ON THE UNITARY PART OF THE STABLE TRACE FORMULA

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ABSTRACT. We introduce a modification of the trace formula by means of certain maps of distribution, which are used to remove the contribution of weighted characters to the spectral side of the trace formula. As a result, we obtain a purely geometric expansion of the unitary part of the stable and endoscopic trace formulas, as a step towards isolating the cuspidal. tempered part of the trace formula as suggested by Langlands' Beyond Endoscopy proposal.

CONTENTS

1. INTRODUCTION

Let G be a connected reductive group over a number field F, and let f be a suitable test function in the Hecke algebra of G. The noninvariant trace formula is a linear form $J(f)$ on G, that is made invariant by modifying the noninvariant distribution

$$I(f) = J(f) - \sum_{M \neq G} |W_0^M||W_0^G|^{-1} \hat{I}_M(\phi_M(f))$$

using certain maps ϕ_M. The resulting invariant linear form $I(f)$ then has geometric and spectral expansions parallel to those of $J(f)$,

$$\begin{aligned} I(f) &= \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M,V,\zeta)} a^M(\gamma) I_M(\gamma, f) \\ &= \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(M,V,\zeta)} a^M(\pi) I_M(\pi, f) d\pi \qquad (1.1) \end{aligned}$$

where γ are conjugacy classes and π are representations of Levi subgroups M of G [Art88b], and $\mathscr{L}$ denotes the set of Levi subgroups of G containing a fixed minimal

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Levi subgroup of G. The contribution of $M = G$ to the spectral expansion (1.1) is the unitary part of the trace formula,

$$I_{\text{unit}}(f) = \int_{\Pi(G,V,\zeta)} a^G(\pi) f_G(\pi) d\pi$$

where $f_G(\pi)$ is the character $\text{tr}(\pi(f))$, and $a^G(\pi)$ is the global spectral coefficient. The contribution to the spectral expansion (1.1), or equivalently $I_{\text{disc}}(f)$, for which the measure $d\pi$ is discrete, on the other hand, can be expressed as

$$I_{\text{disc}}(f) = \sum_{\pi \in \Pi_{\text{disc}}(G,V,\zeta)} a^G_{\text{disc}}(\pi) f_G(\pi),$$

which can also be written in the form

$$\sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{s \in W(M)_{\text{reg}}} |\det(s-1)_{\mathfrak{a}_M^G}|^{-1} \text{tr}(M_P(s,0)\mathscr{I}_P(0,f)).$$

It contains characters in the discrete spectrum of G, and singular points in the continuous spectrum. The term corresponding to $M = G$ represents the contribution of terms that lie in the actual discrete spectrum.

If G is quasiplit over F, the corresponding stabilisation

$$I(f) = \sum_{G'} \iota(G, G') \hat{S}^{G'}(f') \tag{1.2}$$

is a decomposition of $I(f)$ into a finite sum of stable distributions $S^{G'}$ on the elliptic endoscopic groups G' of G. For the case $G' = G^*$, the quasiplit inner form of G, the form $\hat{S}^{G^*}(f^*)$ is regarded as the stable part of $I(f)$. The resulting stable linear form $S(f)$ then comes with geometric and spectral expansions parallel to those of $I(f)$, and provide access to cases of functoriality for endoscopic groups. It takes the form of an identity of stable distributions on G,

$$\begin{aligned} S(f) &= \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\delta \in \Delta(M,V,\zeta)} b^M(\delta) S_M(\delta, f) \\ &= \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Phi(M,V,\zeta)} b^M(\phi) S_M(\phi, f) d\phi. \end{aligned}$$

In the case that G is arbitrary, one obtains an endoscopic linear form $I^{\mathscr{E}}(f)$ instead. This foundation was laid down by Arthur [Art02, Art01, Art03], and extended to the twisted case by Moeglin and Waldspurger [MW16a, MW16b]. Of course, each of these results rests on the solution of the relevant Fundamental Lemmas. It again has the unitary part

$$S_{\text{unit}}(f) = \int_{\Phi(G,V,\zeta)} b^G(\phi) f^G(\phi) d\phi,$$

and with discrete part given by

$$S_{\text{disc}}(f) = \sum_{\phi \in \Phi_{\text{disc}}(G,V,\zeta)} b^G_{\text{disc}}(\phi) f^G(\phi),$$

obtained by a parallel formula (1.2) of $I_{\text{disc}}(f)$.

In order to address functoriality beyond the endoscopic cases, one would like a further decomposition of $S(f)$ into primitive linear forms $P(f)$ as described by Arthur in [Art17, §2]. As Langlands first pointed out, this will require special treatment of the contribution of the tempered automorphic representations on the spectral side [Lan04, §1.6]. In the notation of Arthur, this amounts to isolating $S_{\text{cusp}}(f)$, the linear combination of irreducible characters obtained from removing the contribution to $S_{\text{disc}}(f)$ of characters that are either nontempered or that do not lie in the discrete spectrum of G. If we view $S_{\text{disc}}(f)$ as the difference between the geometric expansion and the spectral terms for which the measure $d\phi$ is continuous, obtaining cancellation between the two quantities is an important concern [Art17, p.9], first emphasized in [FLN10]. In other words, one would like an expression for $S_{\text{cusp}}(f)$ that is as close to a geometric expansion as possible.

With this in mind, the goal of this paper is to convert the contribution of weighted characters in the spectral expansion of $S(f)$,

$$\sum_{\substack{M\in\mathscr{L}\\ M\neq G}}|W_0^M||W_0^G|^{-1}\int_{\Phi(M,V,\zeta)}b^M(\phi)S_M(\phi,f)d\phi,$$

which are the terms associated to proper Levi subgroups $M\neq G$, into geometric distributions, and similarly for the endoscopic linear form $I^{\mathscr{E}}(f)$. This can be viewed as a generalization of the analysis in [Won19b] for the case of $\mathrm{GL}(2)$ to general G, and a partial solution to the desired cancellation in $S_{\text{disc}}(f)$. This method is in contrast to other recent work following ideas of [FLN10], whereby the geometric terms are modified instead by means of a putative Poisson summation formula.

In particular, we obtain modified endoscopic and stable distributions $\tilde{I}^{\mathscr{E}}(f)$ and $\tilde{S}(f)$, whose spectral expansions consist solely of unweighted characters. This direct leads to entirely geometric expansions for the linear forms $I^{\mathscr{E}}_{\text{unit}}(f)$ and $S_{\text{unit}}(f)$, at the slight cost of the parallel structure that is found in the distributions $I^{\mathscr{E}}(f)$ and $S(f)$. This is the content of Corollary 5.3.

