambda, \mathbb{Z}_p)$$

equivariant with respect to the action of the p-adic analytic group $\mathrm{Hom}(G_{\mathbb{Q}}, \Gamma)$ where Γ is the group of 1-units in $Z_p^\times$, arising from twisting by characters.

7.4.2. Eigenforms. Call a classical cuspidal modular form f of type (N, k, ϵ) a weight k cuspidal modular form on $\Gamma_0(N)$ ('of level N') with character ϵ. An **eigenform** is one such that the first Fourier coefficient is 1 and is an eigenfunction of the Hecke operators T_l for $l \not| N$ and Atkin operators U_q for $q|N$.

Assume $p|M$. It is known that $\mathcal{O}_f$, the integral closure of the ring generated by the Fourier coefficients of f is the ring of integers of a number field. Choose a maximal ideal m of $\mathcal{O}_f$ residue characteristic p, denote by $\mathcal{O}_{f,m}$ the completion at m, and $K_{f,m}$ its fraction field. It is a discretely valued field with valuation ord_p normalized so that $\mathrm{ord}_p(p) = 1$. The associated Galois representation $\rho_{f,m}$ is such that for all l not dividing N the Hecke eigenvalue is

$$a_l = \mathrm{tr}(\mathrm{Fr}(\rho_{f,m}))$$

Now if

$$U_p f = \lambda_p f$$

and we call the **slope** of an eigenform f the nonnegative $\mathrm{ord}_p(\lambda_p)$. To any new form φ of type (M, k, ϵ) with $(M, p) = 1$, there are two eigenforms of weight k and level Mp with T_l and U_q eigenvalues equal to those of φ, and whose U_p eigenvalues are the roots of

$$X^2 - T_p(\varphi) + \epsilon(p)p^{k-1}.$$

Their associated Galois representations are equivalent, and their slopes add to $k-1$ since $\lambda_p\lambda'_p = \epsilon(p)p^{k-1}$. If one form has the larger slope it is called the evil twin.

Roughly, given an absolutely irreducible representation $\bar{\rho}$, a deformation to $GL_2(A)$ will be called **modular** in the sense that there is a classical eigenform f with a homomorphism $\mathcal{O}_f \to A$ such that the preimage of m lifts ρ, giving a commutative diagram

$$\begin{array}{ccc} G_{\mathbb{Q}} & \xrightarrow{\rho_{f,m}} & GL_2(\mathcal{O}_{f,m}) \\ & \searrow^{\rho} & \downarrow \\ & & GL_2(A) \end{array}$$

As we will see, such modular representations come in families. Further, a deformation ρ to A is called promodular if A is an inverse limit of rings A_n.

Hida constructed a coefficient T°, a finite flat Λ-algebra as a certain completion of a Hecke algebra with a promodular deformation ρ of a modular absolutely irreducible residual representation $\bar{\rho}$ whose U_p-eigenvalue is a unit in T°. Specializing ρ to integral weights $k = 2, 3 \ldots$, Hida obtained representations associated to classical eigenform. The deformation is ordinary, so there is a surjection

$$R^\circ \to T^\circ$$

The following is a special case:

Theorem 7.4.3. *(Wiles, Taylor-Wiles) Let $K = \mathbb{Q}(\sqrt{q})$ where $q = (-1)^{\frac{p-1}{2}}p$, and $\bar{\rho}$ an ordinary modular residual representation of level p such that $\bar{\rho}|G_K$ is absolutely irreducible. Then $R^\circ \simeq T^\circ$, and the universal deformation ρ° is promodular.*

7.4.3. The infinite fern. Let X be the universal deformation space of of a modular absolutely irreducible residual representation $\bar{\rho}$ of level $N = pM$. Define:

- a p-adic disc of radius ν around k is the open set $D = \{x \in \mathbb{Z}_p : x \equiv k \mod p^\nu\}$ viewed as a p-adic analytic manifold.
- a strictly analytic function from $f : D \to X$ is such that its composition with any τ in $R(\bar{\rho})$ to Z_p can be expressed as avower series in Z_p convergent over D.
- a p-adic analytic family of modular representations (D, z, u) is a disc D with strictly analytics maps $z : D \to X$ and $u : D \to \mathbb{Z}_p$ non vanishing such that for a dense arithmetic progression of integers k in D the representation associated to $z(k)$ is modular.
- a **modular arc** in X is the image $C = z(D)$ in X of a p-adic analytic family above. Each C arises from a unique p-adic analytic family. The slope of C is defined to be the slope of this family.
- the Sen null-space X_0 is the points X whose Hodge-Tate-Sen weights s, r are such that $sr = 0$. Any modular arc C is contained in X_0, and $x \mapsto r_x + s_x$ induces $X \to \mathbb{Z}_p$.
- the **modular locus** is the set-theoretic union in X_0 of all modular arcs of level p in X. It is a countable union of analytic arcs.

Now let $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_p)$ be an absolutely irreducible representation unramified at p, associated to a weight k eigenform of level p with nontrivial character and slope $0 < \alpha < k-1$ ('noncritical'). Assume further that there is a maximal ideal m such that $\mathcal{O}_{f,m} \simeq \mathbb{Z}_p$. By coleman there is an analytic family (D, z, u) with arithmetic progression $\mathcal{K} = P \cap D$ where

$$P = \{r + (p-1)t : t \in \mathbb{Z}, t > \frac{\alpha + 1 - r}{p-1} \text{ and } 0 \leq r < p-1, k-1 \equiv r \bmod p-1\}$$

Next, set

$$\mathcal{K}_0 = \{\kappa \in \mathcal{K} \subset D | \kappa > \alpha = 1, \kappa \neq 2\alpha + 2\}$$

The associated forms f_κ of level p are not newforms, hence coming form level 1 newforms. The evil twins f'_κ have slope $\kappa - \alpha - 1$. This produces another p-adic analytic family (the needle) intersecting the previous modular arc (the spine). This process iterates to produce the image of an infinite fern in X_0.

For pictures, see Mazur's survey paper.

7.5 The eigencurve

The main reference for this section is Kassaei's *The Eigencurve: a brief survey.* Recall that a classical modular form f of weight $k \in \mathbb{Z}$ and character χ satisfies

$$f\Big(\frac{az+b}{cz+d}\Big) = \chi(d)(cz+d)^k f(z)$$

In 1973, Serre introduced the notion of a p-adic modular form beginning with an Eisenstein series of even weight $k \geq 4$

$$E_k(z) = \frac{\zeta(1-k)}{2} + \sum_{n \geq 1} \sigma_{k-1}(n) q^n, \qquad q = e^{2\pi i z}$$

and defining the p-adic Eisenstein series for odd p as

$$E_k^*(z) = E_k(z) - p^{k-1} E_k(pz) = \frac{p^{k-1}\zeta(1-k)}{2} + \sum_{n \geq 1} \sigma_{k-1}^*(n) q^n$$

where σ^* is now the sum of the divisors up to n coprime to p. It is again a modular form (an old-form of level p). In modern parlance, the constant term of a p-adic Eisenstein series is a special value of $\zeta(s)$ interpolated by a p-adic L-function. Let D be the set of even integers $k \geq 4, k \equiv 0(p-1)$. It is dense in $\mathbb{Z}_p$, thus the Fourier coefficients of $E_k^*(z)$ are continuous as k varies in D.

More generally, let D now be a p-adic closed disc in $\mathbb{C}_p$ around a positive integer k_0, and $\mathcal{A}$ the ring of p-adic analytic functions over D. Evaluation at any integer k in D induces a specialization map $\mathcal{A} \to C_p$. A **p-adic family of modular eigenforms** parametrized by D is a formal q-expansion

$$f(q) = \sum_n a_n q^n \in \mathcal{A}[[q]]$$

such that for large enough integers k in D the specialization at k is the q-expansion of an eigenform of weight k. Let's assume further that all eigenforms are of level $\Gamma_1(Np)$ where N is prime to p, called the tame level.

In 1986 Hida introduced families of **ordinary** modular forms, and p-adic analytic families of Galois representations attached to ordinary eigenforms. The slope of an ordinary eigenform is a p-adic unit. These are now called Hida families.

Around 1973 Katz also introduced the notion of an **overconvergent** modular form. Consider p-adic modular forms (limits of classical modular forms) as sections of structure sheaves on the modular curve, minus the superingular locus. If we extend to the supersingular elliptic curves which have a defined canonical subgroup, this excludes badly-behaved p-adic modular forms, and what is left are called overconvergent.

Overconvergent modular forms form p-adic Banach (or Frechet) spaces containing the finite-dimensional spaces of classical modular forms. Using this and the rigid analytic geometry of Tate, Coleman in 1996 proved the existence of families of overconvergent modular forms, which now have finite slope.

In 1998, Coleman and Mazur built the **eigencurve** $C_{p,N}$, parametrizing overconvergent eigenforms of tame level N. The modular forms vary p-adically, thus the weights κ do as well. The weight and the character are put together as the *weight-character*. They form the **weight space**,

$$W_{p,N}(\mathbb{C}_p) = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times, \mathbb{C}_p^\times)$$

As a rigid analytic variety, it is isomorphic to a finite disjoint union of p-adic open discs. The eigencurve then has the natural map

$$C_{p,N} \to W_{p,N}.$$

There are two constructions:

- by multiplicity one for GL_2, replace the eigenform f with its set of eigenvalues. For primes not dividing Np, associate the Galois representation attached to f; the remaining eigenvalues live in a finite dimensional affine space $\mathbb{A}$. Let X_p be the universal deformation space of tame level N modular residual Galois (pseudo)representations (there are finite many of these). Then we find

$$C_{p,N} \subset X_p \times \mathbb{A}$$

  the locus cut out by Fredholm determinants, or characteristic power series of compact Hecke operators. (Overconvergent modular forms have guarantee U_p is compact.) As a result, a p-adic family of Galois pseudorepresentations parametrized by the eigencurve corresponds at a point to an overconvergent modular form f specializing to the Galois representation ρ_f.

- Roughly, the systems of eigenvalues arising from a commutative algebra of operators T can be viewed as points in a space whose structure ring is T. Let $P_\kappa(x)$ be the characteristic power series of the compact operator U_p acting on $M_\kappa^\dagger$, the overconvergent modular forms of character κ. Let L_k be the fibre of the spectral curve over the weight-character κ, i.e., the zero-locus of $P_\kappa(x)$ in the rigid-analytic affine line. Its points correspond to inverses of non-zero eigenvalues of U_p acting on $M_\kappa^\dagger$. We may factorize

$$P_\kappa(x) = F(x)G(x)$$

  where $F(x)$ is a polynomial whose roots are finite subsets of these inverse eigenvalues. By Serre's p-adic Riesz theory, there is a corresponding decomposition

$$M_\kappa^\dagger = M_F \oplus M_G$$

  such that $F(U_p)$ is zero on M_F and invertible on M_G. Now let T_F be the algebra of operators of M_F induced by the Hecke algebra of $M_\kappa^\dagger$. It is the ring of functions on an affinoid rigid analytic space $C_{\kappa,F}$. Varying F, glue together various $C_{\kappa,F}$ along L_κ to get C_κ, it is the fibre of $C_{p,N}$ over κ.

This is the picture for $GL_2(\mathbb{Q})$. More recently there are further constructions of so-called eigenvarieties: Buzzard's so-called eigenvariety machine (2007) and Urban's construction for general reductive groups (2011).

7.6 p-adic Galois representations and p-adic differential equations (Kramer-Miller)

7.6.1. Classical and mod p Riemann-Hilbert correspondence Let X be Riemann surface. The Riemann-Hilbert correspondence is the following equivalence of categories:

$$\{\text{complex representations of } \pi_1(X)\} \leftrightarrow \{\text{Differential equations on } X\} \leftrightarrow \{\text{Complex local systems on } X\}$$

where by a differential equation on X we mean a vector bundle on X with a connection ∇. Here's an example: view $\mathbb{G}_m(\mathbb{C})$ as $\mathbb{P}^1_{\mathbb{C}} - \{0, \infty\}$, and the vector bundle $\mathbb{C}^\times \times \mathbb{C}^2$. If we define a connection by

$$\nabla \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \frac{dz}{z}$$

and set $f'(z) = \frac{1}{z}$, then the solutions to

$$\nabla \begin{pmatrix} f(z)e_1 \\ g(z)e_2 \end{pmatrix} = \begin{pmatrix} f'(z)e_1 \\ g'(z)e_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f(z)e_1 \\ g(z)e_2 \end{pmatrix} \frac{1}{z} = 0$$

are $(0,1)$ and $(1, \ln z)$. From this we get a representation of the fundamental group $\pi_1(\mathbb{G}_m)$ in $GL_2(\mathbb{C})$ by sending the generator γ to

$$\gamma \mapsto \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}$$

Now let's talk about the étale fundamental group. Its definition is motivated by the viewing the classical fundamental group as deck transformations of the universal cover. Unfortunately, the universal covering map is not always algebraic, for example

$$\exp : \mathbb{G}_a = \mathbb{C} \to \mathbb{G}_m = \mathbb{C}^\times$$

So we look at covers that give finite quotients of the fundamental group, for example the n power map

$$x^n : \mathbb{C}^\times \to \mathbb{C}^\times.$$

We also impose the condition that these covers be étale maps, which roughly means that the inverse function theorem holds. That is, a covering

$$\pi : X' \to X$$

is étale if there is a map on tangent spaces $T_{\pi(x)} \to T_x$ for $x \in X'$.

Now if we consider a (smooth, projective..) algebraic curve X over $\mathbb{F}_p$, there is an equivalence between

$$\{\text{continuous } \mathbb{Q}_l\text{-representations of } \pi_1(X)\} \leftrightarrow \{\text{Étale local systems on } X\}$$

Moreover, if $l = p$, these are equivalent to the category of p-adic differential equations, which are known as F-isocrystals. To talk about these, we first point out that there is a lift of $X/\mathbb{F}_p$ to a model $\mathcal{X}/\mathbb{Z}_p$. Then an F-isocrystal is vector bundle on $\mathcal{X}$ with a connection and Frobenius operator F.

Back to our example, we now look at $\mathbb{G}_m(\mathbb{F}_p) = \mathbb{F}_p^\times$, where in honesty it is $\mathrm{Spec}(\mathbb{F}_p[t, t^{-1}]$, and we define a cover $\mathrm{Spec}(\mathbb{F}_p[u, u^{-1}])$ where $u^{p-1} = t$, giving

$$x^{p-1} : \mathbb{G}_m \to \mathbb{G}_m.$$

Then $\pi_1^{\mathrm{et}}(X)$ surjects on to the deck transformations of this cover, which is $\mathbb{Z}/(p-1)\mathbb{Z}$. Sending the generator to the root of unity ζ_{p-1} in $\mathbb{Q}_p^\times$ viewed as $GL_1(\mathbb{Q}_p)$, we get a representation of the étale fundamental group.

Consider Laurent series in an annulus D around $z = 0$. Define the ring

$$\mathcal{O}_{\mathcal{E}} = \{ \sum_{n=-\infty}^{\infty} a_n y^n : a_n \in \mathbb{Z}_p, \lim_{n \to -\infty} |a_n|_p = 0\}$$

giving the local field $\mathcal{E} := \mathcal{O}_{\mathcal{E}}[\frac{1}{p}]$, and such that $\mathcal{O}_{\mathcal{E}}/p = \mathbb{F}_p((y))$. Define a Frobenius-type operator

$$\sigma : \mathcal{O}_{\mathcal{E}} \to \mathcal{O}_{\mathcal{E}}$$

sending $y \mapsto y^p$. This is a lift of the Frobenius on $\mathbb{F}_p((y))$. Also the $\mathbb{Z}_p$-linear derivative

$$\frac{d}{dy} : \mathcal{O}_{\mathcal{E}} \to \mathcal{O}_{\mathcal{E}}$$

that acts as the usual derivative on y^n.

Now look at the $\tilde{\mathcal{E}}^{\mathrm{unr}}$, the p-adic completion of the maximal unramified extension of $\mathcal{E}$. An unramified extension is determined by the extension of their residue fields. In particular the action of the Galois group of $\mathbb{F}_p((y))^{\mathrm{alg}}$ over $\mathbb{F}_p((y))$ acts on $\tilde{\mathcal{E}}^{\mathrm{unr}}$. We get a map from this Galois group to $\pi_1^{\mathrm{et}}(\mathbb{F}_p^\times)$ to $GL_1(\mathbb{Q}_p)$.

7.6.2. F-isocrystals with infinite monodromy Let k be a finite extension of $\mathbb{F}_p$, and X a complete curve over k, U an open affine curve in X. Define $\mathfrak{X}$ to be the formal scheme over $W(k)$, with $\mathfrak{X} \times k = X$, with a lift $\mathfrak{U} \subset \mathfrak{X}$ of U. Also, set $K = W(k)[\frac{1}{p}]$, and define the rigid analytic fiber $\mathcal{X}^{\mathrm{an}} = \mathfrak{X} \times K$ and $\mathcal{U}^{\mathrm{an}} = \mathfrak{U} \times K$.

Theorem 7.6.1 (Tsuzuki)**.** *An F-isocrystal (M, ∇, φ) is overconvergent iff $\rho(I_x)$ is finite for all $x \in X - U$.*

Example 7.6.2. (Igusa) Let $p \not| N$. Consider the ordinary part of the universal elliptic curve E over $X_1(N)$ minus its super singular points, over $\mathbb{F}_p$. Then $R^1\pi_*\mathbb{Z}_p$ has infinite monodromy. In particular, one has for x a supersingular points

$$\rho : \pi_1(X_1(N) - \{\text{s.s.}\}) \to \mathbb{Z}_p^\times$$

such that $\rho(I_x) = \mathbb{Z}_p^\times$.

We want to study F-isocrystals that arise geometrically, so let's look at a family of curves $f : Y \to U$ and the right derived functors of the pushforward of the constant sheaf $R^i f_*(\mathbb{Z}_p)$.

Fontaine showed an equivalence of

$$\{p\text{-adic representations of } \mathrm{Gal}(\mathbb{F}_p((t))^{\mathrm{alg}}/\mathbb{F}_p((t)))\} \leftrightarrow \{\text{modules over } \mathcal{E} \text{ with } \nabla, \varphi\}$$

Here $\mathcal{E}$ is called the Fontaine ring. We have an inclusion, respectively, of

$$\{p\text{-adic representations of } \pi_1(U)\} \leftrightarrow \{F\text{-isocrystals}\}.$$

Now we define an overconvergent version of Fontaine's ring,

$$\mathcal{O}_{\mathcal{E}^\dagger} = \{\sum_{n=-\infty}^{\infty} a_n y^n : a_n \in \mathbb{Z}_p, v_p(a_n) \text{ grows linearly in } -n\}$$

where previously for $\mathcal{O}_{\mathcal{E}}$ we had $v_p(a_n) \gg -\infty$ as $n \to -\infty$. Then we may interpret Tsuzuki's work as

$$\{p\text{-adic representations } \rho \text{ of } \mathrm{Gal}(\mathbb{F}_p((T))^{\mathrm{alg}}) \text{ with } \rho(I) \text{ finite}\} \leftrightarrow \{\text{Modules over } \mathcal{E}^\dagger \text{ with } \nabla, \varphi\}$$

Now consider a finite extension F of $\mathbb{Q}_p$ with ramification index e, and a representation $\rho : G_F \to GL_d(\mathbb{Z}_p)$, and let G be the image of $\rho(G_F)$. We have two filtrations, the Lie filtration

$$G \supset G(1) \supset G(2) \supset \dots$$

where $G(n)$ is defined to be kernel of $\rho_n : G \to GL_d(\mathbb{Z}_p/p^n)$. There is also a ramification filtration with an upped index G^r for any real number r.

Theorem 7.6.3 (Sen)**.** *There exists some $c > 0$ such that $G^{en-c} \subset G(n) \subset G^{en+c}$.*

This tells us that the two filtrations are somewhat close to each other, i.e., it grows linearly. We want to study a similar phenomenon in our setting.

Now let $F = \mathbb{F}_p((T))$, and fix $r > 0$. Define the subcategory of representations $\mathrm{Rep}^r(G_F)$ to be the representations ρ of G_F such that if $G = \mathrm{Im}(\rho)$ there exists $c > 0$ such that

$$G(n) \supset G^{p^{rn}+c}.$$

In particular, we have an exponential growth in how $G(n)$ fits into the ramification filtration. Now define a new subring $\mathcal{O}_{\mathcal{E}^r}$ to be the power series in $\mathcal{O}_{\mathcal{E}}$ with $v_p(a_n) + \log(-n) \gg -\infty$ for $n > 0$. Note that overconvergent power series are contained in these.

Then the key result is the following:

Theorem 7.6.4 (K-M)**.** *There is an equivalence*

$$Rep^r_{G_F} \leftrightarrow \{\text{Modules over } \mathcal{E}^r \text{ with } \nabla, \varphi\}$$

and the two categories include into Fontaine's equivalence.

8 Motives

8.1 Grothendieck motives

8.1.1. Rational cohomology. In Milne's survey article *Motives: Grothendieck's dream*, he translates an excerpt of Grothendieck's introduction to *Récoltes et Semailles*:

> Among all the mathematical things that I have had the privilege discover and bring to the light of day, the reality of motives still appears to me the most fascinating, the most charged with mystery—at the heart even of the most profound identity between 'geometry' and 'arithmetic.' The 'yoga of motives'... is perhaps the most powerful instrument of discovery found by me during the first period of my life as a mathematician.

What are these mysterious 'motives', or 'motifs'? First, a prototype: let X be a compact manifold. Attached to it are the cohomology groups $H^i(X, \mathbb{Q})$ for $i = 0, \ldots, 2\dim X$, which are finite-dimensional $\mathbb{Q}$ vector spaces. If X is complex analytic, then it also has the de Rham cohomology groups $H^i_{\mathrm{dR}}(X)$. Over $\mathbb{C}$, we have

$$H^i_{\mathrm{dR}}(X) \simeq H^i(X, \mathbb{Q}) \otimes \mathbb{C}$$

our first isomorphism theorem. Further, if X is Kähler there is a Hodge decomposition of H_{dR}.

Now, let X be a nonsingular projective variety over an algebraically closed field k. For each prime l not dividing the characteristic of k, there are étale cohomology groups $H^i(X, \mathbb{Q}_l)$ for $i = 0, \ldots, \dim X$, which are finite-dimensional $\mathbb{Q}_l$ vector spaces. Here one also has an algebraic de Rham cohomology over k. In characteristic $p \neq 0$, there is crystalline cohomology instead.

Given an embedding $k \hookrightarrow \mathbb{C}$, we get a complex manifold $X(\mathbb{C})$ such that

$$H^i_{\mathrm{dR}}(X) \otimes_k \mathbb{C} \simeq H^i(X, \mathbb{Q}) \otimes \mathbb{C} \text{ and } H^i_{\text{ét}}(X, \mathbb{Q}_l) \simeq H^i(X, \mathbb{Q}) \otimes \mathbb{Q}_l$$

However, since X is defined by a finite set polynomials, it has a model over k_0 over which k is an infinite extension, and the choice of a model defines an action of $\mathrm{Gal}(k/k_0)$ on $H^i_{\text{ét}}(X, \mathbb{Q}_l)$. If different embeddings of k into $\mathbb{C}$ over k_0 gave the same subspace $H^i(X(\mathbb{C}), \mathbb{Q})$ of $H^i_{\text{ét}}(X, \mathbb{Q}_l)$, then the action of $\mathrm{Gal}(k/k_0)$ would stabilize $H^i(X(\mathbb{C}), \mathbb{Q})$. But infinite Galois groups are uncountable while $H^i(X(\mathbb{C}), \mathbb{Q})$ is countable, implying that the Galois group acts through a finite quotient. This is false in general.

So there is no $\mathbb{Q}$ cohomology theory underlying these various cohomology theories, even though they seem to behave as it they are defined over $\mathbb{Q}$. Furthermore, Weil believed there should exist a cohomology theory satisfying properties like Poincaré duality, Künneth isomorphism, hard and weak Lefshetz. Grothendieck showed that there existed many Weil cohomology theories, and that there should a universal one, thus motives.