Theorem 1. *The unitary part of the stable trace formula has the geometric expansion*

$$S_{\text{unit}}(f)=\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\sum_{\delta\in\Delta(M,V,\zeta)}b^M(\delta)\tilde{S}_M(\delta,f)$$

where $\tilde{S}(\delta,f)$ is a family of modified linear forms on G, and similarly for $I^{\mathscr{E}}_{\text{unit}}(f)$.

This leaves open the task of treating the continuous contribution to $S_{\text{unit}}(f)$, and the contribution of nontempered representations that lie in the discrete spectrum of G in order to isolate the cuspidal tempered contribution $S_{\text{cusp}}(f)$. The latter is certainly the most difficult of all, as the terms cannot be explicitly separated from the trace formula in its current form.

Our method of proof involves introducing families of mappings $\iota^{\mathscr{E}}_M$ and τ_M modeled after Arthur's maps ϕ_M used in the invariant trace formula. We remark that this fits into the analogy described by Arthur between the endoscopic and beyond endoscopic settings, whereby obtaining a geometric expansion of $S_{\text{cusp}}(f)$, the cuspidal tempered contribution to $S_{\text{disc}}(f)$, should be analogous to making the trace formula invariant [Art17, §2]. Additionally, one can deduce from the proof a geometric expansion for the unitary part of the invariant trace formula $I_{\text{unit}}(f)$ also, thought we have not stated it explicitly.

Finally, it appears that what are now called basic functions will play a key role in establishing further refinements of the trace formula. If we are able to remove the contribution of the nontempered spectrum of G to $S(f)$, then the basic functions will allow us to weight the trace formula with automorphic L-functions $L(s,\pi,r)$

that are expected to be holomorphic in the right-half plane $\mathrm{Re}(s) > 1$. Basic functions fall outside of the domain of test functions used in the trace formula, not being compactly supported. Extending the noninvariant trace formula to include them is the subject of [FLM11, FL16], and which we shall generalize to the stable trace formula in [Won], and it is hoped that doing so will provide insight into obtaining a geometric expansion of the primary term $S_{\mathrm{cusp}}(f)$. While Theorem 1 is proved for compactly supported functions f, the proof can be easily applied to noncompactly supported functions once the stable trace formula is proved for such functions.

We conclude with a brief outline of the paper. Section 2 introduces the basic definitions and the families of mappings $\iota_M^{\mathscr{E}}$ and τ_M. These maps are used in Section 3 to modify the distributions appearing on either side of the endoscopic and stable trace formulas respectively. The modified distributions satisfy properties similar to the original ones, and in particular they satisfy the usual descent and splitting properties, which are proved in Section 4. Having established the local theory, Section 5 provides the global expansions which lead to the modified linear forms $\tilde{I}^{\mathscr{E}}$ and $\tilde{S}$.

2. Maps of distributions

2.1. **Preliminaries.** Let G be a connected reductive group over a field F of characteristic zero. We denote by $\mathscr{L}(M)$ to be the collection of Levi subgroups of G containing M, $\mathscr{L}^0(M)$ the subset of proper Levi subgroups in $\mathscr{L}(M)$, and $\mathscr{P}(M)$ the collection of parabolic subgroups of G containing M. Let F be a global field, and V a finite set of places of F. We have the real vector space $\mathfrak{a}_M = \mathrm{Hom}(X(M)_F, \mathbf{R})$, and the set

$$\mathfrak{a}_{M,V} = \{H_M(m) : m \in M(F_V)\}$$

is a subgroup of $\mathfrak{a}_M$, and $F_V = \prod_{v\in V} F_v$. It is equal to $\mathfrak{a}_M$ if V contains an archimedean place, and is a lattice in $\mathfrak{a}_M$ otherwise. The additive character group $\mathfrak{a}^*_{M,V} = \mathfrak{a}^*_M \backslash \mathfrak{a}^\vee_{M,V}$ equals $\mathfrak{a}^*_M$ in the first case, and is a compact quotient of $\mathfrak{a}^*_M$ in the second. Let A_M be the maximal split torus of a Levi subgroup M of G. We then identify the Weyl group of (G, A_M) with the quotient of the normaliser of M by M, thus

$$W^G(M) = \mathrm{Norm}_G(M)/M.$$

If M_0 is a minimal Levi subgroup of G, which we shall assume to be fixed, and denote $\mathscr{L} = \mathscr{L}(M_0), \mathscr{L}^0 = \mathscr{L}^0(M_0)$, and $W_0^G = W^G(M_0)$.

Let Z be a central induced torus of G over F. Then following [Art99, §3] we define the pair (Z, ζ) where ζ is a character of $Z(F)$ if F is local, and an automorphic character of $Z(\mathbf{A})$ if F is global. Given a finite set of places V, we write $G_V = G(F_V)$ and write ζ_V for the restriction of ζ to the subgroup Z_V of $Z(\mathbf{A})$. We then write G_V^Z for the set of $x \in G_V$ such that $H_G(x)$ lies in the image of the canonical map from $\mathfrak{a}_Z$ to $\mathfrak{a}_G$. We shall assume that V contains the places over which G and ζ are ramified.

The stable trace formula requires that we work in fact with G a K-group as defined in [Art99, §1]. Thus

$$G = \coprod_\alpha G_\alpha \qquad \alpha \in \pi_0(G)$$

is a variety whose connected components G_α are reductive groups over F, equipped with an equivalence class of frames

$$(\psi, u) = \{(\psi_{\alpha\beta}, u_{\alpha\beta}) : \alpha, \beta \in \pi_0(G)\}$$

satisfying natural compatibility conditions. Here $\psi_{\alpha\beta} : G_\alpha \to G_\beta$ in an isomorphism over $\bar{F}$, and $u_{\alpha\beta}$ is a locally constant function from $\Gamma = \mathrm{Gal}(\bar{F}/F)$ to the simply connected cover $G_{\alpha,\mathrm{sc}}$ of the derived group of G_α. Any connected reductive group is a component of an K-group that is unique up to weak isomorphism. It comes with a local product structure

$$G_V = \prod_{v\in V} \coprod_{\alpha_v \in \pi_0(G_v)} G_{v,\alpha_v}.$$

The introduction of K-groups is to steamline certain aspects of the study of connected groups, and the definitions for connected groups will extend to K-groups in a natural way. For example, a central induced torus Z of a K-group G will have central embeddings $Z \xrightarrow{\sim} Z_\alpha \subset Z(G_\alpha)$ for each α, and ζ determines a character ζ_α for each α. We shall call a K-group G quasisplit if it has a connected component that is quasisplit over F.

We shall have to pay special attention to the spaces of functions involved. We write $\mathscr{C}(G, V, \zeta) = \mathscr{C}(G^Z_V, \zeta_V)$ for the space of ζ^{-1}-equivariant Schwartz functions on G^Z_V, which contains the Hecke algebra $\mathscr{H}(G, V, \zeta) = \mathscr{H}(G^Z_V, \zeta_V)$ defined with respect to a choice of maximal compact subgroup K_∞ of G_{V_∞}, where V_∞ denotes the archimedean places in V. If F is a local field, we write $\mathscr{C}(G_v, \zeta_v)$ and $\mathscr{H}(G_v, \zeta_v)$ for the corresponding spaces. There are natural decompositions

$$\mathscr{C}(G_v, \zeta_v) = \bigoplus_{\alpha\in\pi_0(G)} \mathscr{C}(G_\alpha, \zeta_\alpha)$$

and

$$\mathscr{C}(G_V, \zeta_V) = \bigotimes_{v\in V} \mathscr{C}(G_v, \zeta_v),$$

and similarly for the Hecke algebra. We shall write $\mathscr{C}(G_v, \zeta_v) = \mathscr{C}(G, \zeta)$ and $\mathscr{H}(G_v, \zeta_v) = \mathscr{H}(G, \zeta)$ when the context is clear. Now, given $\gamma_V \in G^Z_V$, we have a continuous linear form $f \mapsto f_G$ on $\mathscr{C}(G^Z_V, \zeta_V)$, which can be defined as either the ζ-equivariant orbital integral at the conjugacy class γ_V of G^Z_V,

$$\int_{Z_V} \zeta_V(z) f_G(z\gamma_V) dz$$

where

$$f_G(\gamma_V) = |D(\gamma_V)|^{\frac{1}{2}} \int_{G_{\gamma_V} \cap G^Z_V \backslash G^Z_V} f(x^{-1}\gamma_V x) dx,$$

with $D(\gamma_v)$ the discriminant of γ_v, or as the character $f_G(\pi) = \mathrm{tr}(\pi(f))$ where π is an irreducible character of G^Z_V with central character ζ_V. The functions determine each other, so we may use either definition to form the invariant Schwartz space

$$I\mathscr{C}(G, V, \zeta) = I\mathscr{C}(G^Z_V, \zeta_V) = \{f_G : f \in \mathscr{C}(G^Z_V, \zeta)\}$$

and invariant Hecke space

$$\mathscr{I}(G, V, \zeta) = \mathscr{I}(G^Z_V, \zeta_V) = \{f_G : f \in \mathscr{H}(G^Z_V, \zeta)\},$$

equipped with topologies such that the surjective map $f \mapsto f_G$ is open and continuous. We shall write $I(f) = \hat{I}(f_G)$ for any invariant linear form I that lies in the image of the transpose of the map $f \to f_G$.

2.2. **Invariant characters.** We define $\mathscr{F}(G_V^Z, \zeta_V)$ to be the space of finite complex linear combinations of irreducible characters on G_V^Z with Z_V-central character equal to ζ_V, and identify an element $\pi \in \mathscr{F}(G_V^Z, \zeta_V)$ with the linear form

$$f \mapsto f_G(\pi) = \mathrm{tr}(\pi(f)) = J_G(\pi, f)$$

on $\mathscr{H}(G_V^Z, \zeta_V)$. The space $\mathscr{F}(G_V^Z, \zeta_V)$ has a canonical basis $\Pi(G_V^Z, \zeta_V)$ consisting of irreducible characters with Z_V-central character equal to ζ_V, and we can form a chain of subsets

$$\Pi_{\mathrm{temp}}(G_V^Z, \zeta_V) \subset \Pi_{\mathrm{unit}}(G_V^Z, \zeta_V) \subset \Pi(G_V^Z, \zeta_V)$$

consisting of characters that are tempered and unitary, respectively. We consider $\mathfrak{a}_{G,Z}^*$ as the subspace of linear forms on $\mathfrak{a}_G$ that are trivial on the image of $i\mathfrak{a}_Z$ in $\mathfrak{a}_G$. Then there is an action of $\mathfrak{a}_{G,Z}^*$ on $\Pi_{\mathrm{unit}}(G_V, \zeta_V)$, given by $\lambda : \pi \to \pi_\lambda$, whose orbits can be identified with $\Pi_{\mathrm{unit}}(G_V^Z, \zeta_V)$.

The Paley-Wiener space of functions on $\Pi_{\mathrm{temp}}(G_V, \zeta_V) \times \mathfrak{a}_{M,V}$ is a subspace of $\mathscr{I}(G, V, \zeta)$. There is a continuous linear map from $\mathscr{H}(G, V, \zeta)$ to $\mathscr{I}(G, V, \zeta)$ given by $f \to f_G$ where

$$f_G(\pi, X) = \int_{i\mathfrak{a}_{G,V}^*} \mathrm{tr}(\pi_\lambda(f)) \mathrm{e}^{\lambda(X)} d\lambda,$$

for $\pi \in \Pi_{\mathrm{temp}}(G_V, \zeta_V)$ and $X \in \mathfrak{a}_{G,V}$. Let K be a maximal compact subgroup of G_V, Γ a finite subset of $\Pi(K)$, and N a positive integer. Also fix a positive function $||x||$ for $x \in G_V$ as in [Art89a, §11], and define $\mathscr{H}_N(G, V, \zeta)_\Gamma$ to be the space of smooth functions on G_V which are supported on the set of $x \in G_V$ such that $\log ||x|| \leq N$. It has a topology given by the seminorms

$$||f||_D = \sup_{x \in G_V} |Df(x)|, \qquad f \in \mathscr{H}_N(G, V, \zeta)_\Gamma$$

where D is a differential operator on G_{V_∞}, and V_∞ is the subset of archimedean valuations in V. We then define the topological direct limits

$$\mathscr{H}(G, V, \zeta)_\Gamma = \varinjlim_N \mathscr{H}_N(G, V, \zeta)_\Gamma$$

$$\mathscr{I}(G, V, \zeta)_\Gamma = \varinjlim_N \mathscr{I}_N(G, V, \zeta)_\Gamma$$

with $\mathscr{I}_N(G, V, \zeta)_\Gamma$ defined analogously. We shall be interested in the larger spaces

$$\mathscr{H}_{\mathrm{ac}}(G, V, \zeta) = \varinjlim_\Gamma \mathscr{H}_{\mathrm{ac}}(G, V, \zeta)_\Gamma$$

$$\mathscr{I}_{\mathrm{ac}}(G, V, \zeta) = \varinjlim_\Gamma \mathscr{I}_{\mathrm{ac}}(G, V, \zeta)_\Gamma.$$