8.1.2. Algebraic cycles. Let X be a nonsingular projective variety of dimension n over a field k. A *prime cycle* on X is a closed algebraic subvariety $Z \subset X$ that can not be written as a union of two proper, closed algebraic subvarieties. Given prime cycles Z_1, Z_2, then

$$\mathrm{codim}(Z_1 \cap Z_2) \leq \mathrm{codim}(Z_1) + \mathrm{codim}(Z_2).$$

When equality holds we call it a *proper intersection*. Examples of a bad intersection are parallel lines, or a plane containing a line, as moving either would make a proper intersection.

Denote $Z^r(X)$ the group of codimension r algebraic cycles on X, the free abelian group generated by codimension r prime cycles. Proper intersection defines an intersection product $\cdot$, hence a partial map from $Z^r(X) \times Z^s(X)$ to $Z^{r+s}(X)$. To get a map defined on the whole set, we must be able to move cycles. Call two cycles γ, γ' *rationally equivalent* if there is an algebraic cycle on $X \times \mathbb{P}^1$ having γ and γ' as the fibre over a point of $\mathbb{P}^1$ and 0 as the fibre over a second point. Any two algebraic cycles are rationally equivalent to algebraic cycles intersecting properly. Passing to equivalence we get our (bi-additive) map

$$Z^r_{\mathrm{rat}}(X) \times Z^s_{\mathrm{rat}}(X) \to Z^{r+s}_{\mathrm{rat}}(X)$$

and the *Chow ring* $CH^*_{\text{rat}}(X) = \oplus^{\dim X} Z^r_{\text{rat}}(X)$. Alternatively, two algebraic cycles are *numerically equivalent* if $\gamma \cdot \delta = \gamma' \cdot \gamma$ for all algebraic cycle δ of complementary dimension for which the intersection numbers are defined.

We want cohomology theories that are contravariant functors, i.e., a regular map $f : X \to Y$ induces $H^i(f) : H^i(Y) \to H^i(X)$. There are few regular maps of varieties, so one looks for 'many valued maps', or correspondences. Define the *group of correspondences* of degree r from X t oY

$$\text{Corr}^r(X, Y) = Z^{\dim X + r}(X \times Y).$$

For example the graph of f lies in $\text{Corr}^{\dim X}(X, Y)$, and its transpose ${}^t f$ in $\text{Corr}^{\dim X}(Y, X)$. A degree 0 correspondence γ from X to Y defines a homomorphism $H^*(X) \to H^*(Y)$ by

$$x \mapsto \text{pr}_{Y*}(\text{pr}^*_X x \cup \text{cl}(\gamma))$$

where cl is the cycle class map sending intersection products to cup products

$$\text{cl} : C^*_{\text{rat}}(X) \otimes \mathbb{Q} \to H^*(X)$$

8.1.3. Grothendieck motives. Grothendieck's idea was that there should be a universal cohomology theory with values in a $\mathbb{Q}$-category of motives: it should be an abelian, moreover, Tannakian category; Homs should be $\mathbb{Q}$-vector spaces; and every Weil cohomology theory should factor through it.

Let's start with a category with one object hX for every nonsingular projective X over k, with morphisms given by

$$\text{Hom}(hX, hY) = \text{Corr}^0_\sim(X, Y) \otimes \mathbb{Q}$$

To get an abelian category, we add the images of idempotents in $\text{End}(hX) := \text{Corr}^0_\sim(X, X) \otimes \mathbb{Q}$. Define the category of **effective pure motives** $M^{\text{eff}}_\sim(k)_\mathbb{Q}$ (with $\sim$ equivalence and $\mathbb{Q}$ coefficients) to be the category with objects (X, e) and morphisms $e' \circ \text{Corr}^0_\sim(X, Y)) \otimes \mathbb{Q} \circ e$.

Example 1: Let $x : \text{Spec}(k) \to X$ be a k-point and $p : X \to \text{Spec}(k)$ be the structure morphism. The decomposition of id into orthogonal idempotents $p^* x^* + (\text{id} - p^* x^*)$ gives

$$h(X, \text{id}) = h(\text{Spec}(k)) \oplus \tilde{h}(X)$$

where $\tilde{h}(X)$ is called the reduced motive of the pointed variety (X, x).

Example 2: Let $X = \mathbb{P}^1$. The diagonal map Δ_X decomposes into $(1, 0) + (0, 1)$, represented by $\{0\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{0\}$, giving the decomposition

$$h(\mathbb{P}^1, \Delta_X) = h(\mathbb{P}^1, (1, 0)) \oplus h(\mathbb{P}^1, (0, 1)) := \mathbb{I} \oplus \mathbb{L}$$

Finally, for a category with duals we invert the *Lefshetz motive* $\mathbb{L} := \mathbb{I}(-1)$ and introduce integers r, and call the category $M_\sim(k)_\mathbb{Q}$ of **pure motives** whose objects are triples (X, e, r), or $eh(X)(r)$, and morphisms $e' \circ \text{Corr}^{r-r'}_\sim(X, Y)) \otimes \mathbb{Q} \circ e$. We call $M_{\text{rat}}(k)_\mathbb{Q}$ *Chow motives* and $M_{\text{num}}(k)_\mathbb{Q}$ *Grothendieck* or *numerical motives*. Also, $\mathbb{I}(r)$ are called *Tate motives*.

8.2 The Standard Conjectures (Ray Hoobler)

Strategy: Let Var_k be the category of smooth projective varieties over an algebraically closed field k. We want to construct a universal object for cohomology theories. They should form a Tannakian category $\mathcal{C}$, i.e., an abelian category with

1. A bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and identity $\mathbf{1}$
2. It should be associative, commutative with unit, and a rigid category, i.e., objects posses duals $(-)^\vee$
3. Fibre functor: a faithful exact $\otimes, ^\vee$ preserving functor to finite dimensional vector spaces over a field k. Example: cohomology theories like Betti, de Rham, étale, crystalline cohomology.

Let X be a smooth projective variety over k. Let $Z(X)$ be the free abelian group generated by irreducible closed subvarieties of X, graded by codimension, e.g., in codimension 1 we have divisors. Define *adequate* equivalence relations $\sim$ allowing equivalent classes of cycles to intersect properly, e.g., numerical, homological, rational equivalence, and denote $Z_\sim(X) := Z(X)/\sim$.

1. Alter the maps in Var_k to correspondences from X to Y: given Z in $Z_\sim^{\dim X + r}(X \times Y)$, then

$$\mathrm{pr}_{13,*}(\mathrm{pr}_{12}^*(C_{X,Y}) \cdot \mathrm{pr}_{23}^*(C_{Y,Z})) \in Z_\sim^{\dim X + r + s}(X, Z)$$

where $C_{X,Y}$ is a codimension r cycle and $C_{Y,Z}$ is a codimension s cycle.

Weil cohomology: is a contravariant functor $\mathrm{Var}_k^{\mathrm{op}} \to$ graded k-algebras, $X \mapsto H^*(X)$ satisfying

(a) Künneth formula:
$$H^*(X) \otimes_k H^*(Y) \xrightarrow{\sim} H^*(X \times Y)$$

(b) Poincaré duality: $\mathrm{Tr}_X : H^{2d}(X) \xrightarrow{\sim} k$ giving a nonsingular cup product
$$H^i(X) \times H^{2d-i} \to H^{2d} \xrightarrow{\mathrm{Tr}} k$$
One shows that Künneth formula + Poincaré duality = duality between $H^*(X)$ and $H^*(Y)$.

(c) Cycle-class map:
$$\gamma_X : Z^r(X) \to H^{2r}(X)$$
compatible with intersection product and $\gamma_X(\mathrm{pt})$ generates $\mathbb{Z} \xrightarrow{\mathrm{Tr}} k$. This map $\gamma_X : Z(X) \to H(X)$ factors through $Z_\sim(X) \otimes k$. Call $A(X) = \mathrm{Im}(Z_\sim(X) \otimes k)$.

(d) Weak Lefschetz: if $W \subset X$ is a hyperplane intersection then $H^i(X) \to H^i(W)$ is 0 if $i \le d-2$ and injective if $i = d-1$.

(e) Strong Lefschetz: the ample line bundle $L := \gamma_X(W)$ induces $L^{d-i} : H^i(X) \xrightarrow{\sim} H^{2d-i}(X)$, $i \le d$.

The commutative diagram

$$\begin{array}{ccc} H^i(X) & \xrightarrow{L^{d-i}} & H^{2d-i}(X) \\ \downarrow L & & \downarrow L \\ H^{i-2}(X) & \xrightarrow{L^{d-i+2}} & H^{2d-i+2}(X) \end{array}$$

[the left vertical arrow should be pointing up..] implies

$$H^i(X) = LH^{i-2}(X) \oplus P^i(X) = \bigoplus_{j \ge \max(o, i-d)} L^j P^{i-2j}$$

$P^i(X)$ are called the primitive classes.

2. The **Standard Conjectures**:

(a) $B(X)$: the section Λ of L is algebraic

(b) $A(X, L) : L^{r-2i} A^i(X) \xrightarrow{\sim} A^{d-i}(X)$

(c) $C(X)$: the projector $\pi^i : H^*(X) \to H^i(X) \to H^*(X)$ for $0 \le i \le 2d$ is algebraic.

(d) $I(X) : \mathrm{Hdg}(X) : (x, y) \mapsto (-1)^i \langle Lx, y \rangle$ is positive definite on $A^i(X) \cap P^i(X)$. (This is the Hodge index theorem in the complex case, proved by Grothendieck in characteristic p.)

(e) $D(X)$: Numerical equivalence = homological equivalence. Z_1, Z_2 are homologically equivalent if $\gamma_X(Z_1) = \gamma_X(Z_2)$, numerically equivalent if $\langle Z_1 - Z_2, y \rangle = 0$ for all y of appropriate codimension. It is known that hom implies num.

3. Now alter $\mathrm{Var}_{\sim,k}$ by adding images of idempotents to get $M_\sim^{\mathrm{eff}}$ the category of effective motives, whose objets are (X, e) where e is a correspondence $Z_\sim^d(X \times X)$ with $e^2 = e$.

   Example: Given $\phi^2 = \phi : V \to V$, then $V = \ker\phi \oplus \phi(V)$, and $\mathrm{Hom}_k(eV, fW) = f \circ \mathrm{Hom}_k(V, W) \circ e$.

   This is the pseudo-abelianization of $\mathrm{Var}_{\sim,k}$. Jannsen showed that $M_\sim^{\mathrm{eff}}$ is a semisimple category.

   Known:

   (a) $A(X \times X, L \otimes 1 + 1 \otimes L) \Rightarrow B(X) \Rightarrow A(X, L)$ for any $L + C(X)$
   Corollary: $A(X, L)$ holds for all $X, L \Leftrightarrow B(X)$ for all X.

   (b) $B(X), B(Y) \Rightarrow B(X \times Y) + B(W)$ for some hyperplane W.
   Corollary: $A(X \times X, L \otimes 1 + 1 \otimes L) \Leftrightarrow B(X) \Leftrightarrow B(\times X)$.

   (c) Riemann hypothesis for varieties $\Rightarrow C(X)$ if X is defined over $\mathbb{F}_q$.

   (d) $D(X) \Rightarrow A(X, L)$; $D(X) + I(X) \Leftrightarrow A(X, L)$

   (e) $I(X) + B(X) \Rightarrow C(X)$

Note: this construction gives pure motives. One gets mixed motives by considering also singular projective varieties.

8.3 Aside I: Fundamental groups and Galois groups

8.3.1. The topological fundamental group. I'll assume familiarity with the fundamental group from topology. Let X be a connected, locally simply connected topological space with base point x, and $p : Y \to X$ a topological cover of X. Define the fibre functor Fib_x from the category of covers of X to the category of of sets $p^{-1}(x)$, equipped with the monodromy action of $\pi_1(X, x)$. We have the following theorem.

Theorem 8.3.1. *Let X be as above. Then Fib_x is an equivalence of categories, and $\pi_1(X, x) \simeq \mathrm{Aut}(\mathrm{Fib}_x)$.*

There are further analogies with the topological theory that are enlightening, e.g., Galois covers and the Riemann-Hilbert correspondence, but we will move onwards to:

8.3.2. The étale fundamental group. A finite dimensional k-algebra A is *étale over k* if it is isomorphic to a finite direct product of separable extensions of k. Grothendieck's reformulation of the main theorem of Galois theory is the following:

Theorem 8.3.2. *The category of finite étale k-algebras to and the category of finite sets $\mathrm{Hom}_k(-, k^s)$ with left Gal(k^s/k)-action are anti-equivalent.*

Now we introduce some definitions. Let $\phi : X \to S$ be a morphism of schemes, and s in S. Then

1. ϕ is *flat* if the direct image sheaf $\phi_* \mathcal{O}_X(U) := O_X(\phi^{-1}(U))$ is a sheaf of flat $\mathcal{O}_S$-modules.
2. ϕ is *locally of finite type* if S has an affine open covering by $\mathrm{Spec}(A_i)$ so that $\phi^{-1}(\mathrm{Spec}(A_i))$ has an open covering by $\mathrm{Spec}(B_{ij})$, where B_{ij} are finitely generated A_i-modules. (Think compact fibers.)
3. The *fibre of ϕ at s* is the fibre product $X \times_S \mathrm{Spec}\, \kappa(s)$, where $\kappa(s) = A_s/sA_s$ for some affine open $\mathrm{Spec}(A) \subset S$ containing s. The resulting $\mathrm{Spec}(\kappa) \to \mathrm{Spec}(A) \hookrightarrow S$ is independent of choice of A.

Definition 8.3.3. An étale morphism ϕ is a flat, locally of finite type morphism where the fibre over each point has a covering by spectra of finite étale algebras. If ϕ is finite, étale, and surjective, then it is a finite étale cover.

Over algebraically closed fields these are precisely those that induces an isomorphism on tangent spaces.

Example 8.3.4. Let's check that $\phi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ where $B = A[x]/(f)$ with f separable of degree d, is a finite étale cover. A finitely generated module over a local ring is free if and only if it is flat. B is a finitely generated A-module, so ϕ is finite and flat. For $s \in S$, the fibre at s is the spectrum of $B \times_A \kappa(s) \simeq \kappa(s)[x]/(f)$. Finally, f has distinct roots so the fiber is a finite étale $k(s)$-algebra.

Now another set of definitions:

1. A *geometric point* s of S is a morphism $s : \mathrm{Spec}(\Omega) \to S$, where Ω is an algebraically closed field. Its image is a point s (by abuse of notation) in S such that Ω is a closure of $\kappa(s)$.
2. Call Fet_S the category of finite étale covers of S. Given such a cover $X \to S$ and a geometric point s, denote the underlying set of the geometric fiber $X \times_S \mathrm{Spec}(\Omega)$ over s by $\mathrm{Fib}_s(X)$. A morphism $X \to Y$ in Fet_S induces a morphism of the fibres, whence a set-map $\mathrm{Fib}_s(X) \to \mathrm{Fib}_s(Y)$. The set-valued functor Fib_s on Fet_S is called the *fibre functor*.
3. Given a functor F between two categories, an *automorphism* of F is a morphism of functors $F \to F$ with a two-sided inverse. Given C a source object and ϕ an automorphism, ϕ induces a morphism $F(C) \to F(C)$; giving a natural left action of $\mathrm{Aut}(F)$ on $F(C)$ for set-valued F.

Definition 8.3.5. Let s be a geometric point of S. The étale fundamental group of a scheme is the automorphism group of the fibre functor on Fet_S, denoted $\pi_1(X, x) := \mathrm{Aut}(\mathrm{Fib}_s)$.

Example 8.3.6. We claim $\pi_1(S,s) \simeq \mathrm{Gal}(k^s/s)$ when $S = \mathrm{Spec}(k)$. We'll show that $\mathrm{Fib}_s(X) \simeq \mathrm{Hom}_k(L,k^s)$ for all $X = \mathrm{Spec}(L)$. A finite étale S-scheme X is the spectrum of a finite étale k-algebra. Given a geometric point s, Fib_s sends a connected cover $X = \mathrm{Spec}(L)$ to the underlying set of $\mathrm{Spec}(L \times_k \Omega)$, a finite set indexed by k-algebra homomorphism $L \to \Omega$.

The main theorem by Grothendieck is the following:

Theorem 8.3.7. *Let S be a connected.* Fib_s *is an equivalence between* Fet_S *and the category of finite sets with continuous left $\pi_1(S,s)$-action.*

One also has the following comparison theorem:

Theorem 8.3.8. *Let X be a connected scheme of finite type over $\mathbb{C}$. There is an equivalence between the category of finite stale covers of X and finite topological covers of the complex analytic space X^{an}. Then for every $\mathbb{C}$-point $x : \mathrm{Spec}(\mathbb{C}) \to X$, there is an isomorphism relating the profinite completion*

$$\widehat{\pi_1^{\mathrm{top}}(X^{\mathrm{an}}}, x) \simeq \pi_1^{\text{ét}}(X,x).$$

8.3.3. The Tannakian fundamental group. A *(neutral) Tannakian category* is a rigid k-linear abelian tensor category with $\mathrm{End}(1) \simeq k$, together with a faithful, exact, tensor functor ω, called the *fibre functor* (into the category of finite dimensional vector spaces over k.

Theorem 8.3.9. *(Deligne) Let $(\mathcal{C},\omega)$ be a neutral Tannakian category over k. Then $Aut^{\otimes}\omega$ is an affine, flat k-group scheme, and $(\mathcal{C},\omega)$ is equivalent to the category of finite dimensional representations* $\mathrm{Rep}_k(\mathrm{Aut}^{\otimes}\omega)$.

$\mathrm{Aut}^{\otimes}\omega$ is called the *Tannakian fundamental group* of $(\mathcal{C},\omega)$.

References

Szamuely, T., *Galois groups and fundamental groups* (2009)
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8.4 Aside II: Tannakian categories

8.4.1. Categorical definitions

Definition 8.4.1. An *R-linear* category is a category C such that $\mathrm{Hom}(X,Y)$ is an R-module for every X, Y in C, and R is a given commutative ring.

Definition 8.4.2. A *tensor category* is an R-linear category C equipped with

1. an R-bilinear functor $\otimes : C \times C \to C$,
2. functorial isomorphisms

$$a : X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z \qquad \text{(associativity constraint)}$$

   satisfying a certain pentagon axiom for all X, Y, Z in C, and

$$c : X \otimes Y \xrightarrow{\sim} Y \otimes X \qquad \text{(commutativity constraint)}$$

   satisfying a certain hexagon axiom, and

3. an identity object U with an isomorphism $u : U \xrightarrow{\sim} U \otimes U$ such that $U \otimes -$ is an equivalence of categories.

NB: a tensor category is sometimes known as a symmetric monodical category with identity.
Let $(C, \otimes)$ be a tensor category.

- An *invertible* object is an object L in C such that $L \otimes -$ is an equivalence of categories. In particular, there is an L' in C such that $L \otimes L' = 1$.
- The *internal hom*: if the functor $\mathrm{Hom}(- \otimes X, Y)$ is representable, its representing object denoted by $\underline{\mathrm{Hom}}(X,Y)$. The existence of internal hom induces
  1. *dualilty* by $\underline{\mathrm{Hom}}(-,1) := (-)^\vee$,
  2. *evaluation* by $\underline{\mathrm{Hom}}(-,Y) \otimes (-) \to Y$
  3. a *reflexive* object is an object X such that the morphism $X \mapsto X^{\vee\vee}$ is an isomorphism.
  4. An *additive* (resp. *abelian*) tensor category is one where C is additive (resp. abelian) and $\otimes$ is bi-additive.

Example 8.4.3. $\mathrm{Rep}_F(G)$, the category of finite-dimensional F-representations of a group with the usual $\otimes$ is the prototypical example. 1 is given by the trivial representation, and $(-)\vee$ is the contragredient representation.

Example 8.4.4. $\mathrm{Vec}_F^{\mathbb{Z}}$, the category of $\mathbb{Z}$-graded finite-dimensional F-vector spaces is a tensor category. Use the Koszul rule for the commutativity constraint

$$\tilde{c} = \oplus_{r,s}(-1)^{rs} c^{r,s} : V \otimes W \xrightarrow{\simeq} W \otimes V$$

where $c^{r,s} : V^r \otimes W^s \to W^s \otimes V^r$. This condition has as a consequence $\mathrm{rank}(V) = \dim V^{2k} - \dim V^{2k+1}$.

Definition 8.4.5. A *rigid* tensor category is a tensor category where all objects are reflexive and there exist internal homs such that

$$\mathrm{Hom}(X_1, Y_1) \otimes \mathrm{Hom}(X_2, Y_2) \xrightarrow{\sim} \mathrm{Hom}(X_1 \otimes X_2, Y_1 \otimes Y_2)$$

for all X, Y in C. An alternative definition can be given by a certain coevaluation. In particular, all objects in C have duals.

Now let $(C, \otimes)$ be a rigid tensor category.

- The *trace morphism* is the morphism $\mathrm{Tr}_X : \mathrm{End}(X) \to \mathrm{End}(1)$ obtained by applying $\mathrm{Hom}(1,-)$ to the isomorphism $\underline{\mathrm{Hom}}(X,X) \to X^\vee \otimes X \to 1$.

- The *rank* of X is defined to be $\mathrm{Tr}_X(\mathrm{id}_X)$.
- A *tensor functor* between tensor categories is a functor F with a functorial isomorphism $F(X) \otimes F(Y) \to F(X \otimes Y)$ preserving the tensor structure.

Example 8.4.6. Mod_R, the category of finitely generated modules over R is an abelian tensor category with End(1)= R, but it is not rigid in general as not all R-modules are reflexive.

The following is the main feature of Tannakian categories:

Theorem 8.4.7. *Let $(C, \otimes)$ be a rigid abelian tensor category such that End(1)= F, together with $\omega : C \to Vec_F$, an exact, faithful, F-linear tensor functor. Then*

1. *the functor $\underline{Aut}^{\otimes}(\omega)$ of F-algebras is representable by an affine group scheme G.*
2. *the functor $C \to Rep_F(G)$ defined by ω is an equivalence of (tensor) categories.*

Definition 8.4.8. A *Tannakian* category is a rigid abelian tensor category (C, ω) with End(1)= F, where $\omega : C \to \mathrm{Vec}_K$ an exact, faithful, K-linear tensor functor called the fiber functor, for some finite extension K of F. We say C is *natural* if there exists a fiber functor over $K = F$, moreover it is *neutralized* if the ω is chosen.

Example 8.4.9. $\mathrm{Vec}_F^{\mathbb{Z}}$ has a fibre functor $(V^n) \mapsto \oplus V^n$. In this case $G = \mathbb{G}_m$, and (V^n) corresponds to the representation of $\mathbb{G}_m$ on V acting on V^n through the character $v \mapsto v^n$.

Example 8.4.10. $\mathrm{Hod}_{\mathbb{R}}$, the category of real Hodge structures, i.e., finite-dimensional $\mathbb{R}$-vector spaces V with the decomposition $V \otimes C = \oplus V p, q$. Forgetting the decomposition is a fibre functor, with $G = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$. A real Hodge structure corresponds to the representation of G on V where each $z \in G(\mathbb{R}) = \mathbb{C}^\times$ acts on $V^{p,q}$ through $z^p \bar{z}^q$.

More generally, a Tannakian category is a tensor category C with a fibre functor with values in Mod_S, the category of quasicoherent sheaves on an F-scheme. The category $\mathrm{Fib}_S(C)$ of S-valued fibre functors on C is fibered over Spec(F). Then Deligne showed that Fib(C) is an affine gerbe over Spec(F) in the fpqc topology, and C is equivalent to Rep(Fib(C)).

8.4.2. Numerical motives Let k be a field. We will consider smooth projective varieties X over k. Fix a characteristic zero field F, and an adequate equivalence relation $\sim$ on F-linear algebraic cycles (so proper pushforward, flat pullbacks, and intersection products are well-defined modulo $\sim$). Given a variety X and an integer $j \geq 0$, define

$$A^j(X) = \{\text{the group of } F\text{-linear algebraic cycles of codimension } j \text{ on } X\}/\sim$$

Following Jannsen we define the category M_k of (pure) motives over k by objects (X, p, m) where X is a variety, p is a projector ($p^2 = p$) in $A^{\dim(X)}(X \times X)$, and m is an integer; with morphisms

$$\mathrm{Hom}((X, p, m), (Y, q, n)) = qA^{\dim(X)-m+n}(X \times Y)p.$$

Uwe Jannsen proved the following celebrated result:

Theorem 8.4.11. *(Jannsen) The following are equivalent:*

1. *The category M_k of pure motives over k is a semisimple abelian category.*
2. *$A^{\dim(X)}(X \times X)$ is a finite-dimensional, semisimple F-algebra for every variety X.*
3. *The relation $\sim$ is numerical equivalence.*

The main question we will address is:

Are numerical motives Tannakian?

Lemma 8.4.12. *M_k is an F-linear semisimple abelian tensor category.*

Proof. We do this one by one:

- M_k is F-linear because we have chosen F-linear algebraic cycles,
- by construction it is also pseudo-abelian (idempotents have kernels) by the Karoubian completion; it is semisimple abelian by Jannsen,
- the tensor operation is given by

$$(X, p, m) \otimes (Y, q, n) := (X \times Y, p \times q, m + n)$$

  with commutativity and associativity constraints induced by isomorphisms of direct products. The unit object is $1 = (\mathrm{Spec}(k), id, 0)$ and End(1)$= F$. It is rigid, with internal hom given by

$$\underline{\mathrm{Hom}}((X, p, m), (Y, q, n)) = (X \times Y, {}^t p \times q, \dim(X) - m + n)$$

□

Then it only need a fibre functor with values in some extension of F to make it a Tannakian category. Deligne proves the following characterization:

Theorem 8.4.13. *(Deligne) Let T be a tensor category over k of characteristic 0. The following are equivalent:*

1. *T is Tannakian.*
2. *For all X in T, $\dim X$ is a nonnegative integer.*
3. *For all X in T, there exists a nonnegative integer such that $\lambda^n X = 0$.*

Unfortunately, because of this,

Lemma 8.4.14. *M_k is not a Tannakian category.*

Proof. The rank of a motive $h(X) = (X, \mathrm{id}, 0)$ is the Poincaré characteristic, $rk(h(X)) = \sum(-1)^i \dim H^i(X)$ which can be negative. But in the presence of a fibre functor this must be non-negative as the dimension of a vector space. In fact, one computes for the diagonal

$$rk(h(X)) = \langle \Delta \cdot {}^t\Delta \rangle = \sum_{i=0}^{2\dim X} (-1)^i \mathrm{Tr}(\Delta \circ \Delta)|_{H^i(X)}$$

which may be positive or negative. □

Nonetheless,

Corollary 8.4.15. *If the Künneth components of the diagonal (w.r.t. some fixed Weil cohomology $X \mapsto H(X)$) are algebraic for every variety X over k, then the category M_k of motives with respect to numerical equivalence is a semisimple F-linear Tannakian category.*

To understand this, we introduce the following

Definition 8.4.16. The *Künneth decomposition of the diagonal* is the identification of the diagonal

$$\gamma_{X \times X}(\Delta(X)) = \sum_{i=0}^{2d} \Delta_i^{\mathrm{topo}}(X) \in H^{2d}(X \times X) = \bigoplus_{i=0}^{2d} H^{2d-i}(X) \otimes H^i(X)$$

where the i-th Künneth component Δ_i^{topo} is a topological cycle class. This uses the graded isomorphism from the Künneth formula:

$$\mathrm{pr}_X^* \otimes \mathrm{pr}_Y^* : H(X) \otimes H(Y) \to H(X \times Y)$$

The Künneth components of the diagonal are called *algebraic* if the classes

$$\pi^i \in H^{2n-i}(X,\mathbb{Q}) \otimes H^i(X,\mathbb{Q}) \subset H^{2n}(X \times X,\mathbb{Q})$$

appearing in the Künneth decomposition of the diagonal are contained in the image of the cycle class map

$$\gamma_{X\times X} : A^i(X \times X) \to H^{2i}(X \times X).$$

The *Künneth conjecture* is that there exists algebraic cycles Δ_i such that $\gamma_{X\times X}(\Delta_i) = \Delta_i^{\text{topo}}$.

Definition 8.4.17. We say X admits a *Chow-Künneth decomposition* if there exists $p^i(X)$ in $A^d(X \times X)_{\mathbb{Q}} = \mathrm{Cor}^0(X,X)$ for $i = 0,1,\ldots,2d$ such that $\sum p^i(X) = \Delta(X)$, $\gamma_{X\times X}(p^i(X)) = \Delta_i^{\text{topo}}$, and

$$p^i(X)p^j(X) = \begin{cases} 0 & i \neq j \\ p_i(X) & i = j \end{cases}$$

In particular, the $p_i(X)$ are projectors lifting the Künneth components, orthogonal to each other and idempotent in $\mathrm{Cor}(X \times X)$. Warning: the Künneth component Δ_i^{top} is unique, but the the p^i are not unique as cycle classes.

The *Chow-Künneth* conjecture is that every smooth projective variety admits such a decomposition.

Now we are ready to prove the corollary.

Proof. Assume that for every variety X the Künneth components π^i of the diagonal are algebraic, which we identify with their images $p_i(X)$ in $A^{\dim X}_{\text{num}}(X \times X)$. The π_i are central idempotents, with π^i orthogonal to π^j for $i \neq j$. This gives every motive a $\mathbb{Z}$-grading by setting

$$(X,p,m)^r = (X, p\pi^{r+2m}, r)$$

which is respected by all morphisms. The commutativity constraint c can be written as $\oplus c^{r,s}$ as above, and modify it as before with the Koszul sign rule,

$$\tilde{c} = \oplus_{r,s}(-1)^{rs}c^{r,s} : M \otimes N \xrightarrow{\simeq} N \otimes M$$

Then now

$$rk((X,p,m)) = \sum_i \geq 0 \dim_A pH^i(X) \geq 0$$

for every motive (X,p,m) in M_k. By Deligne's theorem above M_k is Tannakian. □

Remark 8.4.18. The assumption of the corollary is satisfied if k is an algebraic extension of a finite field. This is proved by Katz and Messing using the Riemann hypothesis over finite fields, with H either crystalline cohomology or ℓ-adic cohomology with $\ell \neq \mathrm{char}(k)$. Here the Künneth conjecture is true but the Chow-Künneth is not known.

Remark 8.4.19. Moreover, the corollary holds for every (abelian tensor) subcategory of M_k generated by $1(1)$ and a family of varieties whose Künneth components of the diagonal are algebraic. In particular, the category of motives generated by abelian varieties is Tannakian over any field.

Example 8.4.20. Let C be a smooth projective curve with k-rational point p. Take the Künneth components

$$\begin{aligned}\pi^0 &= C \times p \\ \pi^1 &= \Delta(C) - (C \times p) - (p \times C) \\ \pi^2 &= p \times C\end{aligned}$$

giving the decomposition $h(X) = h^0(X) \oplus h^1(X) \oplus h^2(X)$, where $h^i(X) = (C, \pi^i, 0)$.

Example 8.4.21. Let S be a smooth projective irreducible surface with k-rational point p. Then we have π^0, π^4 as above, π^1, π^3 by the Picard and Albanese varieties respectively, and $\pi^2 = \Delta - \pi^0 - \pi^1 - \pi^3 - \pi^4$.

Example 8.4.22. If A is an abelian variety of dimension g, and a_n is the class of the graph of multiplication by n, then we see that in cohomology

$$a_n = \sum_{i=0}^{2g} n^i c_i$$

where c_i is the ith component of the *class* of the diagonal. Do this for $n = 1, 2, 3, ..., 2g, 2g+1$. Then the system of equations

$$a_1 = \sum_{i=0}^{2g} c_i, a_2 = \sum_{i=0}^{2g} 2^i c_i, \dots, a_{2g} = \sum_{i=0}^{2g} (2g)^i c_i, a_{2g+1} = \sum_{i=0}^{2g} (2g+1)^i c_i$$

can be solved for the c_i in terms of a_n (use a Vandermonde determinant to see this). Thus c_i is indeed the class of an algebraic cycle.

Remark 8.4.23. The Chow-Künneth conjecture implies the Künneth conjecture. The former is known for curves, surfaces, and abelian varieties as we have seen above, then also for uniruled threefolds, elliptic modular varieties, complete intersections in projective space, and Calabi-Yau threefolds.

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8.5 The motivic Galois group: examples

The category of pure motives is conjectured to be a Tannakian category. If so, we can associate to it the Tannakian fundamental group, in this setting also called the motivic Galois group.

8.5.1. Background on duality.

1. *Pontrjagin duality*: let G be an abelian locally compact topological group. Its character group $\widehat{G}$ is again locally compact, and there is a canonical continuous isomorphism from G to its bidual, so G is entirely determined by its character group.

   If G is a commutative algebraic group, the same holds for the Cartier dual in place of the character group, i.e., Pontrjagin dual.

2. *Tannaka-Krein duality*: let G be a noncommutative compact group. Its finite dimensional representations are unitary, and G may be recovered from the tensor category $\mathrm{Rep}(G)$ as the tensor-preserving automorphisms of the forgetful functor ω from $\mathrm{Rep}(G)$ to the underlying category of vector spaces.

3. *Neutral Tannakian duality*: let G be an affine group scheme. As before, it is isomorphic to $\mathrm{Aut}^{\otimes}(\omega)$, the natural transformations of the forgetful functor on $\mathrm{Rep}(G)$. The Krein-type theorem asserts that a neutralized Tannakian category $(\mathcal{C}, \omega)$ is equivalent to $\mathrm{Rep}(\mathrm{Aut}^{\otimes}(\omega))$.

   General Tannakian duality: if a fibre functor exists only for some extension of k, replace the group G of natural transformations with the gerbe of fibre functors. A k-valued fibre functor is a tensor functor from $\mathcal{C}$ to Vec_k that is faithful and exact.

Example 8.5.1. Consider the category of $\mathbb{Z}$-graded k-vector spaces. It is equivalent to $\mathrm{Rep}(\mathbb{G}_m)$, the representations of $\mathbb{G}_m$ in Vec_k. Define a grading on V where $\mathbb{G}_m$ acts on V^n thru the character $v \mapsto v^n$.

Example 8.5.2. Consider the category of real Hodge structures, ie., vector spaces V with an isomorphism $V \otimes \mathbb{C} \simeq \bigoplus V^{p,q}$ where $V^{p,q}$ and $V^{q,p}$ are complex conjugate. It is equivalent to $\mathrm{Rep}(S)$, where $S = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$. An element z in $S(\mathbb{R})) = \mathbb{C}^{\times}$ acts on the factor $V^{p,q}$ of $V \otimes \mathbb{C}$ by multiplication by $z^{-p}\bar{z}^{-q}$.

8.5.2. Tannakian duality for motives. Assuming Grothendieck's standard conjectures on algebraic cycles, Mot_k is Tannakian. The realization of any Weil cohomology H is a fibre functor with values in $\mathrm{Vec}_{\mathbb{Q}}$, and $G_M = \mathrm{Aut}^{\otimes}(H)$ is the **motivic Galois group** (or more commonly, G_{mot}. It is a linear pro-algebraic $\mathbb{Q}$-group. The realization H preserves the $\mathbb{Z}$-grading, giving a central cocharacter of weights:

$$w : \mathbb{G}_m \to G_M.$$

Also it turns out that $G_{M(\mathbb{I}(1))} = \mathbb{G}_m$, such that the composition $\mathbb{G}_m \to G_M \to \mathbb{G}_m$ is -2 in $\mathrm{End}(\mathbb{G}_m) = \mathbb{Z}$.

Let $M(E)$ be the smallest Tannakian subcategory containing the motive E. The Galois group $G_{M(E)}$ is a subgroup of $GL_E := GL(H_B(E))$, and $G_M = \varprojlim G_{M(E)}$. Note that for mixed motives (which we have not defined), the corresponding motivic Galois group is an extension of G_M by a pro-unipotent group (or gerbe). Now for some examples:

1. The trivial (Tate) motive $\mathbb{I}$ arising from h^0 of $\mathrm{Spec}(k)$. $G_{M(\mathbb{I})} = \{1\}$.
2. The Tate motive is the inverse of the Lefschetz motive $\mathbb{L} = h^2(\mathbb{P}^1)$. Again $G_{M(\mathbb{L})} = \mathbb{G}_m$.
3. Artin motives are motives over a characteristic zero field k defined by zero-dimensional varieties X. The finite set $X(\bar{k})$ has a continuous left Galois action, then associating to it $\mathbb{Q}^{|X(\bar{k}|}$ with the induced action of $\mathrm{Gal}(\bar{k}/k)$, we obtain an equivalence of categories with the finite dimensional representations of $\mathrm{Gal}(\bar{k}/k)$.

   Alternatively, define Artin motives to be those generated by $h(\mathrm{Spec}(K))$ for finite extensions K/k. They form a Tannakian category with Galois group $G_{M(h(\mathrm{Spec}(K))} = \mathrm{Gal}(\bar{k}/k)$.

4. Artin-Tate motives are the last two together, thus $G_{M(\dots)} = \mathrm{Gal}(\bar{k}/k) \times \mathbb{G}_m$
5. Let E be an elliptic curve without CM, viewed as a motive of weight -1 hence homological dimension 1: $G_{M(E)} = GL(H_B(E)) = GL_2$. (The Mumford-Tate group of (9) is a connected reductive subgroup of GL_2 containing homotheties: $\mathbb{G}_m$, the Cartan, or GL_2, and use $G_{M(E)} = MT(E_{\mathbb{C}})$.)

   Let E be an elliptic curve with CM over a imaginary quadratic field K. If the complex multiplication of E is defined over k, then $G_{M(E)} = T_K = \mathrm{Res}_{K/\mathbb{Q}} G_m$, the two dimensional torus over K. If not, then $G_{M(E)}$ is the normalizer N_K of T_K in GL_2.
6. Let A be an abelian variety of dimension n with $\mathrm{End}_{\bar{k}}(A) = \mathbb{Z}$. For n odd or $n = 2, 6$, $G_{M(A)} = GSp_{2n}$. (When $n = 4$, $G_{M(A)}$ is strictly contained in GSp_{2n}.)
7. The Ramanujan motive is the representation $\rho_l : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\bar{\mathbb{Q}}_l)$ associated to the modular form Δ. Let A be direct sums of Ramanujan motives and Tate motives, then $G_{M(A)}$ is the subgroup of $GL_2 \times \mathbb{G}_m$ of pairs (u, x) where $\det u = 11$.
8. Let k be a number field. Consider the neutral component $G^0_M = G^0_{M,k}$ defined by

$$1 \to G^0_{M,k} \to G_M \to \mathrm{Gal}(\bar{k}/k) \to 1$$

   and let $S = (G^0_M)^{\mathrm{ab}}$ be the maximal commutative quotient of G^0_M. This is Serre's pro-torus. Langlands defines the **Taniyama group** $T = (G_M)^{\mathrm{ab}}$, fitting into the exact sequence

$$1 \to S \to T \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1.$$

   The corresponding Tannakian category is that of potentially CM type $\mathbb{Q}$-motives.
9. Let E be a motive. The Hodge decomposition of $H_B(E) \otimes \mathbb{C}$ can be viewed as a homomorphism

$$h_E : \mathbb{G}_m \times \mathbb{G}_m \to GL_{E/\mathbb{C}}$$

   The **Mumford-Tate group** $MT(H_B(E)) = G^0_{M(E)}$ is the smallest algebraic subgroup of GL_E containing the image of h_E. It is linear and connected, hence easier to calculate than $G_{M(E)}$.

   For example, consider an abelian variety A over k with a fixed embedding of k in $\mathbb{C}$ and its natural Hodge structure on is $H^1(A(\mathbb{C}), \mathbb{Q})$. Then $MT(H^1(-)) = G^0_{M(A)}$.

8.5.3. ℓ-adic representations. Fix a prime l and a motive E over k. The l-adic realization of E carries a Galois action, and in fact a continuous homomophism

$$\rho_{l,E} : \mathrm{Gal}(\bar{k}/k) \to G_{M(E)}(\mathbb{Q}_l) \subset GL_E(\mathbb{Q}_l).$$

Grothendieck and Mumford-Tate conjecture that the image is open and Zariski-dense. Taking the product over finite primes we have a map ρ_E to $G_{M(E)}(\mathbb{A}_f)$.

8.6 The Taniyama group

This section follows Laurent Fargues' 2006 IHEs notes *Motives and automorphic forms: the (potentially) abelian case.*

8.6.1. CM abelian varieties Let A be an abelian variety over $\mathbb{C}$ such that $\mathrm{End}_{\mathbb{Q}}(A)$ is a CM field E, where $[E : \mathbb{Q}] = 2 \dim A$. Its $H^1_B(A, \mathbb{Q})$ is one dimensional over E, inducing

$$H^1_{\mathrm{dR}}(A) \simeq H^1_B(A, \mathbb{Q}) \otimes \mathbb{C} \simeq E \otimes \mathbb{C} = \prod_{\tau : E \to \mathbb{C}} \mathbb{C}$$

A CM field E is a number field with a unique nontrivial involution inducing complex conjugation c in any embedding of E in $\mathbb{C}$. A CM-type of E is a subset Φ such that $\mathrm{Hom}(E,\mathbb{C}) = \Phi \sqcup c\Phi$. The reflex field $K \subset \bar{\mathbb{Q}}$ of (E,Φ) is the field with Galois group $\{\sigma \in \mathbb{G}_{\mathbb{Q}} : \sigma\Phi = \Phi\}$. It is a CM field stable under c. We have the Hodge decomposition

$$H^1_{\mathrm{dR}}(A) = \mathrm{Lie}(A)^\vee \oplus \overline{\mathrm{Lie}(A)^\vee}$$

splitting the embeddings of E into Φ and $c\Phi$. Given $\iota : E \xrightarrow{\sim} \mathrm{End}(A)$, there is a bijections between pairs (A,ι) up to isogeny and pairs (E,Φ). NB: the moduli space of (A,ι,λ,η) where λ is a polarization and η is a level structure forms a Shimura variety.The $\mathbb{Q}$-Hodge structure $H^1_B(A(\mathbb{C}),\mathbb{Q})$ corresponds to the morphisms

$$h : S = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to GL(H^1_B(A,\mathbb{Q}))_{\mathbb{R}}$$

with

$$\mu_h : \mathbb{G}_m(\mathbb{C}) \to GL(H^1_B(A,\mathbb{Q}))_{\mathbb{C}}$$

defining the Hodge filtration. The $\mathbb{Q}$-Hodge structure carries an action of E, so the morphisms factor through the maximal torus $T := \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$. It turns out that

$$\mu_h = \mu_\Phi := \sum_{\tau\in\Phi}[\tau] \in X_*(T)$$

and the Mumford-Tate group of A is the smallest subtorus of T through which μ_Φ factors.

8.6.2. The Serre group Recall the equivalence of categories:

$$\mathrm{Rep}(\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m) \simeq \{\mathbb{R}\text{-Hodge structures}\}.$$

We will now produce the Serre group S whose representations are equivalent to certain CM Hodge structures, i.e., polarizable $\mathbb{Q}$-Hodge structures whose Mumford Tate group is a torus.

Let (V,h) be such a Hodge structure. Its endomorphisms $E = \mathrm{End}(V,h)$ is either a CM field or $\mathbb{Q}$. Consider

$$h : S \to T(\mathbb{R}), \quad w_h : \mathbb{G}_m \to T,$$

and set

$$\mu_h = \sum_\tau a_\tau[\tau] \in X_*(T)$$

then for all τ one has $a_\tau + a_{c\tau} = w \in \mathbb{Z}$ where $w_h = \mu_h^{1+c} = w\sum[\tau]$.

The field of definition of μ_h is the reflex field K, whereby the morphism $\mu_h : \mathbb{G}_m(K) \to T(K)$ induces

$$\mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m \overset{\mathrm{Res}_{K/\mathbb{Q}}\mu_h}{\longrightarrow} \mathrm{Res}_{K/\mathbb{Q}}T(K) \overset{N_{K/\mathbb{Q}}}{\longrightarrow} T.$$

There is a surjection

$$\mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m \twoheadrightarrow S^K \xrightarrow{\phi} T$$

where S^K is the quotient whose characters

$$X^*(S^K) = \{\sum_{\tau:K\to\mathbb{C}} b_\tau[\tau] : b_\tau + b_{c\tau} = w \in \mathbb{Z}\}$$

The group S^K is the **Serre group** attached to the CM field K. Define S to be the inverse limit over all CM fields K over $\mathbb{Q}$.

Lemma 8.6.1. *There is an equivalence of Tannakian categories between finite-dimensional $\mathbb{Q}$-representations of S and CM Hodge structures generated by $H^1_B(A(\mathbb{C}),\mathbb{Q})$ where A is a CM abelian variety over $\mathbb{C}$. The fibre functor is given by the Betti realization.*

Serre's original definition of S^K is as follows: a Hecke character is a character $\chi : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$ such that χ_∞ is continuous and χ_f is unramified almost everywhere. It is **algebraic of weight** ρ given that

1. $\chi_v(x) = \mathrm{sgn}(x)^\epsilon |x|^n$ for $\epsilon = 0, 1, n \in \mathbb{Z}$ if v is a real
2. $\chi_v(z) = z^p \bar{z}^q$ for $p, q \in \mathbb{Z}$ if v is complex, and
3. $\chi|_{(F_\infty^\times)^0} = \rho^{-1} : T(\mathbb{R})^0 \hookrightarrow T(\mathbb{C}) \to \mathbb{C}^\times$. The 'weight' relates to the weight morphism w_h.

Example 8.6.2. χ is algebraic of weight 0 off it is trivial on $(F_\infty^\times)^0$, hence by Artin reciprocity these correspond to finite order characters of $\mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}}$. The idèle norm $|| \cdot || : \mathbb{A}_F^\times / F^\times \to \mathbb{R}_+^\times$ is algebraic of weight $\rho = N_{F/\mathbb{Q}}^{-1}$.

Then we may define S^K as the quotient of $\mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_m$ by the closure of a congruence subgroup

$$\Gamma \subset \mathcal{O}_K^\times \subset K^\times = \mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_m(\mathbb{Q})$$

such that $\rho|_\Gamma = 1$. It turns out that this is independent of choice of Γ.

8.6.3. The Taniyama group is described in the following theorem of Deligne and Langlands

Theorem 8.6.3. *Let K be a Galois CM field. There is an explicit construction of the extension*

$$1 \to S^K \to T^K \to \mathrm{Gal}(K^{\mathrm{ab}}/\mathbb{Q}) \to 1$$

with continuous sections

$$T^K(\mathbb{A}_{\mathbb{Q}_f}) \xrightarrow{(s_\ell)_\ell} \mathrm{Gal}(K^{\mathrm{ab}}/\mathbb{Q}).$$

and

$$\begin{array}{ccc} T^K(\mathbb{C}) & \longrightarrow & \mathrm{Gal}(K^{\mathrm{ab}}/\mathbb{Q}) \\ & {}^{s_\infty}\nwarrow & \uparrow \\ & & W_{\mathbb{Q}} \end{array}$$

Further, $Rep_{\mathbb{Q}}(T)$ is equivalent to the Tannakian category of CM motives over $\mathbb{Q}$ defined by Hodge cycles, generated by $h^1(A)$ where A is potentially CM (becomes CM after a finite extension), and Artin motives.