Here $\mathscr{H}_{\mathrm{ac}}(G, V, \zeta)_\Gamma$ is the space of ζ^{-1}-equivariant functions f on G_V such that for every $b \in C_c^\infty(\mathfrak{a}_{G,V})$ the function

$$f^b(x) = f(x) b(H_G(x))$$

belongs to $\mathscr{H}(G, V, \zeta)_\Gamma$. We may also view it as the space of uniformly smooth ζ^{-1}-equivariant functions f on G_V such that for any $X \in \mathfrak{a}_{G,V}$, the restriction of f to the preimage of X in G_V is compactly supported. By uniformly smooth, we mean that the function f is bi-invarianat under an open compact subgroup of G_V. Similarly, $\mathscr{I}_{\mathrm{ac}}(G,V,\zeta)_\Gamma$ is the space of ζ^{-1}-equivariant functions ϕ on $\Pi_{\mathrm{temp}}(G_V,\zeta_V)\times\mathfrak{a}_{G,V}$ such that for every $b\in C_c^\infty(\mathfrak{a}_{G,V})$ the function

$$\phi^b(\pi,X)=\phi(\pi,X)b(X)$$

belongs to $\mathscr{I}(G,V,\zeta)_\Gamma$. We may also regard an element of $\mathscr{I}_{\mathrm{ac}}(G,V,\zeta)$ as a function on the set of conjugacy classes of G_V by means of orbital integrals. The map $f\to f_G$ then extends to a continuous map from $\mathscr{H}_{\mathrm{ac}}(G,V,\zeta)$ to $\mathscr{I}_{\mathrm{ac}}(G,V,\zeta)$. More generally, there is a function

$$f_M(\pi,X)=\int_{i\mathfrak{a}^*_{M,S}}\mathrm{tr}(\mathscr{I}_P(\pi_\lambda,f))\mathrm{e}^{-\lambda(X)}d\lambda,$$

where $P\in\mathscr{P}(M),\pi\in\Pi_{\mathrm{temp}}(M_V^Z,\zeta_V)$, and $X\in\mathfrak{a}_{M,V}$. Here $\mathscr{I}_P(\pi_\lambda)$ is the representation in $\Pi_{\mathrm{temp}}(G_V,\zeta_V)$ induced from π_λ. Then $f\to f_M$ is a continuous linear map from $\mathscr{H}_{\mathrm{ac}}(G,V,\zeta)$ to $\mathscr{I}_{\mathrm{ac}}(M,V,\zeta)$.

2.3. **Invariant weighted characters.** We now want to define the distributions that occur in the spectral side of the trace formula. Recall the canonically normalised weighted character introduced in [Art98, §2],

$$J_M(\pi,f)=\mathrm{tr}(\mathscr{M}_M(\pi,P)\mathscr{I}_P(\pi,f))$$

where $\mathscr{I}_P(\pi)$ is the induced representation of G obtained from $\pi\in\Pi_{\mathrm{unit}}(M_V^Z,\zeta_V)$ and $\mathscr{M}_M(\pi,P)$ is an operator constructed in a certain way from unnormalized intertwining operators, which we shall describe below. We can then define for any pair (π,X) in $\Pi(M_V^Z,\zeta_V)\times\mathfrak{a}_{M,V}$, the distribution

$$J_M(\pi,X,f)=\int_{i\mathfrak{a}^*_M}J_M(\pi_\lambda,f)\mathrm{e}^{-\lambda(X)}d\lambda,\qquad f\in\mathscr{H}(G,V,\zeta)\tag{2.1}$$

if $J_M(\pi_\lambda,f)$ is regular for $\lambda\in i\mathfrak{a}^*_M$, for example, if π is unitary. Whereas for more general representations $\pi\in\Pi(M_V^Z,\zeta_V)$ we define

$$J_M(\pi,X,f)=\sum_{P\in\ P(M)}\omega_P J_M(\pi_{\varepsilon_P},X,f)\mathrm{e}^{-\varepsilon_P(X)}$$

where for each $P\in\mathscr{P}(M)$, ε_P is a small point in the positive chamber $(\mathfrak{a}_P^*)^+$ and

$$\omega_P=\mathrm{vol}(\mathfrak{a}_P^+\cap B)\mathrm{vol}(B)^{-1},$$

where B is a ball in $\mathfrak{a}_M$ centered at the origin. The two definitions are compatible by a contour shift. More generally, we define the function

$$J_{M,\mu}(\pi,X,f)=J_M(\pi_\mu,X,f)\mathrm{e}^{-\mu(X)}=\int_{\mu+i\mathfrak{a}^*_M}J_M(\pi_\lambda,f)\mathrm{e}^{-\lambda(X)}d\lambda,$$

which is locally constant as a function of $\mu\in\mathfrak{a}^*_M$ on the complement of a finite set of affine hyperplanes.

The invariant weighted characters are then defined inductively by the relation

$$I_M(\pi,X,f)=J_M(\pi,X,f)-\sum_{L\in\mathscr{L}^0(M)}\hat{I}^L_M(\pi,X,\phi_L(f)),$$

where the map

$$\phi_M:\mathscr{H}_{\mathrm{ac}}(G,V,\zeta)\to\mathscr{I}_{\mathrm{ac}}(M,V,\zeta)\tag{2.2}$$

is based on the construction in [Art98, §2] using normalised weighted characters, which we briefly recall here. Suppose first that $\tilde{f}$ belongs to the Schwartz space $\mathscr{C}(G_V,\zeta_V)$. Then $\phi_M(\tilde{f})$ is defined to be the function on $\Pi_{\mathrm{temp}}(M_V,\zeta_V)$ such that

$$\phi_M(\tilde{f},\tilde{\pi}) = \mathrm{tr}(\mathscr{M}_M(\tilde{\pi},P)\mathscr{I}_P(\tilde{\pi},\tilde{f}))$$

for $P\in\mathscr{P}(M)$ and $\tilde{\pi}\in\Pi_{\mathrm{temp}}(M_V,\zeta_V)$. The operator

$$\mathscr{M}_M(\tilde{\pi},P) = \lim_{\Lambda\to 0}\sum_{Q\in\mathscr{P}(M)}(\otimes_{v\in V}\mathscr{M}_Q(\Lambda,\tilde{\pi}_v,P))\theta_Q(\Lambda)^{-1}, \tag{2.3}$$

with

$$\theta_Q(\Lambda) = \mathrm{vol}(\mathfrak{a}_M^G/Z(\Delta_Q^\vee))^{-1}\prod_{\alpha\in\Delta_Q}\Lambda(\alpha^\vee),$$

is defined as part of Arthur's theory of (G,M)-families, where the relevant (G,M)-family is a tensor product of (G,M)-families

$$\mathscr{M}_Q(\Lambda,\pi_v,P) = \mu_Q(\Lambda,\pi_v,P)\mathscr{I}_Q(\Lambda,\pi_v,P), \qquad Q\in\mathscr{P}(M),\ \Lambda\in i\mathfrak{a}_M^*$$

defined for π_v in general position. Here

$$\mu_Q(\Lambda,\pi_v,P) = \mu_{Q|P}(\pi)^{-1}\mu_{Q|P}(\pi_{\frac{1}{2}\Lambda}),$$

and the functions $\mu_{Q|P}(\pi_\Lambda)$ are Harish-Chandra's canonical family of μ-functions. If f and π are restrictions of $\tilde{f}$ and $\tilde{\pi}$ to G_V^Z and M_V^Z respectively, we set

$$\phi_M(f,\pi) = \int_{i\mathfrak{a}_{M,Z}^*}\phi_M(\tilde{f},\tilde{\pi}_\lambda)d\lambda.$$