If the representation ρ corresponds to the motive M, then $\rho \circ (s_\ell)_\ell$ corresponds to the compatible ℓ-adic system $(H_{\mathrm{et}}^*(M, \mathbb{Q}_\ell))_\ell$. Furthermore, the equality $L(s, M) = L(s, \rho \circ s_\infty)$ where $\rho \circ s_\infty$ is the associated continuous finite dimensional representation of $W_{\mathbb{Q}}$.

The proof of the theorem comes in two parts: Langlands constructed an explicit cocyle

$$b^L \in Z^1\Big(G_{K^{\mathrm{ab}}/\mathbb{Q}}, (S^K(\mathbb{A}_{K,f}/K))^{G_{K^{\mathrm{ab}}/\mathbb{Q}}}\Big) = Z^1\Big(W_{K/\mathbb{Q}}, (S^K(\mathbb{A}_{K,f}/K))^{W_{K/\mathbb{Q}}}\Big)$$

defining the extension and its sections; then Deligne checked that this candidate is the one given by Tannakian duality by his theory of Hodge cycles. He begins with an isomorphism

$$\begin{array}{ccccccccc} 1 & \longrightarrow & S^K & \longrightarrow & \mathcal{E}^K & \longrightarrow & G_{K^{\mathrm{ab}}/K} & \longrightarrow & 1 \\ & & \| & & \downarrow \wr & & \| & & \\ 1 & \longrightarrow & S^K & \longrightarrow & T_K^K & \longrightarrow & G_{K^{\mathrm{ab}}/K} & \longrightarrow & 1 \end{array}$$

where $\mathcal{E}^K$ is defined as Serre's extension of S^K, and T_K^K is the pullback of $M^L \twoheadrightarrow S^L$, the motivic extension by $G_{\mathbb{Q}}$. Tannaka duality gives a correspondence of algebraic Hecke characters of $\mathbb{A}_K^\times / K^\times$ with CM motives over K such that $G_{\bar{\mathbb{Q}}/K}$ action on H_{et}^1 is abelian. Then using this it is shown that the sections of the motivic Galois extension and that of Langlands coincide.

8.6.4. Automorphic consequences

Corollary 8.6.4. *To each number field F and algebraic Hecke character χ, there is a rank one CM motive $M(\chi)$ with $\bar{\mathbb{Q}}$ coefficients such that $L(s,\chi) = L(s, M(\chi))$. The λ-adic compatible system associated to χ is given by the étale realizations of $M(\chi)$, and the weight of χ corresponds to the CM-Hodge structure associated to $M(\chi)$.*

In fact, there is a bijection between the two.

Corollary 8.6.5. *Let M be a CM motive over $\mathbb{Q}$ with $\bar{\mathbb{Q}}$ coefficients. Then M is 'virtually autormophic', i.e., there exists a finite set of a_i and $\chi_i : \mathbb{A}^\times/F_i \to \mathbb{C}^\times$ such that*

$$L(s,M) = \prod_i L(s,\chi_i)^{a_i}.$$

In particular, $L(s,M)$ has meromorphic continuation and satisfies a functional equation.

Now, a continuous representation $W_{\mathbb{Q}} \to GL_n(\mathbb{C})$ is called algebraic if the associated representation of $\mathbb{C}^\times = S \hookrightarrow W_{\mathbb{R}} \to W_{\mathbb{Q}}$ is algebraic, i.e., a direct sum of characters $z^p \bar{z}^q$.

Corollary 8.6.6. *Let ρ be object in $Rep_{\mathbb{Q}}(T)$. The correspondence*

$$\rho \mapsto \rho \circ s_\infty$$

defines an equivalence of Tannakian categories between CM motives over $\mathbb{Q}$ with coefficients in $\mathbb{C}$ and algebraic representations of $W_{\mathbb{Q}}$. If K is a Galois CM field, this restricts to an equivalence of CM motives over $\mathbb{Q}$ where the Galois action on H^*_{et} is abelian over K., and algebraic representations of $W_{K/\mathbb{Q}}$.

The extension of S by T is mirrored on the automorphic side by

$$1 \to W^0_{\mathbb{Q}} \to W_{\mathbb{Q}} \to \mathbb{G}_{\mathbb{Q}} \to 1,$$

which is conjectured to be the abelianization of the sequence

$$1 \to L^0_{\mathbb{Q}} \to L_{\mathbb{Q}} \to \mathbb{G}_{\mathbb{Q}} \to 1$$

where $L_{\mathbb{Q}}$ is the conjectural automorphic Langlands group, which we shall next discuss.

8.7 The automorphic Langlands group

In his 1979 Corvallis article, Langlands conjectured a relationship between automorphic forms and motives (in the sense of Grothendieck) in terms of the so-called Tannakian formalism. In his words, 'the attempt [to define Tannakian categories for automorphic representations] may be vain but the prize to be won is so great that one cannot refuse to hazard it.'

8.7.1. Isobaric automorphic representations. Let F be a number field, and $\mathbb{A}_F$ the adéles of F, dropping the subscript when there is no ambiguity. An irreducible representation of $G = GL_n(\mathbb{A}_F)$ is **automorphic** if it is isomorphic to an irreducible subquotient of $L^2(G,\omega)$ where ω is a fixed central character. Denote by $\mathrm{Aut}(n)$ the automorphic representations of $GL_n(\mathbb{A}_F)$, and $\mathrm{Aut} = \coprod_n \mathrm{Aut}(n)$. An automorphic representation π is **cuspidal** if one has

$$\int_{N_F\backslash N_{\mathbb{A}}} f(ng)dn = 0$$

for any f in π and for all unipotent radicals N of parabolics $P = MN \subset G$. By Langlands' work on Eisenstein series, any automorphic representation is isomorphic to a subquotient of the (unitary) induced representation

$$\rho = \mathrm{Ind}^{G_{\mathbb{A}}}_{M_{\mathbb{A}}N_{\mathbb{A}}}(\sigma \otimes 1)$$

where σ is a cuspidal representation of $M_{\mathbb{A}}$. If P is a parabolic of type $(n_1, \ldots, n_r)$ where $n_1 + \cdots + n_r = n$, then we may decompose $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_r$. By a theorem of Jacquet and Shalika, two such σ that are equivalent at almost all finite places up to permuting the partitions are in fact equivalent. This is a form of 'strong multiplicity one' for G.

Let F_v be a local field. The local analogue of this is the Langlands classification: the **cuspidal** representation σ_i are those whose matrix coefficients are square integrable mod centre. Let χ_i be the central characters of σ_i; they are necessarily of the form $|\cdot|^{s_i}$, with s_i real. Up to permutation we may assume that $s_1 \geq \cdots \geq s_r$. The induced representation

$$\rho_v = \mathrm{Ind}_{M_{F_v} N_{F_v}}^{G_{F_v}} (\sigma_1 \otimes \cdots \otimes \sigma_r \otimes 1)$$

contains a unique irreducible subquotient, called the **Langlands subquotient**. Every irreducible representation of $GL_n(F_v)$ is isomorphic to a Langlands subquotient.

Now let N_0 be standard unipotent subgroup of G, consisting of matrices of the form

$$\begin{pmatrix} 1 & x_1 & * \\ & \ddots & x_{n-1} \\ 0 & & 1 \end{pmatrix}$$

and a non degenerate additive character $\theta(n) = \psi(x_1 + \ldots x_{n-1})$. A representation π is **generic** if there is a linear form on the admissible dual that transform by $\theta(n)$. The local components of a cuspidal representation are generic. By results of Kostant and Vogan in the real case and Bernstein and Zelvinsky in the p-adic case, every generic representation can be written as an induced representation above.

Finally, given an automorphic representation $\pi = \otimes \pi_v$ that is a subquotient of $\rho = \otimes \rho_v$, we say it is **isobaric** if π_v is the Langlands subquotient of ρ_v for all v. If σ_i are cuspidal representations of $GL_{n_r}(\mathbb{A})$, denote by

$$\sigma_1 \boxplus \cdots \boxplus \sigma_r$$

the unique isobaric subquotient of $\rho(\sigma_1, \ldots, \sigma_r)$. Denote by $\mathrm{Isob}(n)$ the category of isobaric representations of $GL_n(\mathbb{A})$.

Example 8.7.1. If

$$\rho = \mathrm{Ind}(|\cdot|^{\frac{-n+1}{2}} \otimes |\cdot|^{\frac{-n+3}{2}} \otimes \cdots \otimes |\cdot|^{\frac{n-1}{2}})$$

then the associated isobaric representation is the trivial representation. A more interesting example is the following: let P be a parabolic of type $(b, \ldots, b), n = ab)$ and σ a cuspidal representation of $GL_b(\mathbb{A})$. Moeglin and Waldspurger show that all representations π in the discrete spectrum are isobaric with every π_v the unique irreducible subquotient of

$$\mathrm{Ind}_{M_{F_v} N_{F_v}}^{G_{F_v}} (\sigma_v \otimes \sigma_v |\cdot| \otimes \cdots \otimes \sigma_v |\cdot|_v^{a-1} \otimes 1)$$

8.7.2. Algebraic automorphic representations. For infinite places v, fix an isomorphism of F_v with $\mathbb{R}$ or $\mathbb{C}$ accordingly. The Langlands correspondence associates a representation r_v of W_{F_v} to each π_v. Denote by $\mathrm{Alg}(n)$ the category of isobaric representations π of $GL_n(\mathbb{A})$ such that at every infinite place v, $r_v = \chi_1 \otimes \cdots \otimes \chi_n$ where χ_i is a character of $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m(\mathbb{R}) = \mathbb{C}^\times$ such that $\chi_i |\cdot|^{\frac{1-n}{2}}$ is algebraic.

Example 8.7.2. Let f be a weight $k \in \mathbb{Z}$ modular form and ψ a character of $(\mathbb{Z}/N\mathbb{Z})^\times$. It satisfies

$$f\Big(\frac{az+b}{cz+d}\Big) = \psi(a)(cz+d)^k f(z), \qquad \gamma \in \Gamma(N), z \in \mathbb{H}.$$

When f is holomorphic of weight $|k| > 1$, the associated representation π_∞ corresponds to characters $z^p \bar{z}^q$ and $z^q \bar{z}^p$. It is algebraic if $p, q \in \mathbb{Z} + 1/2$, and their central character on $\mathbb{R}_+^\times$ is $|\cdot|^{p+q}$. If $k = 0, 1$, then $\pi_\infty = \xi_1 \boxplus \xi_2$, where $\xi_i = |\cdot|^{s-i} \mathrm{sun}(x)^{\epsilon_i}$ are characters of $\mathbb{R}^\times$, algebraic if $s_i \in \mathbb{Z} + 1/2$. When f is a Maass form, the associated representation π is not algebraic unless its Laplace eigenvalue is $\lambda = \frac{1}{4}$.

As it is, Alg is not stable under summation. But we could do so by defining the sum of π_1 and π_2 in Isob(n_i) to be

$$\pi_1 \overset{T}{\boxplus} \pi_2 = (\pi_1|\det|^{\frac{1-n_1}{2}} \boxplus \pi_2|\det|^{\frac{1-n_2}{2}})|\det|^{\frac{n_1+n_2-1}{2}}$$

similar to the Tate twist. Then every algebraic representation π can be decomposed as

$$\pi \simeq \pi_1 \overset{T}{\boxplus} \cdots \overset{T}{\boxplus} \pi_r.$$

where π_r are cuspidal algebraic representations.

8.7.3. The Tannakian formalism. Recall that a neutral Tannakian category is an F-linear abelian category with a tensor functor and fibre functor. Langlands expressed the hope that Isob has the structure of a Tannakian category. Its sums are given by $\boxplus$, and products $\boxtimes$ described below. The key obstructions are:

1. The existence of a unique isobaric representation $\pi_n \boxtimes \pi_m$ of GL_{nm} corresponding to isobaric representations π_i of GL_i. The local Langlands gives a candidate at local places, but it remains to prove that the result if *automorphic*. This corresponds to the tensor product of L-groups $GL_n \times GL_m \to GL_{nm}$.
2. An abelian $\mathbb{C}$-linear structure on Isob. This asks for a suitable subcategory of irreducible admissible representations of $GL_n(F_v)$ admitting the structure of a Tannakian category. Clozel constructs the following candidate:

   A **model** of automorphic representations is a realization of representatives of isomorphism classes of (isobaric) automorphic representations in a certain space. For example, the *Whittaker model* for generic π is a function on $G_{\mathbb{A}}$ transforming on the left by $\theta(n)$. The *automorphic model* for a cuspidal representation is a unique submodule of $L^2_{\text{cusp}}(G_F\backslash G_{\mathbb{A}})$. The *principal model* of an isobaric representation $\sigma_1 \boxplus \cdots \boxplus \sigma_r$ is the unique subquotient of the induced representations of the automorphic models of σ_i.

   Clozel defines the category Isob$_1$ of principal models, with hom sets

   $$\text{Hom}(\sigma_1 \boxplus \cdots \boxplus \sigma_r, \tau_1 \boxplus \cdots \boxplus \tau_s = \oplus \text{Hom}(\sigma_i, \tau_j))$$

   such that $\text{Hom}(\sigma, \tau) = 0$ if $\sigma \neq \tau$ and $\text{Hom}(\sigma, \sigma) = \mathbb{C}$. Composition is given as for matrices. One verifies the existence of kernels and cokernels, moreover that Isob$_1$ is abelian. But even admitting (1), tensor products are not functorially defined.
3. The field of rationality of $\pi = \pi_\infty \otimes \pi_f$ is the smallest subfield of $\mathbb{C}$ such that π_f is stable under Aut$(\mathbb{C})$. We call π algebraic regular, or cohomological, if (p_i, q_i) are distinct for all associated χ_i. For such π, it is known that π_f is defined over a number field, but only conjectured for algebraic π. Rationality is not an issue per se, as it may suffice to consider Isob and Mot over $\mathbb{C}$.

8.7.4. The automorphic Langlands group For the moment, let's assume the Tannakian formalism can be applied to Isob$_F$ and proceed formally. The associated Tannakian fundamental group is referred to as the conjectural automorphic Langlands group, denoted L_F. By construction it should be a (pro)algebraic group, whose finite-dimensional representations parametrize isobaric representations.

This Langlands group should fit into an exact sequence

$$1 \to L_F^0 \to L_F \to \text{Gal}(\bar{F}/F) \to 1$$

where the abelianization can be interpreted by global class field theory,

$$1 \to W_F^0 \to W_F \to \text{Gal}(\bar{F}/F) \to 1$$

where W_F is the usual Weil group. Kottwitz later suggested L_F be considered as a locally compact topological group, that is, an extension of W_F by a locally compact group. There would then be a sequence

$$L_F \to W_F \to \mathrm{Gal}(\bar{F}/F)$$

of locally compact groups.

Now the so called local Langlands group L_{F_v} is known, given by W_{F_v} for v archimedean and $W_{F_v} \times SU_2(\mathbb{R})$ for v nonarchimedean. The global group should come with embeddings of the local groups $L_{F_v} \hookrightarrow L_F$ for every place v.

Moreover, one expects a surjection $L_F \twoheadrightarrow G_F$, where we denote by G_F the motivic Galois group of (pure) motives over F with coefficients extended to $\mathbb{C}$. Indeed, one expects a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & L_F^0 & \longrightarrow & L_F & \longrightarrow & L_F^{\mathrm{Gal}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & G_F^0 & \longrightarrow & G_F & \longrightarrow & \mathrm{Gal}(\bar{F}/F) & \longrightarrow & 1 \end{array}$$

where L_F^{Gal} is associated to cuspidal autmorphic representations which at every archimedean place the associated Weil group representation is trivial on $\bar{F}^\times \simeq \mathbb{C}^\times$. Also, the algebraic automorphic representations should give a subgroup L_F^{alg}, conjecturally isomorphic to G_F.

Evidence towards such a surjection is the map from the Weil group to the Taniyama group $W_F \to T_F$ in the previous section.

In terms of the L-group ${}^LG$ of a reductive group G, reciprocity can be described as L-homomorphisms

$$\phi : L_F \to {}^LG$$

commuting with projection onto the Galois group, parametrizing L-packets $\Pi_\phi(F)$ of automorphic representations of $G(\mathbb{A}_F)$. The map should also be compatible with the local Langlands correspondence

$$\phi_v : L_F = W_{F_v} \to {}^LG_v$$

Remark 8.7.3. As a cautionary remark, Langlands makes the following comment on his article *Automorphic representations, motives, and Shimura varieties: ein Märchen* in which L_F is introduced:

> The Märchen was written in the late 1970's when I was still relatively young and impressionable. Having now lived for some decades with various ideas that were new to me then and having had many more years to reflect on the theory of automorphic representations and related matters, I am now inclined to think that although Tannakian categories may ultimately be the appropriate tool to describe the basic objects of algebraic geometry, automorphic representations have a different structure, best expressed by functoriality, in which of course statements formulated in terms of the finite-dimensional representations of the L-group are central.

Indeed, in a 2013 exposition on *Functoriality and reciprocity* Langlands refers to what he calls 'mock Tannakian' categories, to allow for 'a certain latitude and a certain imprecision'.

Nonetheless, he suggests there should be pairs $({}^LH, \pi_H^{\mathrm{st}})$ consisting of an L-group and a stable L-packet, the stability arising from stable conjugacy classes. The generating elements are π_H^{st} that are what Langlands refers to as hadronic or thick, the rest are obtained by functorial transfer

$$\phi : {}^LH \to {}^LG.$$

Then the group obtained from this mock Tannakian category will be an inverse limit over finite sets S of hadronic pairs $\prod_S {}^LH$. Arthur has constructed a candidate along similar lines, as a fibre product of extensions of W_F by a locally compact group.

8.8 Motivic homotopy theory

8.8.1. Voevodsky's intuitive introduction. We want to consider spaces, and invariants associated to them. Let $C(M, N)$ be the set of continuous maps from M to N. If M is a point, then any map is uniquely determined by where the point goes, so a map from a point to N is the same as a point of N. So

$$C(\mathrm{pt}, N) = \{\text{set of points of } N\}$$

On the other hand, $C(M, \mathrm{pt})$ has only one element, since there is only one place any element of N can go. Now a fancier example: consider now $C([0,1], N)$, where any map is determined by where 0 and 1 go, plus all the numbers between. A continuous map, intuitively, means that the path has no breaks:

$$C([0,1], N) = \{\text{set of paths in } N\}$$

One invariant we can associate to a space is the number of pieces of the space. Two points lie in the same space if they can be connected by a path. So to define the number of pieces, π_0 of N, one only needs to know the points and paths between the points of N. Next, we can consider the set π_q of pieces of the space of paths, where we call two paths are connected if they can be connected by a map from the square. We can repeat this procedure from a line to square to cube and so on, and get a sequence of sets π_n.

Now let's use the formalism of category theory to talk about what we have covered: When is a space a point? A space N is a point iff for every other space there is only one morphism to N, i.e., for every space M, $\mathrm{Mor}(M, N)$ has exactly one element. Here is a more complicated question: when is a space connected? The unit interval has two distinguished points 0,1:pt$\to [0,1]$. Then we say two points a, b lie in the same piece of N if there is a path γ in $\mathrm{Mor}([0,1], N)$ such that $a = \gamma(0)$ and $b = \gamma(1)$, viewed as compositions of morphisms.

To define π_0 finally, we need a unit interval, which is not internal to the categorical structure. Similarly for π_0 we need another distinguished object, the square with four distinguished maps from the interval, and so on. So given any category with such a series of distinguished objects and morphisms between them, even if it doesn't work very well. Motivic homotopy theory happens when we apply this procedure to algebraic geometry.

Let us consider $\mathbb{A}^0$. It is the system of no equations and no variables. A morphism from $\mathbb{A}^0$ to another system means finding constants of the second system which satisfy the equations of the second system. Such a morphism is exactly a solution of the second system of equations. Similarly one can verify that every system of equations has exactly one morphism to $\mathbb{A}^0$. Continuing, the objects $\mathbb{A}^n$ gives such a system of distinguished objects as required above.

8.8.2. Cohomology in algebraic topology. Algebraic cycles appear to play the role of singular cycles, but what about singular chains? An algebraic n-simplex Δ^n_k is defined as the hyperplane in $\mathbb{A}^n_k$ given by $t_0 + \cdots + t_n = 1$, and the coordinates t_i give an isomorphism with $\mathbb{A}^n_k$. For a k-variety X, one could work with the set of maps from $\mathbb{A}^n_k$ to X, but there are no such nonconstant maps.

Suslin, following the Dold-Thom correspondence which states that $\tilde{H}_n(X, \mathbb{Z}) = \pi_n(\mathrm{Sym}^\infty X)$, the group of finite cycles

$$C^{\mathrm{Sus}}_n(X) = \langle W \subset \Delta^n \times X : W \text{ irred} \to \Delta^n_k \text{ finite, surjective}\rangle$$

and thus the Suslin homology $H_n(C^{\mathrm{Sus}}_\cdot(X))$. The infinite symmetric product with $X = S^d$ is a $K(\mathbb{Z}, d)$ space, so that one gets singular cohomology $H_n(X) = \pi_{d-n}(\mathrm{Maps}(X, \mathrm{Sym}^\infty S^d))$. Analogously, for a smooth variety X one defines weight d motivic cohomology

$$H^{p,d}(X, \mathbb{Z}) := \pi_{2d-p}(\mathrm{Maps}(X, \mathrm{Sym}^\infty \mathbb{P}^d/\mathbb{P}^{d-1})) = H_{2d-p}(C^{\mathrm{Sus}}_\cdot(\mathbb{P}^d/\mathbb{P}^{d-1})(X))$$

where the dependence on X corresponds to finite surjective maps $W \to \Delta^n \times X$. Also, we denote $H^p(X, \mathbb{Z}(d)) = H^{p,d}(X, \mathbb{Z})$. Note that for the definition to be valid for $p > 2d$, one defines the stabilization $\lim H^{p+2N, d+N}(X \wedge \mathbb{P}^N/\mathbb{P}^{N-1}, \mathbb{Z})$

The Suslin complexes, more generally the complexes for computing $H^{p,d}(X,\mathbb{Z})$, are contained in Voevodsky's triangulated category of k-motives $DM_{-}^{\text{eff}}(k)$. We have to englarge the category of smooth schemes Sm/k to the category of finite correspondences, $\text{SmCor}(k)$. Over this category we define presheaves with transfers $\text{PST}(k)$, which are functors from $\text{SmCor}(k)^{\text{op}}$ to Ab, the category of Abelian groups. In particular, we use the Nisnevich topology, in which a covering family is a collection of étale maps such that for each finitely generated field extension L of k, the map $\coprod U_\alpha(L) \to U(L)$ is surjective.

Finally, we invert maps $f : C \to C'$ which are quasi-isomorphism on stalks of Nisnevich sheaves, also the projection $C_0^{\text{Sus}}(X \times \mathbb{A}^1) \to C_0^{\text{Sus}}(X)$.

8.8.3. Homotopy theory. The construction in homotopy theory is as follows: start with the category of spaces (spectra), and invert weak equivalences to obtain the unstable (stable) homotopy category. This gives the category of generalized cohomology theories, including singular cohomology, topological K-theory, and complex cobordism.

Here is the construction of Morel and Voevodsky: replace spaces with the category of presheaves of spaces on $\text{Sm}(k)$, i.e., functors $\text{Sm}/k^{\text{op}} \to \text{Spc}$. This category receives an embedding of Sm/k by sending X to the presheaf $\text{Hom}_{\text{Sm}/k}(-, X)$, and of Spc by the constant presheaf. Moreover, given pointed presheaves of spaces P, Q we can form the presheaf

$$(P \wedge Q)(X) := P(X) \wedge Q(X)$$

which allows us to define the suspension operator on pointed spaces over k by $\Sigma P := P \wedge S^1$.

For the unstable homotopy category of spaces over k, we have to perform a Bousfield localization: define a *Nisnevich local weak equivalence* $f : P \to Q$ by requiring that f induces a weak equivalence of spaces on each Nisnevich stalk $f_x : P_x \to Q_x$, where $x \in X \in \text{Sm}/k$ and $P_x := \varinjlim P(U)$ where U runs over Nisnevich neighborhoods of x. Invert these weak equivalences to get $\mathcal{H}_{\text{Nis}}$. Now we want to make $\mathbb{A}^1$, our unit interval, contractible, call f an $\mathbb{A}^1$*-weak equivalence* if after $f^* : \text{hom}(Q, Z) \to \text{hom}(P, Z)$ is an isomorphism over $\mathcal{H}_{\text{Nis}}$ for all spaces Z such that $\text{hom}(X, Z) \simeq \text{hom}(X \times \mathbb{A}^1, Z)$ for all X in Sm/k. Then define the unstable homotopy category of spaces over k

$$\mathcal{H}(k) := \mathcal{H}_{\text{Nis}}[WE_{\mathbb{A}^1}(k)^{-1}]$$

For the stable homotopy category of T-spectra, we consider the pointed category Spc., and replace the objects $S^1, \mathbb{P}^1$ with $Th(\mathbb{R}), Th(\mathbb{A}^1)$. The objects are sequences of pointed spaces over k with bonding maps

$$\epsilon_n : E_n \wedge \mathbb{P}^1 \to E_{n+1}$$

Then we have a presheaf of bigraded stable homotopy groups:

$$\pi^s_{a,b}(E)(X) = \varinjlim_N \text{hom}_{\mathcal{H}(\cdot}(S^{a+2N,b+N} \wedge X_+, E_N)$$

for X in Sm/k and the associated Nisnevich sheaf $\pi^s_{a,b}(E)_{\text{Nis}}$. Call a stable weak equivalence a map f of T-spectra, inducing an isomorphism of Nisnevich sheaves for all a, b. This gives the stable homotopy category of T-spectra,

$$\mathcal{SH}(k) := \mathbf{Spt}(k)[sWE^{-1}]$$

The operator $- \wedge \mathbb{P}^1$ on $\mathcal{H}.(k)$ extends to the invertible T-suspension operator on Sm/k. Objects in this stable homotopy category represent bi-graded cohomology theories $E^{a,b}(X) := \pi^s_{-a,-b}(E)(X)$. For example, motivic cohomology, algebraic K-theory, and algebraic cobordism.

For more details, the interested reader is strongly encouraged to consult Voevodsky's Nordfjordeid summer school lectures.

9 Reciprocity

9.1 A historical overview

9.1.1. Residue symbols Before we discuss the reciprocity laws through history, we'll first catalogue here the various symbols introduced. I hope the notation in the next section will be clear by the context.

1. The *Legendre symbol*, for odd primes q, is given by the explicit formula $(\frac{p}{q}) \equiv a^{\frac{p-1}{2}} \bmod q$ with values in $\{0, \pm 1\}$, and is multiplicative in the denominator.

$$\left(\frac{a}{p}\right) = \begin{cases} +1 & \text{if } a \text{ is a nonzero quadratic residue mod } p \\ -1 & \text{if } a \text{ is a quadratic nonresidue mod } p \\ 0 & \text{if } a \equiv 0 \bmod p \end{cases}$$

2. The *biquadratic residue symbol* $[\frac{\pi}{\lambda}]$ for primes π, λ in $\mathbb{Z}[i]$ not dividing 2, gives the unique element in $\{\pm 1, \pm i\}$ such that $[\frac{\pi}{\lambda}] \equiv \pi^{(N\lambda - 1)/4} \bmod \lambda$
3. The *Jacobi symbol* $(\frac{a}{n})$ generalizes Legendre's symbol to positive odd n, followed by the *Kronecker symbol* which allows n to be any integer.
4. The l-th *power residue symbol* $(\frac{\alpha}{\mathfrak{p}}) \equiv \alpha^{N\mathfrak{p}-1/l} \bmod \mathfrak{p}$. More generally, for a number field k containing a primitive ζ_n and n comprime to $\mathfrak{p}$ in $\mathfrak{o}_k$

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_n = \begin{cases} +1 & \text{if } \alpha \notin \mathfrak{p} \text{ and } \alpha \equiv x^n \bmod \mathfrak{p} \text{ for some } x \text{ in } \mathfrak{o}_k \\ -1 & \text{if } \alpha \notin \mathfrak{p} \text{ and there is no such } x \\ 0 & \text{if } \alpha \in \mathfrak{p} \end{cases}$$

This is the Legendre symbol when $n = 2$.

5. The quadratic Hilbert or norm residue symbol for a local field k is

$$\left(a, b\right) = \begin{cases} +1 & \text{if } z^2 = ax^2 + by^2 \text{ has a nonzero solution in } k^3 \\ -1 & \text{otherwise} \end{cases}$$

and generalizes to m-th power symbols.

6. The *Frobenius symbol* for a finite normal extension K/L and prime ideal $\mathfrak{p}$ in $\mathfrak{o}_L$, $[\frac{K/L}{\mathfrak{p}}]$ is defined to be the unique automorphism ϕ in $\mathrm{Gal}(K/L)$ such that

$$\phi(\alpha) \equiv \alpha^{N\mathfrak{p}'} \bmod \mathfrak{p}$$

where $\mathfrak{p}'$ is the prime ideal in $\mathfrak{o}_K$ below $\mathfrak{p}$.

7. The *Artin symbol*, given for a finite normal abelian extension K/L, whence the Frobenius symbol is independent of choice of $\mathfrak{p}|\mathfrak{p}'$, so that we may set $(\frac{K/L}{\mathfrak{p}}) = [\frac{K/L}{\mathfrak{p}'}]$ for any $\mathfrak{p}$. Moreover, since the extension is abelian, the symbol extends multiplicatively to all ideals $\mathfrak{a} = \prod \mathfrak{p}$ prime to $\mathrm{disc}(L/K)$ by

$$(\frac{K/L}{\mathfrak{a}}) = \prod (\frac{K/L}{\mathfrak{p}})$$

Note that for $L = \mathbb{Q}(\zeta_p)$ and primes $q \neq p$, the automorphism $\sigma_q : \zeta_p \mapsto \zeta_p^q$ is precisely $(\frac{L/\mathbb{Q}}{q})$; for quadratic fields $L = \mathbb{Q}(\sqrt{d})$ and primes $p \nmid d$, then

$$\left(\frac{L/\mathbb{Q}}{(q)}\right) = \begin{cases} 1 & \text{if } (\frac{d}{q}) = 1 \\ \sigma & \text{if } (\frac{d}{q}) = -1 \end{cases}$$

and so identified with the Kronecker symbol $(\frac{d}{q})$ by $(\frac{L/\mathbb{Q}}{(q)})\sqrt{d} = (\frac{d}{q})\sqrt{d}$ which also generalizes to the n-th power residue symbols.

9.1.2. Reciprocity laws What is a reciprocity law? Euler's way of looking at it was the following: the quadratic character of a mod p only depends of the residue class of p mod a.

Theorem 9.1.1 (Euler's criterion, 1772)**.** *For integers a and odd primes p not dividing a we have*

$$a^{\frac{p-1}{2}} \equiv \begin{cases} +1 \bmod p & \text{if } a \text{ is a quadratic residue mod } p \\ -1 \bmod p & \text{if } a \text{ is a quadratic nonresidue mod } p \end{cases}$$

It was Legendre who coined the term reciprocity,

Theorem 9.1.2 (Legendre's quadratic reciprocity law)**.** *Let p, q be distinct odd primes. Then*

$$\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}} \left(\frac{q}{p}\right)$$

Moreover we have the first and second supplementary laws

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}, \qquad \left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$$

Legendre's proof of this was not considered rigorous, in it he assumes the theorem on primes in arithmetic progressions which was not proven then. It was Gauss who noticed that to formulate the theory of biqudratic residues one would need to consider the Gaussian integers $\mathbb{Z}[i]$, hence

Theorem 9.1.3 (Gauss' quartic reciprocity law)**.** *Let π, λ be distinct primary primes in $\mathbb{Z}[i]$, i.e., $\pi \equiv \lambda \equiv 1$ mod $(2+2i)$. Then*

$$\left[\frac{\pi}{\lambda}\right] = (-1)^{\frac{N\pi-1}{4}\frac{N\lambda-1}{4}} \left[\frac{\lambda}{\pi}\right]$$

with analogues of the first and second supplementary laws.

We pause here to point out that Weil was inspired to make his conjectures for varieties over finite fields by studying Gauss' work on quadratic residues.

Theorem 9.1.4 (Eisenstein's reciprocity law)**.** *Let l be an odd prime and α primary in $\mathbb{Z}[\zeta_l]$, i.e., congruent to an integer mod $(1-\zeta_l)^2$. Then*

$$\left(\frac{\alpha}{a}\right)_l = \left(\frac{a}{\alpha}\right)_l$$

for all integers a prime to l.

Theorem 9.1.5 (Kummer's reciprocity law)**.** *Consider $\mathbb{Q}(\zeta_l)$ such that l does not divide the class number of $\mathbb{Q}(\zeta_l)$. Then*

$$\left(\frac{\mathfrak{a}}{\mathfrak{b}}\right)_l = \left(\frac{\mathfrak{b}}{\mathfrak{a}}\right)_l$$

for all relatively prime integral ideas $\mathfrak{a}$ and $\mathfrak{b}$ prime to l.

Theorem 9.1.6 (Hilbert's reciprocity law)**.** *Let k be an algebraic number field containing the m-th roots of unity. Then for all a, b in $k^\times$,*

$$\prod_v \left(\frac{a, b}{v}\right) = 1$$

the m-th power norm residue symbol mod v, the product extended over all places v of k.

Hilbert's 9th problem was to find the most general reciprocity law for an arbitrary number field.

Theorem 9.1.7 (Artin's reciprocity law)**.** *(Global) Let K be a finite extension of k an algebraic number field. Then the global norm residue symbol $(\frac{K,k}{\cdot})$ induces an isomorphism*

$$C_k/N_{K/k}C_K \simeq \mathrm{Gal}(K/k)^{ab}$$

(Local) Let K be a finite extension of k a finite extension of $\mathbb{Q}_p$. Then the local norm residue symbol induces an isomorphism

$$k^\times/N_{K/k}K^\times \simeq \mathrm{Gal}(K/k)^{ab}$$

Theorem 9.1.8 (Artin's reciprocity à la Tate)**.** *Let K/k be a normal extension, and let u in $H^2(\mathrm{Gal}(K/k), C_K)$ be the fundamental class of K/k. Then the cup product with u induces an isomorphism for all integers q*

$$u\smile: H^q(\mathrm{Gal}(K/k),\mathbb{Z}) \to H^{q+2}(\mathrm{Gal}(K/k), C_K)$$

where the case $q = -2$ is again Artin's reciprocity law.

9.2 Explicit reciprocity laws and the BSD conjecture (Florez)

Recall HIlbert's recicprocity law: given $K/\mathbb{Q}$ contaning μ_n, then $\prod_v (a,b)_{v,n} = 1$ for any $a, b \in K^\times$, where the Hilbert symbol is given by

$$(a,b)_{v,n} = \frac{\psi_v(x)(\sqrt[n]{y})}{\sqrt[n]{y}} \in \mu_n$$

where ψ_v is the local reciprocity map. Coates and Wiles extended this pairing to elliptic curves. Giving explicit descriptions of the global reciprocity is called explicit reciprocity laws.

History: Kummer $K_v = \mathbb{Q}_p(\zeta_p)$ (1858); Artin-Hasse (1928); Iwasawa Main Conjecture (1968); Wiles for elliptic curves and obtained results for BSD (1978); Kato proved IMC for modular forms (1999). Also Kolyvagin's work between Wiles and Kato.

9.2.1. Iwasawa Main Conjecture relates cyclotomic units to p-adic zeta functions. Consider the real subfield $F_n = \mathbb{Q}(\zeta_{p^n})^+$ and the union $F_\infty = \cup_{n\geq 1} F_n$, set $G = \mathrm{Gal}(F_\infty/\mathbb{Q})$, $U_n^\perp$ the units congruent to 1 in $\mathbb{Q}_p(\zeta_{p^n})^+$ and $U_\infty^\perp = \lim U_n^\perp$, and V_n the closure of cyclotomic units in $U_n^\perp$. Then

$$U_\infty^\perp V_\infty^\perp \simeq \Lambda/I(G)\zeta_p$$

where $\Lambda = \mathbb{Z}_p[G]$, $I(G)$ the augmentation ideal, ζ_p the p-adic zeta function.

Take $K_v = \mathbb{Q}_p(\zeta_{p^n})$, $v = p$. then

$$(x,y)_{v,p^n} = \zeta_p^{\mathrm{Tr}_{K_v/\mathbb{Q}_p}(\delta(x),\log(y))}$$

where $x = g_x(\pi_n)$ where $\pi_n = \zeta_{p^n} - 1$, where $g_x(T) \in \mathbb{Z}_p[[T]]$. Define Kummer's p-adic logarithm

$$\delta(x) = \frac{1}{\log'(T)}\frac{d}{dT}(\log g_x(T))\Big|_{T=\pi_n} = \frac{1}{1+T}\frac{d}{dT}(\log g_x(T))\Big|_{T=\pi_n}$$

Define $D = (1+T)\frac{d}{dT}$, Wiles looked at

$$\delta_1(x) = D(\log g_x(T))\Big|_{T=0}$$

What we will see is

$$\delta_1(\frac{\zeta_{p^n}^{n/2} - \zeta_{p^n}^{-n/2}}{\zeta_{p^n} - \zeta_{p^n}^{-1}}) = (1-n^k)\zeta(1-k) \qquad k = 2, 4, 6, \ldots$$

and in general δ_1(cyclotomic units) = special zeta values, where these cyclotomic units form an Euler system.

9.2.2. BSD Conjecture Let E be an elliptic curve over a number field K, then the conjecture states that

$$\text{ord}_{s=1} L(E/K, s) = \text{rank}_{\mathbb{Z}} E(K)$$

where by Mordell-Weil $E(K) \simeq \mathbb{Z}^r \times E(K)_{\text{tor}}$. Coates and Wiles in 1977: K imaginary quadratic, E CM by $\mathfrak{o}_K$, i.e., $\text{End}(E) \simeq \mathfrak{o}_K$, if $E(K)$ infinite then $L(E/K, 1) = 0$. So if we have nonvanishing then the Mordell-Weil group is finite, so analytic rank = algebraic rank = 0. In this case one uses elliptic units for the Euler system, which only exist for CM elliptic curves.

Proof sketch: K class number 1, p a prime in K, π a generator of p, K_p the completion at p, $E[p^n] = \{q \in E(\bar{K}) | [\pi^n]q = 0\}$ Fix p_0 not dividing $6f$, f the conductor of E. Define $K_n = K(E[p_0^n])$, and ψ_p the local rec map for $K_{n,p}$. Then define

$$(\cdot, \cdot)_{n,p} : K_{n,p}^\times \times E(K_{n,p}) \to E[p_0^n]$$

by $(x, p)_{n,p} = \psi_p(x)(q) - q$ where q is any solution to $[\pi^n]q = p$ which gives an analog of Hilbert reciprocity

$$\sum_p (x, p)n, p = 0, \qquad x \in K_n^\times, p \in E(K_n)$$

Idea now is to pick a point x of infinite order, which will also be an elliptic unit, $(x, p)_{n,p} = 0, p \neq p_0$ which implies that $(x, p)_{n,p_0} = 0$.

$$(x, p)_{n,p_0} = \text{Tr}_{K_n/K}(\delta(x) \log_E p)[w_n] = 0$$

where $\log_E$ is the logarithm of the ellpitic curve, w_n is a generator of $E[p_0^n]$. Since p has infinite order its logarithm is nonzero. Thus we deduce that $\delta_1(x) =: \text{Tr}_{K_n/K}(\delta(x)) = 0$

9.3 Non-abelian reciprocity laws on Riemann surfaces (Horozov)

9.3.1. Introduction. Gauss proved quadratic reciprocity for odd primes p, q in $\mathbb{Q}$:

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}, \qquad \left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \mod p$$

Followed by the Hilbert symbol (also work of Takagi):

$$(a, b)_p = \begin{cases} 1 & ax^2 + by^2 = z^2 \text{ has nontrivial solutions in } \mathbb{Q}_p \\ -1 & \text{otherwise} \end{cases}$$

giving the Hilbert reciprocity law $\prod_p (a, b)_p = 1$. This specializes to analogues of quadratic reciprocity:

$$(a, b)_p = (-1)^{\alpha\beta\epsilon(p)} \left(\frac{u}{p}\right)^\beta \left(\frac{v}{p}\right)^\alpha$$

where $a = p^\alpha u, b = p^\beta v, \epsilon(p) = (-1)^{(p-1)/2}, \alpha = \text{ord}_p(a), \beta = \text{ord}_p(b)$.

Weil reciprocity: f, g rational functions on a Riemann surface such that the divisors $|(f)| \cap |(g)| = \varnothing$. Define for a divisor $D = a_1 P_1 + \cdots + a_n P_n$ the function

$$f(D) := \prod_i f(P_i)^{a_i}$$

Then Weil reciprocity is the statement that $f((g)) = g((f))$. We can also express this as $\prod_p (f, g)_p = 1$ where

$$(f, g)_p = (-1)^{\alpha\beta} \frac{f_0(P)^\beta}{g_0(P)^\alpha}$$

where $f = z^\alpha f_0, \alpha = \text{ord}_p f$, $g = z^\beta g_0, \beta = \text{ord}_p g$ and z is a local 'uniformizer' at P.

9.3.2. Iterated integrals We may express:

$$\zeta(2) = \int_0^1 \int_0^{z_2} \frac{dz_1}{1-z_1}\frac{dz_2}{z_2} = \int_0^1 \int_0^{z_2} (1 + z_1 + z_1^2 + \dots) dz_1 \frac{dz_2}{z_2}$$

$$= \int_0^1 (z_2 + \frac{z_2^2}{2} + \frac{z_2^3}{3} + \dots)\frac{dz_2}{z_2} = \Big[z_2 + \frac{z_2^2}{2^2} + \dots \Big]_0^1.$$

And similarly

$$\zeta(3) = \int_{0<z_1<z_2<z_3=1} \frac{dz_1}{1-z_1} \wedge \frac{dz_2}{z_2} \wedge \frac{dz_3}{z_3}$$

Now consider

$$\int_{0<z_1<z_2<z_3<1} \frac{dz_1}{1-z_1} \wedge \frac{dz_2}{1-z_2} \wedge \frac{dz_3}{z_3} = \int_{0<z_2<z_3<1} (\sum_m \frac{z_2^m}{m}) \frac{dz_2}{1-z_2} \wedge \frac{dz_3}{z_3}$$

$$= \int\int (\sum_m \sum_n \frac{z_2^m z_2^{n-1}}{m} dz_2)\frac{dz_3}{z_3} = \int_0^1 \frac{z_3^{m+n}}{m(m+n)} \frac{dz_3}{z_3} = \sum_{m,n} \frac{1}{m(m+n)^2} = \sum_{m<n} \frac{1}{mn^2} = \zeta(1,2)$$

But notice now that by a change of variables $z_i \mapsto 1 - z_i$ we have the relation

$$\zeta(1,2) = \zeta(3).$$

Now for a Riemann surface. Let w_1, w_2 be 1-forms on X a Riemann surface, and $\gamma : [0,1] \to X$ a path. Consider

$$\int_{0<t_1<t_2<1} w_1(\gamma(t_1)) \wedge w_2(\gamma(t_2))$$

Associate a generating sequence F to each γ define $dF = F(A_1 w_1 + \dots + A_n w_n)$, so then

$$F_0 = 1, dF_1 = F_0(A_1 w_1 + \dots + A_n w_n),$$

$$F_1 = 1 + \sum A_i \int_\gamma w_i \quad dF_2 = F_1(A_1 w_1 + \dots + A_n w_n)$$

$$F_2 = 1 + \sum A_i \int w_i + \sum_{i,j} A_i A_j \int_\gamma w_i \circ w_j$$

Such F's satisfy the properties $F_{\gamma_1\gamma_2} = F_{\gamma_1} F_{\gamma_2}$ and $F_\gamma F_{\gamma^{-1}} = 1$, and F_γ is homotopy invariant with respect to γ, i.e., up to fixing ends points.

Now we may express the homotopy group $\pi_1(X, \mathbb{Q}) = \langle \sigma_{P_1}, \dots, \sigma_{P_n}, \alpha_1, \beta_1, \dots, \alpha_g, \beta_g \rangle$ where σ_i denotes the simple loops around P_i. We have the relation

$$\delta = \prod_i \sigma_i \prod [\alpha_j, \beta_j] \sim 1$$

where $[\alpha, \beta]$ is the usual commutator bracket. So

$$1 = F_\delta = F_{\sigma_1} \dots F_{[\alpha_g, \beta_g]}$$

Lemma 9.3.1. *Composition of paths:* $F_{\gamma_1\gamma_2} = F_{\gamma_1} F_{\gamma_2}$*, and*

$$\int_{\gamma_1\gamma_2} w_1 \circ w_2 = \int_{\gamma_1} w_1 \circ w_2 + \int_{\gamma_2} w_1 \circ w_2 + \int_{\gamma_1} w_1 \int_{\gamma_2} w_2$$

The coefficients of $A - 1, A_2$ in F_{σ_1}, where $dF = F(A_1 df_1/f_1 + A_2 df_2/f_2)$

Theorem 9.3.2. *The following identity is true:*

$$\exp(\frac{1}{2\pi i}\int_{\sigma_1}\frac{df_1}{f_1}\circ\frac{df_2}{f_2}) = (f_1, f_2)_{P_1}$$

where

$$\int\frac{df_1}{f_1}\circ\frac{df_2}{f_2} = a\log f_2 - b\log f_1\Big|_Q^P$$

where $f_1 = z^a g_1, f_2 = z^b g_2, df_1/f_1 = adz/z + dg_1/g_1, df_2/f_2 = bdz/z + dg_2/g_2$.

9.4 An n-dimensional Langlands correspondence

In this section we describe parts of Kapranov's paper 'Analogies between the Langlands correspondence and topological quantum field theory.' In particular, we will ignore the 'analogies with TQFT', and instead focus on the perspectives on the Langlands correspondence.

First, recall that a local field is a complete discrete valued field with finite residue field, i.e., a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$. Then an n-dimensional local field is one whose residue field is an $(n-1)$ dimensional local field, e.g., $\mathbb{Q}_p((t))$ and $\mathbb{F}_p((t_1))((t_2))$. Parshin points out that these arise naturally as completions of fraction fields on schemes of absolute dimension n.

9.4.1. The formal structure of the Langlands correspondence. Let F be a 1-dimensional local field. The Langlands correspondence is betwen l-adic representations of W_F (or complex representations of WD_F of dimension m and l-adic or complex admissible representations of $GL_m(F)$, satisfying certain properties. Following Langlands, rather than these Galois representations Kapranov considers the (conjectural) abelian category of M_F mixed motives over F—it is unconditional if we consider pure motives under numerical equivalence. So to every quasiprojective variety X over F we associated objects $h^i(X)$.

If A is any commutative ring, we assume the existence of the category of motives M_F with coefficients in A, or with complex multiplication by A. One first defines an additive category $M_F\otimes A$ with

$$\mathrm{Hom}_{M_F\otimes A}(V\otimes A, W\otimes A) = \mathrm{Hom}_{M_F}(V, W)\otimes_{\mathbb{Z}} A$$

and tries to complete it to an abelian category. This presents the same obstructions as in the case of motives.

Definition 9.4.1. Now we call a short exact sequence of motives

$$0\to V'\to V\to V''\to 0$$

admissible if there are no subquotients W' of V' and W'' of V'' such that $W'\simeq W''\otimes\mathbb{Z}(i)$ for $i>0$. A filtration $V_1\subset\cdots\subset V_n$ of motives is called admissible if any short exact sequence $0\to Vi\to V_j\to V_j/V_i\to 0$ is.