From [Art02, §2] and [Art89a, §3] we know that ϕ_M maps $\mathscr{H}(G,V,\zeta)$ continuously to $\mathscr{I}(M,V,\zeta)$. If π is tempered, the inductive definition shows that

$$I_M(\pi,X,f) = \begin{cases} f_G(\pi,X), & M=G \\ 0 & M\neq G. \end{cases} \tag{2.4}$$

Moreover, if we consider only tempered π, then we may take f to be Schwartz, in which case we have that ϕ_M maps $\mathscr{C}(G_V^Z,\zeta_V)$ continuously to $I\mathscr{C}(M_V^Z,\zeta_V)$.

If π is nontempered, on the other hand, then the linear form $I_M(\pi,X,f)$ is a finite sum of residues of weighted characters in the complex domain. More generally, we set

$$I_{M,\mu}(\pi,X,f) = I_M(\pi_\mu,X,f)\mathrm{e}^{-\mu(X)}, \qquad \mu\in\mathfrak{a}_M^*$$

satisfying

$$I_{M,\mu} = \sum_{P\in\mathscr{P}(M)}\omega_P I_{M,\mu+\varepsilon_P}(\pi,X,f).$$

It is again locally constant on the complement of a finite set of affine hyperplanes [Art88a, Lemma 3.2]. Let us also agree to write

$$J_M(\pi,f) = J_M(\pi_\mu,0,f)$$

and

$$I_M(\pi,f) = I_M(\pi_\mu,0,f)$$

for $f\in\mathscr{H}_{\mathrm{ac}}(G,V,\zeta)$.

2.4. **Stable weighted characters.** We say a distribution on G_V^Z is stable if it lies in the closed linear span of strongly regular, stable orbital integrals

$$f^G(\delta_V) = \sum_{\gamma_V} f_G(\gamma_V)$$

where δ_V is any strongly regular, stable conjugacy class in G_V^Z, and the sum is taken over the finite set of conjugacy classes in the stable conjugacy class δ_V containing γ_V. Let us write $S\mathscr{F}(G_V^Z, \zeta_V)$ for the subspace of ζ_V-equivariant stable distributions in $\mathscr{F}(G_V^Z, \zeta_V)$, and $\mathscr{S}(G_V^Z, \zeta_V)$ for the subspace of $\mathscr{I}(G_V^Z, \zeta)$ spanned by stable orbital integrals.

Referring to [Art02, §6], there is an endoscopic basis $\Phi^{\mathscr{E}}(G_V^Z, \zeta_V)$ of $\mathscr{F}(G_V^Z, \zeta_V)$, and a subset

$$\Phi(G_V^Z, \zeta_V) = \Phi^{\mathscr{E}}(G_V^Z, \zeta_V) \cap S\mathscr{F}(G_V^Z, \zeta_V)$$

that forms a basis of $S\mathscr{F}(G_V^Z, \zeta_V)$. They are defined in terms of abstract bases $\Phi_{\mathrm{ell}}(M_V^Z, \zeta_V)$ of the space $\mathscr{S}_{\mathrm{cusp}}(M_V^Z, \zeta_V)$ of stable orbital integrals of cuspidal ζ_V-equivariant functions on M_V^Z, where a function is called cuspidal if it vanishes on any properly induced element. The basis $\Phi_{\mathrm{ell}}(M, \zeta)$ plays the formal role of cuspidal Langlands parameters. Let $i\mathfrak{a}_G^*$ be the imaginary dual space of $\mathfrak{a}$. The action $\phi \to \phi_\lambda$ of $i\mathfrak{a}_G^*$ then makes $\Phi_{\mathrm{ell}}(M_V^Z, \zeta_V)$ into a disjoint union of compact tori of the form $i\mathfrak{a}_{G,\phi}^* = i\mathfrak{a}_M^*/\mathfrak{a}_{M,\phi}^\vee$, where $\mathfrak{a}_{M,\phi}^\vee$ is the stabiliser of ϕ in $i\mathfrak{a}_M^*$. We note that $\mathscr{S}_{\mathrm{cusp}}(M_V^Z, \zeta_V)$ is the Paley-Wiener space on $\Phi_{\mathrm{ell}}(M_V^Z, \zeta_V)$, in the sense that its elements are supported on finitely many connected components, and on the component of any ϕ, pull back to a finite Fourier series on $i\mathfrak{a}_{M,\phi}^*$.

Let $\mathscr{E}(G)$ be the set of isomorphism classes of endoscopic data $(G', \mathcal{G}', s', \xi')$ for G over F that are locally relevant to G, which is to say that for every v, $G'(F_v)$ has a strongly G-regular element that is an image of some strongly G-regular conjugacy class in G_v. We shall write $\mathscr{E}(G, V)$ for the subset of elements $G' \in \mathscr{E}(G)$ that are unramified outside of V, and also $\mathscr{E}_{\mathrm{ell}}(G)$ and $\mathscr{E}_{\mathrm{ell}}(G, V)$ for the subsets of $\mathscr{E}(G)$ and $\mathscr{E}(G, V)$ respectively that are elliptic over F. Finally, if G' belongs to $\mathscr{E}_{\mathrm{ell}}(G)$, we fix a central extension $\tilde{G}'$ and an L-embedding $\tilde{\xi}' : \mathcal{G}' \to {}^L\tilde{G}'$ satisfying the conditions of [Art96, Lemma 2.1].