Now for a finite extension F of $\mathbb{Q}_p$, define a *Langlands correspondence* for F to be the system associating:

1. To any motive V, a complex vector space $L(V)$.
2. To any two isomorphic motives V, W and F-linear isomorphism g of realizations, an isomorphism of vector spaces $L(g): L(V)\to L(W)$
3. For every admissible exact sequence $0\to V'\to V\to V''\to 0$, a homomorphism $L(V')\otimes L(V'')\to L(V)$

behaving functorially with respect to composition of morphisms. The first two conditions describe GL_* representations associated to Galois representations, while the third incorporates parabolic induction, where given any exact sequence one has

$$\mathrm{Ind}_P^{GL(V_F)}(L(V')\otimes L(V''))\to L(V).$$

For a global correspondence, we consider the tensor product of all the realizations of V over all places of F.

What would a 0-dimensional correspondence look like? A 0-dimensional local field is simply $\mathbb{F}_p$, rather than a vector space $L(V)$ we associate an element of some ring R, and condition (3) is then $L(V')L(V'')=L(V)$. Take $R=\mathbb{Q}[t]$, and define $L(V)=\det(1-t\mathrm{Frob}|V)$ the characteristic polynomial of the Frobenius action on the l-adic realization. Then the 'Langlands correspondence' in this sense associates to a motive over $\mathbb{F}_p$ its L-function. Furthermore, for the 1-dimensional scheme $\mathrm{Spec}(\mathfrak{o}_F)$, the correspondence should associate the global L-function $\prod L_p(V,s)$.

9.4.2. The formal construction of the Langlands correspondence. In this section we assume the existence of 'fantastically vague and ill-defined' moduli spaces of motives with prescribed realizations, not only those given by abelian varieties, which we know about.

Let $\mathbb{A}_f$ denote the finite adeles of a given number field F. For any free $\mathbb{A}_f$-module E let $P(E)$ be the moduli space of pairs (V,ψ) where V is motive over F with complex multiplication by F, and $\psi: W_F\to E$ is an isomorphism of $\mathbb{A}_f$-modules. For example, if $F=\mathbb{Q}$ and $E=\mathbb{A}_f^2$ then any elliptic curve with full level structure defines a point of $P(E)$, hence $P(E)$ contains the profinite modular curve $M_\infty=\varprojlim M_N$ where M_N is the moduli space of elliptic curves with level structure N. In general, $P(E)$ is an ill-defined pro-algebraic variety over F containing infinitely many components in different dimensions: taking any variety X over F with rank $H^i(X,\mathbb{A}_f)=$ rank E and any identification $H^i(X,F)\to E$ we get a point of E. Now taking multiplicities (or Hom in M) can be seen a a kind of integration

Now let $h^i(P(E))$ be a motive of the 'variety' $P(E)$. Given a motive V we define an F-vector space

$$L_f^i(V)=\ \mathrm{Hom}_M(V,h^i(P(V_{\mathbb{A}_f}))$$

which carries an action of $GL(V_{\mathbb{A}_f})$, and consider the virtual representation given by the alternating sum of $L_f^i(V)$. This is, very roughly, the formal structure of the standard method for constructing the correspondence. More precisely, it is traditionally defined as the multiplicity space of the profinite modular curve, on which $GL(V_{\mathbb{A}_f})$ acts, in the cohomology of a (profinite) Shimura variety.

9.4.3. Charades and Waldhausen spaces. We begin with a definition:

Definition 9.4.2. Let $f:\mathcal{A}\to\mathcal{B}$ be an exact functor of abelian categories, and $\mathcal{E}$ a class of short exact sequences in $\mathcal{A}$ in which one of the terms is zero, and containing any sequence isomorphic to a sequence from $\mathcal{E}$. Let k be a field. Then a k-linear charade over f w.r.t $\mathcal{E}$ is a collection Λ of the following:

1. For every object A of $\mathcal{A}$, k-vector space $\Lambda(a)$
2. For any isomorphism in $\mathcal{B}$, $g: f(A)\to f(A')$, a linear isomorphism $\Lambda(g):\Lambda(A)\to\Lambda(A')$
3. For every exact sequence $0\to A'\to A\to A''\to 0$ in $\mathcal{A}$ belonging to $\mathcal{E}$, a linear operator $\Lambda(A')\otimes\Lambda(A'')\to\Lambda(A)$.

The Langlands correspondence is an example of a k-linear charade over the realization functor $M\to\mathrm{Vect}_F$, where $F=\mathbb{C}$ or $\overline{\mathbb{Q}}_l$ with respect to $\mathcal{E}$. Another example: the determinant Det forms a k-linear charade over the identity functor of the category of finite dimensional vector spaces.

Now the Waldhausen S-construction is as follows: given $f:\mathcal{A}\to\mathcal{B}$ as before we associate a simplicial category (i.e., a functor from Δ to a given category) $S_\bullet(f)$. For any m define $S_m(f)$ to be the category whose objects are filtrations of length m in $\mathcal{A}$, i.e, monomorphisms $A_1\subset\cdots\subset A_m$, and morphisms are collections of isomorphisms $f(A_i) \to f(A'_i)$ such that the diagram of inclusions commutes. The simplicial face maps ∂_i omits the i-th term in the filtration, for $i \neq 0$, and for $i = 0$ replaces the filtration $A_1 \subset \cdots \subset A_n$ with $A_2/A_1 \subset \cdots \subset A_n/A_1$. Finally, if $\mathcal{E}$ is a class of short exact sequences as above, then denote by $S_m(f, \mathcal{E})$ the subcategory of admissible filtrations.

Remark 9.4.3. When f is the identity and $\mathcal{E}$ consists of all short exact sequences, Waldhausen showed that the homotopy groups of $S(\mathcal{A})$ give the K-theory of $\mathcal{A}$, i.e., $\pi_i(S(\mathcal{A}) = K_{i-1}(\mathcal{A})$.

Proof. Given a k-linear charade, we construct the following stack:

1. To the only 0-cell we associate Vect_k,
2. To 1-cells, which correspond to A in $\mathcal{A}$, we associate the functor $\Lambda(A) \otimes -$

(2a) To a 2-cell square corresponding to the isomorphism $g : f(A) \to f(A')$ with two degenerate faces, we associate the natural transformation induced by $\Lambda(g) : \Lambda(A) \to \Lambda(A')$;

(2b) And to a 2-cell triangle corresponding to an exact sequence, we associate the natural transformation induced by $\Lambda(A') \otimes \Lambda(A'') \to \Lambda(A)$. From the axioms of charades we also get compatibility with 3-cells. And the converse construction is straightforward.

□

Proposition 9.4.4. *Let $F : \mathcal{A} \to \mathcal{B}$ and $\mathcal{E}$ be as above. There is a bijection between k-linear charades over f and combinatorial stacks on $S_\bullet(f, \mathcal{E})$ associating to the unique point $S_0(f, \mathcal{E})$ the category Vect_k, and to 1-cells functors $\mathrm{Vect}_k \to \mathrm{Vect}_k$ of the form $V \mapsto X \otimes V$.*

In particular, the Langlands correspondence associates a stack on the Waldhausen space of M, the realization functor, and admissible exact sequences. Here a combinatorial stack is a sheaf of categories on a bisimplicial set, where to every 0-cell is associated a category, to every 1-cell a functor between categories, to every 2-cell a natural transformation of functors, satisfying compatibility conditions as a sheaf.

9.4.4. A framework for a 2-dimensional Langlands correspondence. Now let F be a 2-dimensional local field, and M the category of motives over F with complex multiplication. Consider the exact functor $f : M \to \mathrm{Vect}_F$ that is two copies of the realization functor, $f(V) = V_F \oplus V_F$. To the resulting Waldhausen space we would like to associate stacks of 2-categories.

Definition 9.4.5. A module category over Vect_k, where k is a field, is a category C with a bifunctor $\oplus : C \times C \to C$ and $\otimes : \mathrm{Vect}_k \times C \to C$ satisfying natural axioms. By definition, a 2-vector space is such a module category.

Here is Kapranov's proposal for a 2-dimensional Langlands correspondence: as in the 1-dimensional case, we restrict our attention to 2-stacks whose value at the 0-cell of $S(f, \mathcal{E})$ is 2-Vect_k, where k is either $\mathbb{C}$ or $\overline{\mathbb{Q}}_l$. Then 1-cells of $S(f, \mathcal{E})$ correspond to motives of M, and the correspondence should associate some 2-functor.

9.5 A survey of the relationship between algebraic K-theory and L-functions (Glasman)

9.5.1. Spectra: 'topologists' chain complexes'. A *spectrum* X is a sequence of pointed spaces $X_0, X_1, \ldots$ together with homotopy equivalences to to based loop spaces

$$X_n \xrightarrow{\sim} \Omega X_{n+1}$$

that (1) are very coherently commutative (in the sense of concatenating loops) (2) come with graded abelian groups

$$\pi_i X =: \pi_{n+i} X_n, i \in \mathbb{Z}$$

these act like chain complexes.

9.5.2. Algebraic K-theory. Let R be a commutative ring, P, Q, projective R-modules, then

$$\mathrm{Aut}_R(P) \times \mathrm{Aut}_R(Q) \to \mathrm{Aut}_R(P \oplus R)$$

The matrices

$$\begin{pmatrix} \phi & \\ & \psi \end{pmatrix}$$

are not commutative, but

$$\begin{pmatrix} \phi & \\ & \psi \end{pmatrix}, \begin{pmatrix} \psi & \\ & \phi \end{pmatrix}$$

are conjugate by a known element of $\mathrm{Aut}_R(P \oplus R)$.

Recall that the classifying space of a group G is the unique space BG with $\Omega BG = G$ and $\pi_n BG = G$ if $n = 1$ and 0 otherwise. $[X, BG]$ classifies G-torsors on X. Now define

$$X = \coprod_{P/\sim} B\mathrm{Aut}_R(P)$$

where P are finitely generated projective R modules. X is a topological monoid, i.e., we have a morphism

$$\mu : X \times X \to X$$

that is coherently homotopy commutative. Call M a *commutative monoid space*, if $\pi_0 M$ is a commutative monoid. We say M is *grouplike* if $\pi_0 M$ is a group.

Theorem 9.5.1 (Quillen)**.** *There is a topological group completion functor such that gives ordinary group completion on π_0, but the effects on higher homotopy are complex and unpredictable.*

Definition 9.5.2. Define $K(R)$ to be the topological group completion of X, and $K_n(R) = \pi_n K(R)$.

Algebraic K-theory arises as the zero space of a spectrum.

For the K-theory of schemes, remark that K satisfies a topological form of Zariski descent, use some kind of Waldhausen construction on the category of perfect complexes on a scheme X (not necessarily affine) to get $K(X)$.

For the K theory of a finite field, we introduce BU the classifying space of the infinite unitary group U. There's a self map $\psi^r : BU \to BU$ for $r = 1, 2, \ldots$, called the r-th Adams operations. This acts like a 'Frobenius surrogate'. By Bott periodicity,

$$\pi_n BU = \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

And ψ^r acts by r^n on $\pi_{2n} BU$. This is reminiscent of Fontaine's period rings, say, B_{dR}. There's a homotopy fiber sequence (something that gives a long exact sequence on homotopy groups),

$$K(\mathbb{F}_q) \to BU \overset{\psi^q - 1}{\to} BU$$

which is a hard calculation by Quillen. So we can read off data for $K(\mathbb{F}_q)$:

$$K_n(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/(q^k - 1) & n = 2k - 1 \\ 0 & \text{otherwise} \end{cases}$$

We think of the fiber of the map $\psi^q -$ as looking for the fixed points, which is some equalizer in the category of spectra.

Theorem 9.5.3. *If we define $\zeta(Spec\mathbb{F}_q, s) = (1-q^{-s})^{-1}$, then for $n > 0$ we have*

$$\zeta(Spec\mathbb{F}_q, s) = \frac{|K_{2n}(\mathbb{F}_q|}{|K_{2n-1}(\mathbb{F}_q)|}$$

More generally, there are analogues of this for varieties, related to Beilinson's conjectures.

One of the main tools for K-theory calculations is the motivic spectral sequence. The K-theory spectrum has a filtration (Grayson, Levine, et al.), i.e., the weight filtration and coniveau filtration (proved to be the same), whose associated graded are motivic cohomology spectra. Here it is:

$$E_2^{p,q} = H^q_{\text{mot}}(X, \mathbb{Z}(p) \Rightarrow K_{2p-q}(X)$$

where X is a scheme.

Conjecture 9.5.4 (Quillen-Lichtenbaum)**.** *Suppose l is invertible on X, then*

$$H^q_{mot}(X, \mathbb{Z}(p))^\wedge_p \simeq H^1_{et}(X, \mathbb{Z}_l(p))$$

for q sufficiently large (larger than the étale cohomological dimension of X).

In the motivic spectral sequence $H^i(\text{Spec}(\mathbb{F}_q), \mathbb{Z}_l(j)) = 0$ for $i = 1, j = 1, 2, 3$. What's the connection? Well, we have

$$|H^1_{\text{et}}(\text{Spec}(\mathbb{F}_q), \mathbb{Z}_l(n)) = (q^n - 1)_l = \det(1 - q^n\text{Frob})_l$$

where the Frobenius action is on $H^0_{\text{et}}(\mathbb{F}_q, \mathbb{Z}_l)$, and the subscript l meaning the l-part.

9.5.3. K-theory of curves Refer to 'On the K-theory of curves over finite fields' by Kevin Coombes.

Theorem 9.5.5 (Coombes)**.** *Let X be a smooth projective curve over $\mathbb{F}_q$. Then*

$$K_{2n}(X)_l \simeq T_l Pic(X)/(1 - q^n Frob)$$

where T_l denotes the l-adic Tate module, $T_l Pic(X) = H^1(X, \mathbb{Z}_l(1))$. Then

$$|T_l Pic(X)/(1 - q^n Frob)| = \det(1 - q^n Frob)|_{T_l Pic(X))}$$

This gives the contribution of $|K_{2n}(X)|$ to $\zeta(X, -n)$. But this is not in the Quillen-Lichtenbaum range.

Question 9.5.6. *Is this related to $H^2_{et}(X, \mathbb{Z}_l(n+1)) \Rightarrow K_{2n}(X)$? This is in the Quillen-Lichtenbaum range.*

Question 9.5.7. *What about the values of $\zeta(X, s)$ at positive integers?*

Answer: $K(1)$-localization in chromatic homotopy theory. Here are some properties:

1. This is a categorical localization, similar to the completion of modules at an ideal (but here there is no ideal). The localization functor

$$L_{K(1)} : Sp \to Sp$$

   commuting with $K(1)$-local spectra. (This $K(1)$ is Morava K-theory).
2. $K(1)$-localization depends on a prime l, in the sense that $K(1)$-local objects are l-complete.
3. $L_{K(1)}X$ depends only on the germ of X at ∞. That is to say, if $f : X \to Y$ induces $\pi_\infty(X) \simeq \pi_n()$ for n large enough, then

$$L_{K(1)}f : L_{K(1)}X \xrightarrow{\sim} L_{K(1)}Y$$

4. Typically have homotopy in arbitrarily negative degree.

Theorem 9.5.8 (Thomason)**.** *$K(1)$-localized K-theory is what you get by forcing Quillen-Lichtenbaum to hold in all degrees.*

Claim: $K(1)$-localization is like analytic continuation. Why? There are certain topological dualities on $K(1)$-local spectra, i.e., Spanier-Whitehead, Brown-Comenetz, and Gross-Hopkins, which give rise to functional equations

$$(\pi_n L_{K(1)} K(\mathbb{F}_q))^\vee \simeq \pi_{2-n} L_{K(1)} K(\mathbb{F}_q)$$

for $n \neq 0, 1$.

Remark 9.5.9. No time to mention: Connes-Consani and Hesselholt's approach to Riemann zeta function using periodic cyclic homology.

9.6 The Stark conjectures

9.6.1. Dirichlet's regulator. Dirichlet's Class Number Formula gives the leading coefficient of the Taylor expansion of the Dedekind zeta function at $s = 0$,

$$\zeta_k(0) = -\frac{hR}{e} s^{r_1+r_2-1} + O(s^{r_1+r_2})$$

where h is the class number, R the regulator, and e the number of units in k. There is also an S version where S is a finite set of primes of k including the archimedean places, in which case $r_1 + r_2$ is replaced by $|S|$.

We define the regulator more precisely: let $u_1, \ldots, u_{|S|-1}$ be the fundamental units of $\mathcal{O}_S^\times$, that is, generators modulo roots of unity, or representatives for a basis of the maximal torsion-free quotient. This follows from Dirichlet's S-unit theorem that

$$\mathcal{O}_k^\times \simeq \mathbb{Z}^{|S|-1} \times \{\text{torsion}\}.$$

Then let $\sigma_1, \ldots, \sigma_{|S|}$ be embeddings of k, the complex embeddings taken up to conjugation, and form the $|S| - 1 \times |S||$matrix

$$A = (\log |\sigma_i(u_j)|).$$

Then if we defined A_i to be the $|S| - 1 \times |S| - 1$ matrix where the i-th column of A is removed, then the regulator of k is defined to be

$$R = |\det(A_i)|$$

and is independent of i.

9.6.2. Stark's regulator S will still be as above. Let K be a finite Galois extension of k with Galois group G, and S_K the set of primes in K above S. Then define a hyperplane

$$X = \{\sum_{v \in S_K} n_v v : \sum_{v \in S_K} n_v = 0\}.$$

We have an isomorphism of $\mathbb{C}[G]$-modules,

$$\varphi : X \otimes \mathbb{C} \to \mathcal{O}_k^\times \otimes \mathbb{C}$$

In fact, they are isomorphic as $\mathbb{Q}$-modules, though not canonically. Also, the logarithmic embedding

$$\lambda(u) = \sum_v \log |u|_v v$$

of $\mathcal{O}_k^\times$ into $X \otimes \mathbb{R}$ induces an isomorphism of $\mathbb{C}[G]$-modules, so that the composition gives an automorphism of $X \otimes C$.

Now let V be a finite-dimensional complex representation of G with character χ, and denote $V^* = \mathrm{Hom}(V, \mathbb{C})$, then $\lambda \circ \varphi$ induces a $\mathbb{C}$-linear automorphism of $\mathrm{Hom}_G(V^*, X \otimes \mathbb{C})$ sending any map

$$f \mapsto \lambda \circ \varphi \circ f$$

Finally define the *Stark regulator* to be the determinant of this automorphism, denoted

$$R(\chi, \varphi) = \det(\lambda \circ \varphi)$$

In the case where $K = k$, the Galois group is trivial and V is the trivial representation, we have

$$\mathrm{Hom}_G(V^*, X \otimes \mathbb{C}) \simeq X \otimes \mathbb{C}$$

and so the determinant of the automorphism is indeed the determinant of $\lambda \circ \varphi$. Note that this notation follows that of Dasgupta rather than Stark's.

9.6.3. The Stark conjecture To state the conjecture, which should be thought of as a class number formula for Artin L-functions, we first introduce some notation: Define $c(\chi)$ to be the first nonzero coefficient in the Taylor series of $L(s, \chi)$. If χ is a character of a finite group G, then denote by $\mathbb{Q}(\chi)$ the extension $\mathbb{Q}(\chi(\sigma_1), \ldots, \chi(\sigma_n)$ for all σ_i in G. Finally, if $\alpha : K \to F$ is a field homomorphism, denote by $\chi^\alpha = \alpha \circ \chi$.

Conjecture 9.6.1. *Let K be a finite Galois extension of k with Galois group G. Let χ be the character of a finite dimensional complex representation V of G, and let $\varphi : X \otimes \mathbb{Q} \to O_k^\times \otimes \mathbb{Q}$ be a homomorphism of $\mathbb{Q}[G]$-modules. Then*

$$\frac{R(\chi, \varphi)}{c(\chi)} \in \mathbb{Q}(\chi)$$

and

$$\left(\frac{R(\chi, \varphi)}{c(\chi)}\right)^\alpha = \frac{R(\chi^\alpha, \varphi)}{c(\chi^\alpha)}$$

for all α in Gal$(\mathbb{Q}(\chi)/\mathbb{Q})$.

Let us consider some special cases. If the order of vanishing of $L(s, \chi)$ at $s = 0$ is $r(\chi)$, then one has the formula

$$r(\chi) = \sum_{v|\infty} \dim V^{G_w} - \dim V^G$$

where V is the vector space defining χ, and G_w is the decomposition group at a place w of K lying over an infinite place v of k. From this it follows that if K/k is abelian and χ is faithful, then $r(\chi) = 1$ if and only if the decomposition group is trivial at exactly one v. In this case the regulator is 1×1, and

$$L'(0, \chi) = -\frac{1}{e} \sum_\sigma \chi(\sigma) \log |\sigma(\epsilon)|$$

for a suitable element ϵ in K called the Stark unit, e denotes the number of roots of unity in K, and the absolute value is taken with respect to the infinite place lying over v where the decomposition group is trivial. The Stark unit is such $K(\epsilon^{1/e})$ is abelian over k. The case $K = \mathbb{Q}$ is well known, while when K is imaginary quadratic it is an elliptic unit.

Finally, we note that this refines the class number formula because $\zeta_k(s) = \prod_\chi L(s, \chi)$.

9.6.4. Tate's reformulation for rank one abelian case. We first state Tate's reformulation of the rank one abelian Stark conjecture, that is, the conjecture for $r(\chi) = 1$ and K/k abelian. Define a meromorphic function on $\mathbb{C}$ with values in $\mathbb{C}[G]$

$$\theta_S(s) = \sum_{\chi \in \hat{G}} L_S(s, \chi) e_{\bar{\chi}}$$

where $\hat{G}$ denotes the Pontrjagin dual, and e_χ is a central idempotent associated to irreducible characters χ

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \bar{\chi}(\sigma)\sigma \in \mathbb{C}[G]$$

which acts as a projection onto the χ component in the canonical decomposition of G.

The function $\theta(s)$ satisfies

$$\chi(\theta_S^{(n)}(s)) = L_S^{(n)}(s, \bar{\chi})$$

for any $s \in \mathbb{C}$ and character χ of G, and $^{(n)}$ denotes the n-th derivative. Also, for $\mathrm{Re}(s) > 1$, we have the absolutely convergent expression

$$\theta_S(s) = \prod_{\mathfrak{p} \notin S} (1 - \mathrm{Frob}_{\mathfrak{p}} N\mathfrak{p}^{-s})^{-1}.$$

Then Tate's reformulation of the rank one abelian case is:

$$\theta'_S(0) X \subset \frac{1}{e} \lambda(\mathcal{O}_{K/k}^{\times,\mathrm{ab}}).$$

9.6.5. The Brumer-Stark conjecture Now Let T be a finite set of primes of k containing the archimedean places and primes which ramify in K, and T_K the set of primes in K lying above those in T. Define

$$k_T = \begin{cases} \{x \in k : |x|_w = 1 \text{ for all } w \in T_K\} & |T| \geq 2 \\ \{x \in k : |x|_{\sigma w} = |x|_w \text{ for all } \sigma \in G\} & T = \{w\}, w|v \end{cases}$$

Define $S = T \cup \{\mathfrak{p}\}$ where $\mathfrak{p}$ is a finite prime of k that splits completely in K. Consider the Stark conjecture for $v = \mathfrak{p}$ and $w = \mathfrak{P}$ a prime lying above $\mathfrak{p}$. Note that in $N\mathfrak{p} = N\mathfrak{P}$. The Stark unit ϵ is a unit away from the primes dividing $\mathfrak{p}$, and we write

$$(\epsilon) = \prod_{\sigma \in G} (\mathfrak{P}^\sigma)^{n_\sigma}, \quad n_\sigma = -\frac{\log|\epsilon|_{\mathfrak{P}^\sigma}}{\log N\mathfrak{P}} \in \mathbb{Z}.$$

If we define $\lambda = \sum n_\sigma \sigma$ so that $(\epsilon) = \mathfrak{P}^\lambda$, one can show that

$$\lambda = \sum_{\sigma \in G} -\frac{\log|\epsilon|_{\mathfrak{P}^\sigma}}{\log N\mathfrak{P}} \sigma = \sum_{\sigma \in G} \frac{e\zeta'_S(0, \sigma^{-1})}{\log N\mathfrak{P}} \sigma = \frac{e\theta'_S(0)}{\log N\mathfrak{P}} = e\theta_T(0).$$

Theorem 9.6.2 (Deligne-Ribet)**.** *Let A be the annihilator of the $\mathbb{Z}[G]$-module $\mu(K)$. Then $A \cdot \theta_T(0) \subset \mathbb{Z}[G]$, and in particular $e\theta_T(0) \subset \mathbb{Z}[G]$.*

Now Brumer's idea that $e\theta_T(0)$ annihilates the class group of $\mathfrak{O}_K$ generalizes Stickelberger's theorem, which led to what Tate called the Brumer-Stark conjecture:

Conjecture 9.6.3 (Brumer-Stark)**.** *Let I be a fractional ideal of K. Then there exists an ϵ in K_T such that $(\epsilon) = I^{e\theta_T(0)}$ and $K(\epsilon^{1/e})$ is an abelian extension of k.*

9.7 Semistable abelian surfaces of given conductor (Kramer)

9.7.1. Introduction This is joint with A. Brumer. Start with a genus 2 curve,

$$C : y^2 + Q(x)y = P(x)$$

with $f = Q^2 + 4P$ of deg 5 or 6, and discriminant $\Delta_C \neq 0$. The primes $p|\Delta_C$ iff bad reduction for the curve.

Definition 9.7.1. Semistable: Let A be an abelian surface, with p a bad prime for A.

$$0 \to T_p \to \tilde{A}_p^0 \to B_p \to 0$$

where B_p is the abelian variety over $\mathbb{F}_p$, $\tilde{A}_p^0$ the Neron model for A, and a torus T_p with dimension $t_p =$ 1 or 2. Then the conductor of A is $\prod p^{t_p}$.

Example 9.7.2. $C_{277} : y^2 + y = x^5 - 2x^3 + 2x^2 - x, \Delta_C = 277$. Here $\bar{f}_{277} = 4(x + 12)^2$, a cubic with multiplicative reduction, and $A = J(C_{277})$, $\mathrm{cond}(A) = 277$. This is an S-unit problem to find the curves with good reduction outside S.

Example 9.7.3. From LMFDB: $C : y^2 = -3(x^6 - 6x^4 + 2x^3 - 3x^2 - 18x - 19), \Delta_C = 3^2 2 - 587$. Over $\mathbb{F}_q$, $J(C)$ looks like a product of two ellpitic curves, but $\mathbb{Q}_3(J(C)[\ell^\infty]), \ell \neq 3$ is unramified, so $A = J(C)$ is good at 3. Call 3 mild prime if the curve has bad reduction at 3 but the Jacobian does not.

Remark 9.7.4. Warning: in general, it is not guaranteed to have a Jacbian in an isogeny class of an abelian surface.

Theorem 9.7.5 (BK)**.** *If A is an abelian surface of prime conductor, then there is a Jacobian in its isogeny class.*

Idea is to look for Galois modules in the isogeny class, and find a principally polarized abelian variety.

9.7.2. Uniqueness of the isogeny class for certain conductors. Go back to $N = 277$, B of conductor 277. Find that

$$\mathrm{Gal}(\mathbb{Q}(B[2])/\mathbb{Q}) \hookrightarrow Sp(\mathbb{F}_2) \simeq S_6.$$

Previous nonexistence results imply $B[2] \simeq A[2], A = J(C_{277})$.

Remark 9.7.6. From now on fix A to be a semistable abelian surface of prime conductor N. Let $E = A[p], p \neq N$ prime, it is a finite flat group scheme over $\mathbb{Z}[1/N]$, such that that E is locally absolutely irreducible at p, and $\mathrm{cond}(E) = N$. Also fix B an abelian variety of dimension $2d$, such that $B[p]$ is filtered by E's. The semisimplification $B[p]^{\mathrm{ss}} \simeq E^d$.

Question 9.7.7. *What does it take to make B isogenous to A^d?*

Let C be the category of p-primary finite flat group schemes V over $\mathbb{Z}[1/N]$ satisfying

1. Composition factors of V all are E
2. Semistability: Inertia at $v|N$ is tame, generator σ_v and $(\sigma_v - 1)^2$ annihilates V.
3. Artin conductor of V is N^m where m is the multiplicity of E in V. (This is the key condition.)

Theorem 9.7.8. *Assume the above, also that cond$(B) = N^d$ and Ext$^1_{[p],C}(E, E) = 0$. Then B is isogenous to A^d.*

Proof idea. Asummptions of irreducibility of E and no extensions killed by p, then

$$0 \to E \to A[p^2] \xrightarrow{p} E \to 0.$$

This uses Schoof's general result that shows p-divisible groups of B and A^d are isomorphic, plus Faltings.

How to check Ext^1 is 0? Let $p = 2$, pick a favourable abelian surface A and $F := \mathbb{Q}(E)$. Want $\mathrm{Gal}(\mathbb{F}/\mathbb{Q}) \simeq S_5$. (Note: consider $y^2 + y =$ monic quintic, find f with discriminant $\pm 2^8 N$.)

Use a *stem field* K for F, i.e., a field K whose the Galois closure of $K/\mathbb{Q}$ is $F/\mathbb{Q}$. Choose $K = \mathbb{Q}(r_1 + r_2)$, where r_q, r_2 are a pair of roots of quintic f, so $[K : \mathbb{Q}] = 10$. □

Theorem 9.7.9. *There is a unique prime $\mathfrak{p}|2$ in K such that $Ext^1_{[2],C}(E, E) = 0 \Leftrightarrow$ (*) there is at most one quadratic extension of K of ray class conductor dividing $\mathfrak{p}^4\infty$ and none of the ray class conductors dividing $\mathfrak{p}^2\infty$.*

Ingredients: Take a nontrivial extension

$$0 \to E \to W \to W \to 0$$

and set $L = \mathbb{Q}(W), G = \mathrm{Gal}(L/\mathbb{Q})$. Define a representation $\rho_W(g)$ to be $\rho_E(g)$ on the diagonal, 0 on the sub diagonal and arbitrary on the super diagonal. This is an 8 by 8 parabolic.

The Galois group of the extension L/F is an elementary two group, and $F/\mathbb{Q}$ is S_5. Now use number theory for better control of L:

1. Over N. Consider $\sigma_v, v|N$. Then $(\sigma_v - 1)^2 = 0 \Leftrightarrow \sigma_v$ is an involution. Rank $(\sigma_v - 1) = 2$.
2. Over 2. Fontaine has conductor bounds for p-primary group schemes over $\mathbb{Z}_p$. Needed to improve these using Honda systems, which classify W's. Then had to make subtle computations, to control abelian conductor for $\mathbb{Q}_p(W)/\mathbb{Q}_p(E)$.

Lemma 9.7.10. *G is generated by involutions σ such that the rank$(\sigma - 1) = 2$.*

Proof. Let M be the fixed field of all the $\sigma_v, v|N$. The Fontaine bound gives $M \subset \mathbb{Q}(i)$, which does not fit in our conductor bound at 2. Conclude that $M = \mathbb{Q}$. □

From the conductor bound, we get a short list of possible groups G. Find a normal subgroup $\mathrm{Gal}(L/K') = H'$ of $\mathrm{Gal}(L/K) = H$ such that H/H' is abelian, by the condition that $\cap gH'g^{-1} = \{1\}$, and K'/K quadratic. Note that L could have degree up to $2^{15}(5!)$ over $\mathbb{Q}$!

General result of Rene Schoof about patching:

$$\mathrm{Ext}^1_{\mathbb{Z}[1/N]}(E, E) \to \mathrm{Ext}^1_{\mathbb{Z}_p}(E, E) \times \mathrm{Ext}^1_{\mathbb{Z}[1/pN]}(E, E) \to \mathrm{Ext}^1_{\mathbb{Q}_p}(E, E)$$

For us the left factor is killed by the 2 and classified by Honda systems, the right factor is the Galois module from the field L (the 8 by 8), so we find the group scheme on the far left.

Corollary 9.7.11. *Let A be a favourable abelian surface, $p = 2$, and criterion (*) holds. If B is an abelian variety of dimension $2d$ and conductor N^d, $B[2]$ filtered by $E = Q[2]$, then B is isogenous to A^d.*

Remark 9.7.12. Why is this interesting? According to paramodular conjecture, by this result there should be a unique Siegel modular form of level 277. By Poor and Yuen, there is a unique eigenform of the right type in the paramodular conjecture. Uses Gritsenko lifts.

10 Stable homotopy theory

10.1 Manifolds and modular forms

For the moment we will consider all our manifolds M to be compact, oriented, and differentiable.

Definition 10.1.1. We say a manifold M *bounds* if there exists a manifold N such that $\partial N = M$. For example, $S^n = \partial B^{n+1}$, so S^n bounds. Denote by $-N$ the manifold N given the opposite orientation. Then if $M + (-N)$ bounds we say that M and N are *cobordant.* This is an equivalence relation on manifolds.

Definition 10.1.2. Denote by Ω^n the set of manifolds M up to cobordism. Then $(\Omega^n, +)$ is a finitely generated abelian group, where $+$ is given by set union. Moreover,

$$\Omega = \oplus_{n=0}^{\infty} \Omega^n$$

forms a graded commutative ring, with the unit given by the equivalence class of a point. It has one generator in each degree not equal to $2^k - 1, k = 1, 2, \ldots$. This is the *cobordism ring.*

Theorem 10.1.3. *(Thom) $\Omega^n \otimes \mathbb{Q} = 0$ for $4 \not| n$, the rank of Ω^{4k} is equal to the number of partitions of k, and*

$$\Omega \otimes \mathbb{Q} = \mathbb{Q}[P_2(\mathbb{C}), P_4(\mathbb{C}), \ldots]$$

that is to say, after tensoring with $\mathbb{Q}$ we obtain a $\mathbb{Q}$ algebra generated by $P_2k(\mathbb{C})$.

Definition 10.1.4. A *genus* is a ring homomorphism $\varphi : \Omega \otimes \mathbb{Q} \to R$ where R is any integral domain over $\mathbb{Q}$, such that $\varphi(1) = 1$.

Definition 10.1.5. Consider the even power series

$$Q(x) = 1 + \sum_{n=1}^{\infty} a_{2n} x^{2n}$$

with coefficients in R. Then for indeterminates $x_1, \ldots, x_n$, the product

$$Q(x_1) \ldots Q(x_n) = 1 + a_2 \sum_{i=1}^{n} x_i^2 + \ldots$$

is a symmetric function in x_i^2. The weight $4r$ term is a homogenous polynomial of weight $4r$ in elementary symmetric functions p_i of x_i. Thus

$$Q(x_1) \ldots Q(x_n) = 1 + a_2 \sum_{i=1}^{n} x_i^2 + \cdots = 1 + K_1(p_1) + K_2(p_1, p_2) + \ldots K_n(p_1, \ldots, p_n) + K_{n+1}(p_1, \ldots, p_n, 0) + \ldots$$

Then we define the genus associated to Q as follows: given a manifold M^{4n},

$$\varphi_Q(M) = K_n(p_1, \ldots, p_n)[M] \in R$$

where $p_i = p_i(M) \in h^{4i}(M, \mathbb{Z})$, which are Pontrjagin classes, and $\varphi_Q = 0$ if $4 \not| n$. Moreover, we call Q the characteristic power series of $Q(x)$.

Example 10.1.6. Let $Q(x) = \dfrac{x}{\tanh(x)}, f(x) = \dfrac{x}{Q(x)} = \tanh x$. Then if $g = f^{-1}$, called the logarithm of φ_Q, we see that $f'(x) = 1 - f(x)^2$ and $g'(y) = 1/(1 - y^2)$. This leads to the *L-genus*, which take the value 1 on all $P_{2k}(\mathbb{C})$.

Example 10.1.7. Let $Q(x) = \dfrac{x}{\sinh(x)}, f(x) = \dfrac{x}{2Q(\frac{x}{2})} = \sinh x$.This is related to the Dican operator, and leads to the *L-genus*, which take the value 1 on all $P_{2k}(\mathbb{C})$.Call this the $\hat{A}$-genus.

Definition 10.1.8. We say φ is *elliptic* if its character power series satisfies any one the equivalent definitions below

1. $f'4 = 1 = 2\delta f^2 + \epsilon f^4$,
2. $f(u+v) = \dfrac{f(u)f'(v) + f(u)'f(v)}{1 - \epsilon f(u)^2 f(v)^2}$,
3. $f(u+v) = \dfrac{2f(u)f'(u)}{1 - \epsilon f(u)^4}$.

where $\delta, \epsilon \in \mathbb{C}$.

Example 10.1.9. The L-genus is given by $\delta = \epsilon = 1$; and $\hat{A}$-genus by $\delta = -1/8, \epsilon = 0$. The required identities are given by the addition laws of tanh and sinh.

Remark 10.1.10. If we consider stably almost complex manifolds, we allow for $Q(x)$ to be any power series, and there is a bijection between $Q(x)$ and $\varphi(M)$.

Theorem 10.1.11. *Considere the Weierstraß $\mathcal{P}$ function, with periods ω_1, ω_2. Then the function*

$$f(x) = \frac{1}{\sqrt{\mathcal{P}(z) = e_1}},$$

is an elliptic genus, and $\delta = -3/2e$, $\epsilon = (e_1 - e_2)(e_1 - e_3)$.

Definition 10.1.12. The Jacobi quartic $y^2 = 1 - 2\delta x^2 + \epsilon x^4$ is an elliptic curve, unifromized by the function

$$s(u) = -2\frac{\mathcal{P}(u) - e_3}{\mathcal{P}'(u)}.$$

where $\delta = \theta_1^4 = -\theta_2^4, \epsilon = \delta^2 = \theta_3^8$. Here the Jacobi theta functions are given by

$$\theta_1(\tau) = 2q^{1/4}\sum_{n=0}^{\infty} q^{n(n+1)}, \quad \theta_2(\tau) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} \quad \theta_3(\tau) = 1 + 2\sum_{n=1}^{\infty} q^{n^2},$$

they are modular forms of level 2, that is, on $\mathbb{H}$ with respect to the group $\Gamma_0(2)$, the two by two matrices with the lower right entry $c \equiv 0 \bmod 2$. In particular, the numbers δ, ϵ are in fact modular forms.

Definition 10.1.13. A *formal group law* over a commutative ring R with unit is a power series $F(x, y)$ such that

1. $F(x, 0) = F(0, x) = x$ (identity)
2. $F(x, y) = F(y, x)$ (reflexivity)
3. $F(F(x, y), z) = F(x, F(y, z))$ (associativity)

and inverse is immediate from inverting power series. This means that $F(x, y)$ acts like a group.

Example 10.1.14. To each genus we may associate a formal group law as follows: let $f(x)$ be an odd power series (which is true if $Q(x)$ is even). Then define for classes u, v in $H^2(M^{2n}, \mathbb{Z})$, which by Thom represent codimension two submanifolds through Poincaré duality. Then

$$f(u+v) = f(g(f(u)) + g(f(v)) = f(g(x) + g(y)) = \sum a_{rs}x^r y^s.$$

is the formal group law of f.

Example 10.1.15. The formal group law of the Jacobi quartic, or equivalently the elliptic genus, is

$$F(x, y) = \frac{x\sqrt{P(y)} + y\sqrt{P(x)}}{1 - \epsilon x^2 y^2.}$$

Landweber and Stong constucted a universal elliptic genus on Ω with values in $[\mathbb{Q}]$, more precisely, $\mathbb{Z}[\frac{1}{2}][\delta, \epsilon]$. This genus turns out to coincide with that the *Witten genus*, which is defined as follows: let L be a lattice in $\mathbb{C}$, and $\sigma_L(x)$ the Weierstraß σ function associated to L. Then the characteristic power series

$$Q(x) = \frac{x}{\sigma_L(x)} = \exp\Big(\sum_{n=2}^{\infty} \frac{2}{(2k)!} G_{2k}(\tau) x^{2k}\Big)$$

Theorem 10.1.16 (Zagier). *Let M^4k be a manifold (+ conditions). Then the Witten genus*

$$\varphi_W(M) = q^{-4k/24} \hat{A}(\dots)\Delta^{4k/24}$$

where Δ is the Ramanujan form, is a modular form of weight $2k$ with integral Fourier coefficients, and $G_{2k}(\tau)$ is the weight wk Eisenstein series with parameter.

In particular, a genus φ is a function assigning to each manifold M a modular form. But which modular forms arise this way?

Theorem 10.1.17 (Chudnovsky and Chudnovsky).

$$\varphi(\Omega^{SO}) = \mathbb{Z}[\delta, 2\gamma, \dots, 2\gamma^{2^k}, \dots]$$

where SO indicates real oriented cobordism.

We finally note the following, which is important to ellipitic cohomology, which follows from the next section:

Theorem 10.1.18 (Landweber-Ravenel). $\Omega \otimes \mathbb{Z}[\frac{1}{2}][\delta, \epsilon, \Delta^{-1}]$ *is a homology theory.*

10.2 Stable homotopy groups of spheres

Our story on complex cobordism continues: it is deeply connected to the stable homotopy groups of spheres. The starting point of the latter is the following:

Theorem 10.2.1 (Freudenthal suspension theorem). *The suspension homomorphism $\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ is surjective for $k = n-1$ and bijective for $k < n-1$. As a corollary, $\pi_{n+k}(S^n)$ depends only on k if $n > k+1$, in which case we denote by π_k^S, the k-th stable stem.*

From the previous section we discovered a map from the complex cobordism ring to the ring of modular forms,

$$\varphi : \Omega^U \otimes \mathbb{Q} \to \mathbb{Z}[\frac{1}{2}][\delta, \epsilon, \Delta^{-1}],$$

where $\Delta = \epsilon(\delta^2 - \epsilon^2)$ is the discriminant of the Jacobi quartic. We now develop the left hand side, and in the next section explore the right hand side.

10.2.1. The Thom spectrum. The first thing to see is that Ω^U can be upgraded to an extraordinary cohomology theory, represented by the Thom spectrum MU^*. So, for example, we have $\Omega_*^U(X) = \pi_*(X \wedge MU)$.

Definition 10.2.2. A graded multiplicative cohomology theory functor $X \mapsto E^*(X)$ concentrated in even degrees can be viewed as a functor taking values in the category of sheaves over a ringed space E. Also note that $E^*(\mathrm{pt}) = E$.

Example 10.2.3. We will call a cohomology theory E complex oriented if $E^*(\mathbb{C}P^\infty) = E^*(\mathrm{pt})[[x]]$ with x in $E^2(\mathbb{C}P^\infty)$. Given such an E, we have

$$E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = E^*(\mathrm{pt})[[x \otimes 1, 1 \otimes x]].$$

Now the map on $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty \times$ inducing the tensor product on complex line bundles induces a contravariant map

$$E^*(\mathrm{pt})[[x]] \to E^*(\mathrm{pt})[[x \otimes 1, 1 \otimes x]]$$

and if we denote $F_E(x \otimes 1, 1 \otimes x)$ the image of x under this map, one verifies that F_E is a formal group law over E^*. In particular, MU^* is complex oriented, thus we have a formal group law F_{MU^*}.

Note that this x is given by the Euler class of line bundles on $\mathbb{C}P^\infty$, and the formal logarithm for the Ω^U is given by Miščenko:

$$\log_{\Omega^U}(x) = \sum_{n=1}^{\infty} [\mathbb{C}P^{n-1}] \frac{x^n}{n}.$$

Remark 10.2.4. There is a natural homomorphism from MU^* to MSO^* from complex cobordism theory to real oriented cobordism theory. Localizing at the prime 2, i.e., tensoring with $\mathbb{Z}[1/2]$, the two theories are equivalent.

10.2.2. The Lazard ring. It is not hard to see that there exists a universal formal group, i.e., a formal group law over a given ring satisfying the universal property. But the structure of this ring is more difficult to discover, and was shown to be:

Theorem 10.2.5 (Lazard)**.** *There exists a formal group law $G(x, y) = \sum a_{ij} x^i y^j$ over a ring L, such that for any formal group law $F(x, y)$ over R, there exists a unique ring homomorphism $\theta : L \to R$ so that*

$$F(x, y) = \sum_{i,j} \theta(a_{ij}) x^i y^j,$$

and L is a polynomial algebra of the form $\mathbb{Z}[x_1, x_2, \dots]$.

The next theorem of shows the key role that complex cobordism plays:

Theorem 10.2.6 (Quillen)**.** *The formal group law F_{MU} associated to complex cobordism MU is isomorphic to $G(x, y)$, hence $L \simeq MU$.*

One can ask if one can associate an oriented cohomology theory to a given formal group law. We have the following criterion:

Theorem 10.2.7 (Landweber Exact Functor Theorem)**.** *For every prime p, there are elements $v_1, v_2, \dots$ in MU^* such that given any graded MU^*-module M where $(p, v_1, \dots, v_n)$ is a regular sequence in M for all p and n, then*

$$E^*(X) := MU^*(X) \otimes_{MU^*} E^*$$

is a cohomology theory.

Remark 10.2.8. Ravenel's comment is worth noting here: 'Once Quillen's theorem is proved, the manifolds used to define complex bordism theory become irrelevant, however pleasant they may be. All of the applications we will consider follow from purely algebraic properties of formal group laws. This leads one to suspect that the spectrum MU can be constructed somehow using formal group law theory and without using complex manifolds or vector bundles.'

Definition 10.2.9. View $\mathrm{Spec}(A)$ by the Yoneda Lemma as a functor

$$\mathrm{Hom}(A, -) : \mathbf{Comm} \to \mathbf{Sets}$$

where **Comm** is the category of commutative rings. Then given a commutative algebra k, $\mathrm{Hom}(L, k)$ is the set of formal group laws $F \in k[[x, y]]$. Thus we consider $\mathrm{Spec}(L)$ as a moduli space of (one-dimensional) formal group laws, and denote it by Λ.

Definition 10.2.10. Now, operations in complex cobordism can be described most easily by the functor

$$\Gamma(A) = \{g(X) = \sum_{i=0}^{\infty} g_i X^{i+1} : g_0 \in A^\times\}$$

sending a commutative ring A to invertible formal power series with coefficients in A. The latter set is a group, and the functor is representable, thus Γ is a group scheme. It acts naturally on formal group laws by

$$F_g(x, y) = g^{-1} F(g(X), g(Y)),$$

giving another formal group law.

10.2.3. The Adams-Novikov spectral sequence. To make the connection to chromatic homotopy theory, we briefly describe the ANSS. The first spectral sequence to arrive on scene for computing homotopy groups of spheres was that of Serre's.

Definition 10.2.11. The Serre spectral sequence is associated to certain fibrations

$$F \to E \to B$$

where we take $E = S^n$ and $B = K(\pi, n)$, and rely on a result of Hurewicz that if X is simply connected with $H_i(X) = 0$ for $i < n$ for some $n > 1$, then $\pi_n(X) = H_n(X)$. We get a long exact sequence

$$\cdots \to H_n(F) \to H_n(E) \to H_n(K) \to H_{n-1}(F) \to \ldots$$

Definition 10.2.12. The Adams spectral sequence is a variation of Serre's, where one works only in the stable range and only on the p-component. One replaces $K(\pi, n)$ with $\prod K(H^n(X, \mathbb{F}_p), n)$, and define spaces X_i and K_i inductively so that X_{i+1} is the fiber of the map $X_i \to K_i$. We get a short exact sequence in the stable range

$$0 \leftarrow H^*(X_i, \mathbb{F}_p) \leftarrow H^*(K_i, \mathbb{F}_p) \leftarrow H^*(\Sigma X_{i+1}, \mathbb{F}_p) \leftarrow 0.$$

Moreover $H^*(K_i)$ is a free module over the mod p Steenrod algebra A, giving a free A-resolution of $H^*(X)$. Each fibration $X_{i+1} \to X_i \to K_i$ gives a long exact sequence of homotopy groups, and the associated spectral sequence is that of Adams.