Now let M be a fixed Levi subgroup of the K-group G over a global field F, and let M' represent an elliptic endoscopic datum $(M', \mathcal{M}', s'_M, \xi'_M)$ of M [Art02, §6]. We shall assume that $\mathcal{M}'$ is an L-subgroup of ${}^LM$ and that ξ'_M is the identity embedding of $\mathcal{M}'$ into M'. We write $\mathscr{E}_{M'}(G)$ for the set of endoscopic data $(G', \mathcal{G}', s', \xi')$ for G over F, taken up to translation by s' in $Z(\hat{G})^\Gamma$, in which s' lies in $s'_M Z(\hat{M})^\Gamma$, $\hat{G}'$ is the connected centralizer of s' in $\hat{G}$, $\mathcal{G}'$ equals $\mathcal{M}'\hat{G}'$, and ξ' is the identity embedding of $\mathcal{G}'$ into ${}^LG$.

The invariant weighted characters are then stabilised as follows. Given a pair (ϕ, X) in $\Phi^{\mathscr{E}}(M_V^Z, \zeta_V) \times \mathfrak{a}_M^Z$, define the invariant linear form

$$I_M(\phi, X, f) = \sum_{\pi \in \Pi(M_V^Z, \zeta_V)} \Delta_M(\phi, \pi) I_M(\pi, X, f)$$

where $\Delta_M(\phi, \pi)$ is the spectral transfer factor defined in [Art02, §5], and for f in $\mathscr{H}(G, V, \zeta)$. If G is general, define for $\phi' \in \Phi((\tilde{M}'_V)^{\tilde{Z}'}, \tilde{\zeta}'_V)$ the linear form

$$I_M^{\mathscr{E}}(\phi', X, f) = \sum_{G' \in \mathscr{E}^0_{M'}(G)} \iota_{M'}(G, G') \hat{S}_{\tilde{M}'}^{\tilde{G}'}(\phi', f') + \varepsilon(G) S_M^G(M', \phi', X, f) \tag{2.5}$$

with the requirement that

$$I_M^{\mathscr{E}}(\phi', X, f) = I_M(\phi, X, f)$$

in the case that G is quasisplit. Here $\varepsilon(G)$ is equal to 1 if G is quasisplit and 0 otherwise. If ϕ' and M' are locally relevant to M, Proposition 6.4 of [Art02] allows us to define the endoscopic form $I_M^{\mathscr{E}}(\pi, X, f)$ by inversion of the formula

$$I_M^{\mathscr{E}}(\phi', X, f) = \sum_{\pi \in \Pi(M_V^Z, \zeta_V)} \Delta_M(\phi', \pi) I_M^{\mathscr{E}}(\pi, X, f).$$

The distributions $I_M^{\mathscr{E}}(\pi, X, f)$ and $S_M^G(M', \phi', X, f)$ are then the main objects appearing in the spectral side of endoscopic and stable trace formulas respecitvely. In the case that G is quasisplit and $M' = M^*$,

$$S_M^G(\phi, X, f) = S_M^G(M^*, \phi^*, X, f).$$

These stable and endoscopic forms satisfy the usual descent and splitting formulas proved in [MW16b, X.4], parallel to those satisfied by the invariant forms [Art88a, §8–9]. Also, we shall set

$$I_M^{\mathscr{E}}(\pi, f) = I_M^{\mathscr{E}}(\pi_\mu, 0, f)$$

and

$$S_M(\phi, f) = S_M(\phi_\mu, 0, f)$$

as before.

2.5. **Maps of distributions.** We shall now introduce two new families of maps which will be endoscopic and stable analogues of ϕ_M. We first define the space

$$\mathscr{S}_{\mathrm{ac}}(G(F_V)) = \varinjlim_{\Gamma} \mathscr{S}_{\mathrm{ac}}(G(F_V))_\Gamma$$

parallel to the definition of $\mathscr{I}_{\mathrm{ac}}(G, V, \zeta)$ above by replacing $\Pi_{\mathrm{temp}}(G(F_V))$ with $\Phi_{\mathrm{temp}}(G(F_V))$. Then if G is arbitrary, we would like to define for each $M \in \mathscr{L}$ a map

$$\iota_M^{\mathscr{E}} : \mathscr{H}_{\mathrm{ac}}(G, V, \zeta) \to \mathscr{I}_{\mathrm{ac}}(M, V, \zeta)$$

such that

$$\iota_M^{\mathscr{E}}(f, \pi, X) = I_M^{\mathscr{E}}(\pi, X, f), \qquad \pi \in \Pi(M(F_V)), X \in \mathfrak{a}_{M,V},$$

and if G is quasisplit, we would like similarly a map

$$\tau_M : \mathscr{H}_{\mathrm{ac}}(G, V, \zeta) \to \mathscr{S}_{\mathrm{ac}}(M, V, \zeta)$$

such that

$$\tau_M(f, \phi, X) = S_M(\phi, X, f), \qquad \phi \in \Phi(M(F_V)), X \in \mathfrak{a}_{M,V}.$$

Recall that the map ϕ_M is defined initially for tempered representations π and then extended by analytic continuation. The definitions of the weighted characters for general elements π and ϕ above implicitly rely on this property.

We shall also define in this case the more general maps $\iota_{M,\mu}^{\mathscr{E}}$ and $\tau_{M,\mu}$ for $\mu \in \mathfrak{a}_M^*$ given by

$$\iota_{M,\mu}^{\mathscr{E}}(f, \pi, X) = I_M^{\mathscr{E}}(\pi_\mu, X, f) \mathrm{e}^{-\mu(X)}$$

$$\tau_{M,\mu}(f, \phi, X) = S_M(\phi_\mu, X, f) \mathrm{e}^{-\mu(X)}.$$

If $\mu = 0$, we shall omit the subscript μ and simply write $\phi_{M,0} = \phi_M$, for example. We shall provide inductive constructions of these maps using $\phi_{M,\mu}$ in the following proposition.

Proposition 2.1. *For each $\mu \in \mathfrak{a}_M^*$, there exist continuous linear maps $\iota^{\mathscr{E}}_{M,\mu}$ and $\tau_{M,\mu}$ from $\mathscr{H}_{\mathrm{ac}}(G,V,\zeta)$ to $\mathscr{I}_{\mathrm{ac}}(M,V,\zeta)$ and $\mathscr{S}_{\mathrm{ac}}(M,V,\zeta)$ respectively.*

Proof. To study the properties of these maps, it will be useful for us to define an invariant form of the map ϕ_M,

$$\iota_M : \mathscr{H}_{\mathrm{ac}}(M,V,\zeta) \to \mathscr{I}_{\mathrm{ac}}(M,V,\zeta)$$

whose value at a point (π, X) in $\Pi(M(F_V)) \times \mathfrak{a}_{M,V}$ is given by

$$\iota_M(f,\pi,X) = I_M(\pi,X,f), \qquad f \in \mathscr{H}_{\mathrm{ac}}(G,V,\zeta),$$

and which can be equivalently defined by the inductive formula

$$\iota_M(f,\pi,X) = J_M(\pi,X,f) - \sum_{L \in \mathscr{L}^0(M)} \hat{I}^L_M(\pi,X,\phi_L(f)),$$

exactly as for the invariant weighted character. It follows that ι_M vanishes if π belongs to $\Pi_{\mathrm{temp}}(M(F_V))$ and $M \neq G$, and for nontempered representations it is defined by analytic continuation. Setting

$$\iota_{M,\mu}(f,\pi,X) = I_M(\pi_\mu,X,f)\mathrm{e}^{-\mu(X)}, \tag{2.6}$$

it follows from the inductive definition and properties of the transform ϕ_M that ι_M also maps $\mathscr{H}_{\mathrm{ac}}(M,V,\zeta)$ continuously to $\mathscr{I}_{\mathrm{ac}}(M,V,\zeta)$. Indeed, this can also be shown directly using the expansion of $I_{M,\mu}(\pi,X,f)$ in terms of residues of $J_M(\pi_\lambda,f)$ described in [Art89b, Theorem 4.1], and examining the continuity argument of $\phi_{M,\mu}$ in [Art89a, Theorem 12.1].

Now, given $\phi' \in \Phi((\tilde{M}'_V)^{\tilde{Z}'}, \tilde{\zeta}'_V)$ we can then construct the map $\iota^{\mathscr{E}}_M$ by the inductive definition

$$\iota^{\mathscr{E}}_M(f,\phi',X) = \sum_{G' \in \mathscr{E}^0_{M'}(G)} \iota_{M'}(G,G')\hat{S}^{\tilde{G}'}_{\tilde{M}'}(f',\phi',X) + \varepsilon(G)\tau_M(M',f,\phi',X)$$

with the supplementary requirement that

$$\iota^{\mathscr{E}}_M(f,\phi',X) = \iota_M(f,\phi,X)$$

in the case that G is quasisplit. Certainly this agrees with our definition of $\iota^{\mathscr{E}}_M$ above. Also, if $G' = G^*$ we again set

$$\tau_M(f,\phi,X) = \tau_M(M^*,f,\phi^*,X).$$

If ϕ' and M' are locally relevant to M, we define the endoscopic form $I^{\mathscr{E}}_M(f,\pi,X)$ by the relation

$$\iota^{\mathscr{E}}_M(f,\phi',X) = \sum_{\pi \in \Pi(M^Z_V,\zeta_V)} \Delta_M(\phi',\pi)\iota^{\mathscr{E}}_M(f,\pi,X)$$

and the inversion formula

$$\sum_{\pi \in \Pi(M^Z_V,\zeta_V)} \Delta_M(\phi,\pi)\Delta_M(\pi,\phi_1) = \delta(\phi,\phi_1), \qquad \phi,\phi_1 \in \Phi^{\mathscr{E}}(M^Z_V,\zeta_V)$$

described in [Art02, §5], where $\delta(\phi,\phi_1) = 1$ if $\phi = \phi_1$ and 0 otherwise. It follows then from the inductive definitions that the maps $\iota^{\mathscr{E}}_{M,\mu}(f,\pi,X)$ and $\tau_{M,\mu}(f,\pi,X)$ are continuous. □

As with ϕ_M, if we consider only tempered π and ϕ respectively, then we can allow for f belonging to the larger space $\mathscr{C}(G_V^Z, \zeta_V)$. In that case $\iota_M^{\mathscr{E}}$ would be a continuous map from $\mathscr{C}(G_V^Z, \zeta_V)$ to $I\mathscr{C}(G_V^Z, \zeta_V)$, and τ_M from $\mathscr{C}(G_V^Z, \zeta_V)$ to the stably invariant Schwarz space $S\mathscr{C}(G_V^Z, \zeta_V)$ generated by stable ζ-equivariant orbital integrals of functions $f \in \mathscr{C}(G_V^Z, \zeta_V)$.

3. Modified distributions

3.1. **Weighted characters.** The maps $\iota_M^{\mathscr{E}}$ and τ_M that we have just constructed allow us to modify the endoscopic and stable weighted characters in a manner similar to the invariant linear forms $I_M(\pi, X, f)$. Define inductively the modified linear forms

$$\tilde{I}_M^{\mathscr{E}}(\pi, X, f), \qquad f \in \mathscr{H}_{\text{ac}}(G, V, \zeta)$$

by setting

$$\tilde{I}_M^{\mathscr{E}}(\pi, X, f) = I_M^{\mathscr{E}}(\pi, X, f) - \sum_{L \in \mathscr{L}^0(M)} \hat{\tilde{I}}_M^{L,\mathscr{E}}(\pi, X, \iota_L^{\mathscr{E}}(f)), \tag{3.1}$$

if G is arbitrary, and

$$\tilde{S}_M^G(\phi, X, f), \qquad f \in \mathscr{H}_{\text{ac}}(G, V, \zeta)$$

by setting

$$\tilde{S}_M^G(\phi, X, f) = S_M^G(\phi, X, f) - \sum_{L \in \mathscr{L}^0(M)} \hat{\tilde{S}}_M^L(\phi, X, \tau_L(f)), \tag{3.2}$$

in the case that G is quasisplit. As usual, if $X = 0$ we shall simply write $\tilde{I}_M^{\mathscr{E}}(\pi, f)$ and $\tilde{S}_M^G(\phi, f)$. We shall also define more generally $\tilde{S}_M(M', \phi', X, \tau_L(f))$ in a similar manner, with

$$\tilde{S}_M^G(\phi, X, f) = \tilde{S}_M^G(M^*, \phi^*, X, \tau_L(f)).$$

Where the context is clear, we shall omit the superscript and write simply $\tilde{S}_M = \tilde{S}_M^G$.We then have the following analogue of (2.4).

Lemma 3.1. *We have that*

$$\tilde{I}_M^{\mathscr{E}}(\pi, X, f) = \begin{cases} f^G(\pi, X), & \text{if } M = G, \\ 0, & \text{if } M \neq G \end{cases}$$

if G is not quasiplit, and

$$\tilde{S}_M(\phi, X, f) = \begin{cases} f^G(\phi, X), & \text{if } M = G, \\ 0, & \text{if } M \neq G \end{cases}$$

if G is quasisplit.

Proof. First, suppose that G is arbitrary. If $M = G$, then $\tilde{I}_M^{\mathscr{E}}(\pi, X, f)$ reduces to $I_G^{\mathscr{E}}(\pi, X, f) = f^G(\pi, X)$ by definition. If $M \neq G$, the definitions imply that

$$I_M^{\mathscr{E}}(\pi, X, f) = \iota_M^{\mathscr{E}}(f, \pi, X) = \hat{\tilde{I}}_M^{\mathscr{E}}(\pi, X, f),$$

and the required idenitty follows inductively from (3.1).