Theorem 10.2.13 (Adams)**.** *There is a spectral sequence converging to the p-component of* $\pi_{n+k}(S^n)$ *with* $k < n - 1$*, and*

$$E_2^{s,t} = Ext_A^{s,t}(\mathbb{F}_p, \mathbb{F}_p) := H^{s,t}(A)$$

and $d_r : E_r^{s,t} \to E_r^{s+r, t+r-1}$.

Definition 10.2.14. Novikov's Adams spectral sequence replaces H^* with MU^*, and Eilenberg-Maclane spaces with Thom spaces. Rather than describe the full spectral sequence, we only introduce the groups

$$\mathrm{Ext}^s_{\Lambda/\Gamma}(\Omega^U(X), \Omega^U(\Sigma^t Y))$$

for stable maps between spaces X and Y. In particular,

$$E_2^{s,t} = \mathrm{Ext}_B^{s,t}(\mathbb{Z}, L) = H^s_\Gamma(\Lambda, \Omega^U(S^t)) \Rightarrow \pi^S_{s-t}$$

where B is a certain Hopf algebra.

10.2.4. The Greek letters. Now what we'd like to do is produce nontrivial elements in $E_2^{s,t}$. The cohomology groups $H^*(\Gamma, L)$ is bigraded because L itself is graded, and is in fact the E_2-term of the ANSS. It is concentrated in dimension zero, i.e.,

$$H^0(\Gamma, L) = \mathbb{Z}$$

and corresponds to the 0-stem in stable homotopy theory. This tells us that the only Γ-invariant principal ideals in L are generated by the integers, and one finds that

Theorem 10.2.15 (Morava, Landweber)**.** *There are elements v_n in L of dimension $2(p^n - 1)$ such that $I_n = (p, v_1, \ldots, v_n)$ is a Γ-invariant prime ideal for all $n > 0$, $H^0(\Gamma, L/I_n) = \mathbb{F}_p[v_n]$, and there is a short exact sequence*

$$0 \to \Sigma^{2(p^n-1)} L/I_n {}^{v_n}{\to} L/I_n \to L/I_{n+1} \to 0$$

with connecting homomorphism $\delta : H^i(\Gamma, L/I_{n+1}) \to H^{i+1}(\Gamma, L/I_n)$. In fact, the only Γ-invariant prime ideals are I_n for all $n > 0$ and all p.

Definition 10.2.16. Using this we may define the Greek letter elements $\alpha_t^{(n)} = \delta_0 \ldots \delta_{n-1}(v_n^t) \in E_2^{n,*}$ for any $t, n > 0$. The $\alpha^(n)$ denotes the n-th Greek letter, for our knowledge of these elements are far lesser than the Greek aplhabet.

The elements α_i have a complete description:

Theorem 10.2.17. *Let a_2k be the denominator of $\zeta(1 - 2k) = B_{2k}/4k$. Then*

$$H^{1,t}(\Gamma, L) = \begin{cases} 1 & t \text{ odd} \\ \text{cyclic of order } 2 & t = 2 \cdot \text{odd} \\ \text{cyclic of order } a_{2k} & t = 4k \end{cases}$$

Remark 10.2.18. While on the one hand, $\zeta(1-2k)$ appears as the constant coefficient of an Eisenstein series associated to the Weierstraß $\mathcal{P}$-function, we warn the reader that the theory of formal A-modules where $A = \mathcal{O}_K$ for some number field K, the analogous $H^{1,t}(\Gamma_A, L_A)$ does not seem to be related to the Dedekind zeta function of K.

The β family also has a classification in terms of congruences of modular forms, but we do not have the language to describe this as yet.

Definition 10.2.19. Let $J \subset L$ be a Γ-invariant regular prime ideal, that is, an ideal $(x_0, \ldots, x_{n-1})$ such that x_i is not a zero divisor modulo $(x_0, \ldots, x_{i-1})$. For example, ideals of the form $(p^{i_0}, v_1^{i_1}, \ldots, v_n^{i_n})$. Then we define generalized Greek elements using connecting homomorphisms for $H^0(\Gamma, L/J)$. For example, $\alpha_{s/t}$ is the image of $v_1^s \in H^0(\Gamma, L/(p^t))$ and $\beta_{s/t}$ is the image of $v_2^s \in H^0(\gamma, L/(p, v_1^t))$.

It is an important problem to show that these elements are nontrivial, which is known for certain α, β and γ.

10.3 The Adams conjecture

We pause our narrative for a brief historical inderlude.

10.3.1. The J-homomorphism. The image of the Hopf-Whitehead J-homomorphism

$$J : \pi_k(GL_n(\mathbb{C})) \to \pi_{2n+k}(S^{2n}),$$

which is defined by the action of $GL_n(\mathbb{C})$ acting on the $2n$ sphere viewed as the compactified space $\mathbb{C}^n$. Adams made use of the natural fiber-preserving map

$$\eta \xrightarrow{F} \eta^k$$

defined on vectors v by $v \mapsto v \otimes \cdots \otimes v$, and on each fiber the map can be thought of as $z \mapsto z^k$ on the complex plane. Thus if we remove the zero sections of η and η^k, F induces a fiber homotopy equivalence modulo primes dividing k. Adams did this by forming the stable bundle

$$(\eta^k - \eta) \oplus \cdots \oplus (\eta^k - \eta)$$

of N copies, where N is the dimension of the base. Adams proved that the corresponding sphere bundle was fiber homotopy trivial for N large enough, giving an upper bound on the order of the image of J.

At this point Adams conjectured that for all elements ξ in the K-theory of a finite complex the stable bundle $k^N(\psi^k\xi - \xi)$ has a fiber homotopy trivial associated sphere bundle, for N large enough. We reformulate this using the finite completion of K-theory defined for finite complexes by

$$\hat{K}(X) = \operatorname{proj}\lim K(X) \otimes \mathbb{Z}/n.$$

Now each element γ in $\hat{K}(X)$ has a well-defined stable fiber homotopy type. This follows by continuity from the fact that the group of stable spherical fiber homotopy types over a finite complex is finite. Then for integers k we define an isomorphism ψ^k on the p-part for $(p, k) = 1$ and trivial otherwise.

Further, computations of Adams imply that these operations extend by continuity to an action of $\hat{Z}^\times$. Sullivan refers to this symmetry group $\hat{\mathbb{Z}}^\times$ which contains the isomorphic part of the Adams operations the Galois group.

Theorem 10.3.1 (Adams' conjecture)**.** *In real of complex $\hat{K}(X)$ theory, the stable fiber homotopy type is constant on orbits of $\hat{\mathbb{Z}}^\times$. Thus any element in $\hat{K}(X)$ which is fiber homotopy trivial is of the form $\eta^\sigma - \eta$ with $\eta \in \hat{\mathbb{Z}}^\times$ and $\eta \in \hat{K}(X)$.*

The theorem follows from the fact that the Galois symmetry in $\hat{K}(X)$ arises in homotopy symmetry in the finite completions of the Grassmanian approximations to the classifying space of ordinary K-theory.

10.3.2. The arithmetic square. Recall the localization of a ring R at a multiplicatively closed subset S. Notice $R_S \hookrightarrow R_{S'}$ if $S \subset S'$. One can check that

$$\begin{array}{ccc} R_{S\cup S'} & \longrightarrow & R_S \\ \downarrow & & \downarrow \\ R_{S'} & \longrightarrow & R_{S\cap S'} \end{array}$$

commutes. If $R = Z$ and S is a subset of primes, we can use this to localize abelian groups $G_S = G \otimes_{\mathbb{Z}} \mathbb{Z}_S$.

Example 10.3.2. If G is a finitely generated abelian group then $G_S \simeq Z_S \oplus \cdots \oplus \mathbb{Z}_S \oplus$ (S-torsion). Also, $(\mathbb{Q}/\mathbb{Z})_S = \oplus_{p\in S}\mathbb{Z}/p^\infty$.

More generally, we can think of G_S as being formed by making multiplication by integers prime to l in S into isomorphisms,

$$G_S = \varinjlim_{(n,l)=1} \{G \xrightarrow{n} G\},$$

which shows that localization preserves exactness, finite products, and commutes with taking homology.

Consider now the ring of finite adeles

$$\mathbb{A}_S^f = \prod_{p\in S} \mathbb{Q}_p \times \prod_{p\notin S} \mathbb{Z}_p$$

The arithmetic square is then

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \hat{Z} \\ \downarrow & & \downarrow \\ \mathbb{Q} & \longrightarrow & \mathbb{A}_S^f \end{array}$$

10.4 Chromatic homotopy theory

10.4.1. Brown-Peterson theory. The work of Adams taught us to work mod p, and Novikov taught us to use MU, or equivalently, L. Now we will work p-locally.

Definition 10.4.1. Brown and Peterson showed that the spectrum MU, after localizing at a prime p, splits into an infinite wedge suspension of identical smaller spectra BP, where

$$\pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$$

where $\dim(v_n) = 2(p^n - 1)$. We will denote this coefficient ring by BP_*, and note that it is much smaller than MU_* which has a generator in every even dimension.

Theorem 10.4.2 (Cartier)**.** *Every formal group law F over a torsion free $\mathbb{Z}_{(p)}$-algebra R is canonically isomorphic to a formal group law L_T such that if $\log_F(x) = \sum a_i x^i$, then $\log_{L_T}(x) = \sum a_{p^i} x^{p^i}$. We will refer to F_{L_T} as a p-typical formal group law.*

Theorem 10.4.3 (Quillen)**.** *The p-typical Lazard ring L_T is isomorphic to BP_*.*

Now there is a BP analogue of the ANSS, which is simply tensoring by $\mathbb{Z}_{(p)}$, hence its E_2-term is the p-part of $H^*(\Gamma, L)$ converging to the p-part of π_*^S. But the p-typical analogue of the group Γ is instead a groupoid.

Theorem 10.4.4. *The p-part of the E_2-term in the Adams-Novikov spectral sequence converging to π_*^S is $\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*)$.*

10.4.2. The orbit stratification. It is a consequence of Lie theory that over a field of characteristic zero, all one-dimensional formal group laws are isomorphic to the additive group law. The orbits of the action of Γ on Λ will be called formal groups, so the stack Λ/Γ is essentially a category with one object. But in characteristic p the picture becomes more interesting.

We start with a classification theorem for formal group laws. But first, a definition:

Definition 10.4.5. If a formal group law F is define over a field of characteristic p, then the power series $[p]_F(x)$ defined by F by $[0]_F(x) = 0, [1]_F(x) = x$ and $[m]_F(x) = F(x, [m-1]_F(x))$ is a power series over x^{p^n} with leading term ax^{p^n} for some $n > 0$. The integer n is called the height of F. The height of the additive formal group law is defined to be ∞.

Example 10.4.6. The multiplicative formal group law $F(X,Y) = X + Y + XY$ has $[p]_F(x) = (1 + x)^p - 1 \equiv x^p \bmod p$, thus has height one. All positive integers occur: Honda's logarithm

$$\log_{H_n}(x) = \sum_{k=0}^{\infty} p^{-k} x^{p^{nk}}$$

defines a formal group law H_n over $\mathbb{Z}_p$ whose reduction mod p has $[p]_{F_n}(x) = x^{p^n}$.

Theorem 10.4.7 (Lazard)**.** *Two formal group laws over $\bar{\mathbb{F}}_p$ are isomorphic if and only if they have the same height. If F is not additive, its height is the smallest n such that $\theta(v_n) \neq 0$ where $\theta : L \to K$ is the homomorphism of Lazard's theorem, and K is a finite field.*

Height is a complete invariant of one-dimensional formal groups over a separably closed field. In characteristic p, the set of geometric points of Λ stratifies into orbits indexed by $\mathbb{N} \cup \infty$.

Definition 10.4.8. The orbit of a formal group law F is a homogeneous space $\Gamma/\mathrm{Aut}(F)$. We call $\mathrm{Aut}(F)$ the Morava sabilzer group of F.

Example 10.4.9. Let H_n be the Honda formal group law of height n. Then denote $S_n = \mathrm{Aut}(H_n) \simeq D_{1/n}^{\times}$ where

$$D_{1/n} \simeq W(\mathbb{F}_{p^n})\langle F \rangle / (F^n - p)$$

is a p-adic division algebra of rank n^2 over $\mathbb{Q}_p$. Here $W(k)$ denotes the ring of Witt vectors of k and $F(x) = x^p$ is the Frobenius endomorphism of H_n, satisfying $a^\sigma F = Fa$ for $a \in W(\mathbb{F}_q)$ and $\sigma \in \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$. Note that $W(\mathbb{F}_p) = \mathbb{Z}_p$.

Now let $\Gamma_{\bar{\mathbb{F}}_q}$ be the group of power series in $\bar{\mathbb{F}}_q[[x]]$ with leading term x. It acts on $\Lambda(\bar{\mathbb{F}}_q) = \mathrm{Spec}(L \otimes \bar{\mathbb{F}}_p)$, the stack of formal group laws over $\mathbb{F}_p$, which we can think of as an infinite-dimensional vector space over $\bar{\mathbb{F}}_p$, with topological basis given by the generators of L.

Theorem 10.4.10 (Morava change of rings theorem)**.** *There is one $\Gamma_{\bar{\mathbb{F}}_p}$-orbit of $\Lambda(\bar{\mathbb{F}}_p)$ at each height n. The height n orbit Λ_n is the locus where the polynomials defined by $v_1, \ldots, v_{n-1} = 0$ and v_n is invertible; it is the closed points in Spec($L_n \otimes \bar{\mathbb{F}}_p$) where $L_n = v_n^{-1}L/I_n$. We have the change of rings*

$$H^*(\Gamma_{\bar{\mathbb{F}}_p}, L_n \otimes \bar{\mathbb{F}}_p) = H^*(S_n, \bar{\mathbb{F}}_p) \tag{10.1}$$

and also $H^*(\Gamma, L \otimes \mathbb{F}_p) \otimes \bar{\mathbb{F}}_p \simeq H^*(\Gamma_{\bar{\mathbb{F}}_p}, L \otimes \bar{\mathbb{F}}_p)$.

In fact, a form this change of rings isomorphism holds over $\mathbb{F}_p$ itself, where on the right we have instead the cohomology of a Hopf algebra called the n-th Morava stabilizer algebra $\Sigma(n)$.

Definition 10.4.11. The cohomology groups in 10.1 are v_n-periodic in the sense that they are modules over the rings

$$K(n)^* = \mathbb{F}_p[v_n^{\pm 1}]$$

which are the coefficient ring of the n-th Morava K-theory: $K(0)^*$ denotes rational cohomology, $K(1)^*$ mod p complex K-theory, and

$$K(n) = \varinjlim \Sigma^{-2i(p^n-1)} k(n)$$

where $k(n)$ are the connective analogues obtained from BP by killing of all the generators but v_n, so that $\pi_*(k(n)) = \mathbb{F}_p[v_n^{\pm 1}]$. This leads to certain p-adic Adams operations, which we will not discuss.

10.4.3. The chromatic theory. Start with the chromatic resolution, which is a long exact sequence of Γ-modules,

$$0 \to L \otimes \mathbb{Z}_{(p)} \to M^0 \to M^1 \to \ldots$$

where $M^0 = L \otimes \mathbb{Q}$, and if N^1 is the cokernel in

$$0 \to L \otimes \mathbb{Z}_{(p)} \to M^0 \to N^1 \to 0$$

then M^n and N^n are defined inductively by

$$0 \to N^n \to M^n \to N^{n+1} \to 0$$

where $M^n = v_n^{-1} N^n$. Hence we have

$$N^1 = L \otimes \mathbb{Q}/\mathbb{Z}_{(p)} = \varinjlim L/(p^i) = L/(p^\infty)$$

and $N^{n+1} = L/(p^\infty, v_1^\infty, \ldots, v_n^\infty)$. The short exact sequences are spliced together to form the chromatic spectral sequence, which we describe loosely:

Theorem 10.4.12. *There is a spectral sequence converging to* $\mathrm{Ext}(L \otimes \mathbb{Z}_{(p)})$ *with* $E_1^{s,t} = \mathrm{Ext}_B^t(\mathbb{Z}, M^s)$.

Let's study the E_1-term. Define a family of Γ-modules $M_i^n, i = 0, 1, \ldots, n$ by $M_0^n = M^n$ and

$$0 \to M_i^n \to M_{i-1}^n \stackrel{v_{i-1}}{\to} M_{i-1}^n \to 0$$

with $v_0 = p$. This gives $M_n^n = L_n$, so the $\mathbb{F}_p$-analog of the change of rings theorem describes $\mathrm{Ext}_B(\mathbb{Z}, M^n)$ in terms of the cohomology of S_n. It is known that $H^(S_n, \mathbb{F}_p)$ is periodic if $p - 1 | n$, in which case the S_n has a cyclic subgroup of order p to whose cohomology the periodic element restricts nontrivially, so the cohomology can be used to detect elements in the E_2-term of high degree. If $p - 1 \not| n$, then S_n has cohomological dimension n^2, and $E_1^{s,t} = 0$ for $t > n^2$.

Definition 10.4.13. From BP we construct the Johnson-Wilson spectra $BP\langle n\rangle$ by killing off the ideal $(v_{n+1}, v_{n+2}, \dots) \subset \pi_*(BP)$ to get

$$\pi_*(BP\langle n\rangle) = \mathbb{Z}_{(p)}[v_1, \dots, v_n].$$

One has fibrations

$$\Sigma^{2(p^n-1)} BP\langle n\rangle^{v_n}{\to} BP\langle n\rangle \to BP\langle n-1\rangle,$$

so that $BP\langle 0\rangle$ is $H_{(p)}$, Eilenberg-Mac Lane spectrum localized at p, and $BP\langle 1\rangle$ is a summand of $bu_{(p)}$, connective complex K-theory spectrum localized at p. Iterating the v_n map we form the direct limit

$$E(n) = \varinjlim \Sigma^{-2(p^n-1)} BP\langle n\rangle$$

where now $E(1)$ is periodic complex K-theory localized at p. Johnson and Wilson showed that

$$E(n)_*(X) = BP_*(X) \otimes E(n)_*$$

so $E(n)$ is a homology theory. This can also be obtained from the Landwebwer theorem knowing that

$$\pi_*(E(n)) = \mathbb{Z}_{(p)}[v_1, \dots, v_n, v_n^{-1}]$$

satisfies the necessary hypotheses.

Definition 10.4.14. Start with the Honda formal group law H_n over $\bar{\mathbb{F}}_p$. We consider the Lubin-Tate universal deformation of H_n, and associate by Landweber the Morava E-theory spectrum E_n, with coefficient ring

$$(E_n)_* = W(\bar{\mathbb{F}}_p)[[u_1, \dots, u_{n-1}]][u^{\pm 1}]$$

and let $G_n = S_n \rtimes \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ be the extended Morava stabilizer group. Using the change of rings theorem the ANSS now takes the form

$$H_c^*(G_n, (E_n)_*/(p^\infty, \dots, u_{n-1}^\infty) \Rightarrow \pi_* M_n S$$

$$H_c^*(G_n, (E_n)_*) \Rightarrow) \Rightarrow \pi_* S_{K(n)}$$

where S is the sphere spectrum and M_n is the monochromatic layer of the homotopy fibers

$$M_n X \to X_{E(n)} \to X_{E(n-1)}$$

of the chromatic tower of X, of Bousfield localizations

$$\cdots \to X_{E(2)} \to X_{E(1)} \to X_{E(0)}$$

with respect to the p-primary Johnson-Wilson spectra.

Remark 10.4.15. We conclude this section with the following relations: Goerss and Hopkins showed that E_n is an E_∞-ring spectrum (which we'll define later) and G_n acts on E_n by E_∞-ring maps. Devinatz and Hopkins refined Morava's change of rings theorem to show that

$$S_{K(n)} \simeq E_n^{hG_n}$$

where on the right we have taken homotopy fixed points. In other words, the homotopy type of the $K(n)$-local sphere is given as the hypercohomology of G_n with coefficients in the spectrum E_n. Indeed, at $n = 1$ the formal completion of $\mathbb{G}_m/\bar{\mathbb{F}}_p$ is isomorphic to H_1. The J-spectrum is given by the fiber sequence

$$J \to KO_p{}^{\psi^k}{\to} KO_p$$

where k is the topological generator of $\mathbb{Z}_p^\times$ and ψ^k the k-th Adams operation. Then there is a natural equivalence $S_{K(1)} \simeq J$

10.5 Topological modular forms

11 Appendix

11.1 Structure theory of algebraic groups

11.1.1. Definitions Basic facts about algebraic groups, summarized from J. Milne's notes on Reductive Groups: An **affine algebraic group** over a field k is a functor G rom k-algebras to sets with a natural transformation $m : G \times G \to G$ such that (a) $m(R)$ is a group structure on $G(R)$ for all k-algebras R and (b) G is representable as a finitely generated k-algebra.

1. *Connected*: G has no nontrivial étale quotient.
2. *Diagonalizable*: G is a functor from R to $\mathrm{Hom}(M, R^\times)$ with coordinate ring $k[M]$, with M a finitely generated commutative group.
3. *Torus:* G is smooth, connected, and diagonalizable over an extension field.
4. *Unipotent:* Every nonzero representation of G has a nonzero fixed vector, or equivalently, every representation of $r : G \to GL(V)$ has a basis of V such that $r(G)$ is upper triangular with 1 on the diagonal.
5. *Normal:* $H(R)$ is normal in $G(R)$ for all R.
6. *Radical:* Largest solvable subgroup of G among smooth connected normal subgroups of G.
7. *Unipotent radical:* Largest unipotent subgroup of G among smooth connected normal subgroups.
8. *Geometric (unipotent) radical:* (Unipotent) radical of $G(k^{al})$.
9. *Semisimple (reductive):* Geometric (unipotent) radical of G is trivial. A torus in a reductive group is maximal if and only if it is equal to its own centralizer.
10. *Split:* Reductive G containing a split maximal torus (smooth conn diagonalizable alg subgroup).
11. *(Co)character group:* ($X_*(G)$ =$\mathrm{Hom}(\mathbb{G}_m, G)$) $X^*(G)$ =$\mathrm{Hom}(G, \mathbb{G}_m)$ with $\chi(\lambda(t)) = t^{(\chi,\lambda)} \in \mathbb{G}_m(R)$.
12. *Lie algebra:* The kernel of the homomorphism $G(k[X]/(X^2)) \to G(k)$, denoted $\mathrm{Lie}(G) = \mathfrak{g}$.
13. *Roots:* Nonzero characters in $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus \mathfrak{g}_\chi$ by the action of T diagonalizable Ad: $T \to GL_{\mathfrak{g}}$.
14. *Weyl group:* $W(G,T) = N_G(T)(k)/Z_G(T)(k)$.
15. *Coroots:* Cocharacters $\alpha^\vee$ such that $(\alpha, \alpha^\vee) = 2$ for roots α, and $s_\alpha(\chi) = \chi - (\chi, \alpha^\vee)\alpha$ is again a root for all χ, α and generates a finite automorphism group of $X^*(G)$, equal to $W(G,T)$.

12 Some references

Relevant references are scattered throughout the notes, but the following are some references to the general ideas of the program. The standard reference is the Corvallis 1979 AMS Proceedings. There is also the Edinburgh 1997 instructional conference which has some good introductory articles. Finally, there are the proceedings of the 1991 Seattle conference on Motives.

1. A. Borel, Automorphic L-functions
2. B. Casselman, The L-Group
3. S. Gelbart, An Elementary Introduction to the Langlands Program
4. B. Gross, On the Satake Isomorphism
5. T. Knapp, Introduction to the Langlands Program
6. J. Milne, Reductive Groups
7. J. Tate, Number Theoretic Background