Similarly, if G is quasiplit, then $\tilde{S}_M(\phi, X, f)$ reduces to $S_G(\phi, X, f) = f^G(\phi, X)$ if $M = G$. Otherwise, we have that

$$S_M(\phi, X, f) = \tau_M(f, \phi, X) = \hat{\tilde{S}}_M(\phi, X, f)$$

and the result then follows inductively from (3.2). □

This simple result will be the basis for our modification of the endoscopic and stable trace formulae, allowing us to isolate the unitary part of the spectral expansions. Recall that the analogous property for the invariant weighted characters $I_M(\pi, X, f)$ in (2.4) holds only for tempered representations π, whereas if π is not tempered they are defined by analytic continuation. In contrast, the lemma here for modified distributions above holds in the nontempered case also as implicitly we have depended on the definition of $I_M(\pi, X, f)$ for general π.

As with the stable and invariant weighted characters, the distributions $\tilde{I}^{\mathscr{E}}_M(\pi, X, f)$ and $\tilde{S}_M(\phi, X, f)$ do not assume too many values. Set

$$\tilde{I}^{\mathscr{E}}_{M,\mu}(\pi, X, f) = \tilde{I}^{\mathscr{E}}_M(\pi_\mu, X, f)\mathrm{e}^{-\mu(X)}$$

if G is arbitrary, and

$$\tilde{S}_{M,\mu}(\phi, X, f) = \tilde{S}_M(\phi_\mu, X, f)\mathrm{e}^{-\mu(X)}$$

if G is quasisplit, where $\mu \in \mathfrak{a}^*_M$.

Lemma 3.2. (a) *As functions of μ, $\tilde{I}^{\mathscr{E}}_{M,\mu}(\pi, X, f)$ and $\tilde{S}_{M,\mu}(\phi, X, f)$ are locally constant on the complement of a finite set of hyperplanes of the form $\mu(\alpha^\vee) = N$ for $N \in \mathbf{R}$ and α a root of (G, A_M).*

(b) *For each $P \in \mathscr{P}(M)$, given a small point ϵ_P in the chamber $(\mathfrak{a}^*_P)^+$, then*

$$\tilde{I}^{\mathscr{E}}_{M,\mu} = \sum_{P \in \mathscr{P}(M)} \omega_P \tilde{I}^{\mathscr{E}}_{M,\mu+\epsilon_P}(\pi, X, f)$$

and

$$\tilde{S}_{M,\mu} = \sum_{P \in \mathscr{P}(M)} \omega_P \tilde{S}_{M,\mu+\epsilon_P}(\phi, X, f).$$

Proof. Extending the definitions (3.1) and (3.2) to

$$\tilde{I}^{\mathscr{E}}_{M,\mu}(\pi, X, f) = I^{\mathscr{E}}_{M,\mu}(\pi, X, f) - \sum_{L \in \mathscr{L}^0(M)} \hat{\tilde{I}}^{L,\mathscr{E}}_{M,\mu}(\pi, X, \iota^{\mathscr{E}}_L(f)) \tag{3.3}$$

and

$$\tilde{S}_{M,\mu}(\phi, X, f) = S_{M,\mu}(\phi, X, f) - \sum_{L \in \mathscr{L}^0(M)} \hat{\tilde{S}}^{L}_{M,\mu}(\phi, X, \tau_L(f))$$

respectively, where

$$I^{\mathscr{E}}_{M,\mu}(\pi, X, f) = I^{\mathscr{E}}_M(\pi_\mu, X, f)\mathrm{e}^{-\mu(X)}$$

and

$$S_{M,\mu}(\phi, X, f) = S_M(\phi_\mu, X, f)\mathrm{e}^{-\mu(X)},$$

we see from the inductive definitions that the result will follow if we can establish the corresponding statement of (a) for $I^{\mathscr{E}}_{M,\mu}(\pi, X, f)$ and $S_{M,\mu}(\phi, X, f)$, which in turn follows from the inductive definition (2.5) and the corresponding statement for $I_{M,\mu}(\pi, X, f)$ (see also [MW16b, XI.6]).

On the other hand, assume inductively that (b) holds if G is replaced by any element $L \in \mathscr{L}^0(M)$. Then

$$\hat{\tilde{I}}^{L,\mathscr{E}}_{M,\mu}(\pi, X, \iota^{\mathscr{E}}_L(f)) = \sum_{R \in \mathscr{P}^L(M)} \omega_R \hat{\tilde{I}}^{L,\mathscr{E}}_{M,\mu+\epsilon_R}(\pi, X, \iota^{\mathscr{E}}_L(f))$$

and applying assertion (a) to L we see that this may be written as

$$\sum_{P\in\mathscr{P}(M)} \omega_P \hat{\tilde{I}}^{L,\mathscr{E}}_{M,\mu+\epsilon_P}(\pi, X, \iota^{\mathscr{E}}_L(f)).$$

The assertion then follows from the corresponding statement for $I^{\mathscr{E}}_M(\pi_\mu, X, f)$ and (3.3). The proof for $\tilde{S}_{M,\mu}$ follows by the same argument. □

Let us also define inductively the modified invariant linear form

$$\tilde{I}_M(\pi, X, f), \qquad f \in \mathscr{H}_{\mathrm{ac}}(G, V, \zeta)$$

by setting

$$\tilde{I}_M(\pi, X, f) = I_M(\pi, X, f) - \sum_{L\in\mathscr{L}^0(M)} \hat{\tilde{I}}^L_M(\pi, X, \iota_L(f)).$$

Then we have the following property as an immediate consequence of Local Theorem $2'$ of [Art02], which will be needed for the global expansion.

Lemma 3.3. *Let V be a finite set of valuations such that G and ζ are unramified outside of V.*

(a) *if G is arbitrary,*

$$\tilde{I}^{\mathscr{E}}_M(\pi, X, f) = \tilde{I}_M(\pi, X, f), \qquad \pi \in \Pi(M^Z_V, \zeta_V),\ f \in \mathscr{H}(G, V, \zeta)$$

(b)*Suppose that G is quasisplit, and that ϕ' belongs to $\Phi(\tilde{M}', \tilde{\zeta}')$ for some $M' \in \mathscr{E}_{\mathrm{ell}}(M)$. Then the linear form*

$$f \to \tilde{S}^G_M(M', \phi', X, f), \qquad f \in \mathscr{H}(G, V, \zeta)$$

vanishes unless $M' = M^*$, in which case it is stable.

3.2. **Supplementary maps.** In order to study these distributions, we shall have to define a family of supplementary maps. For any set of hyperplanes Φ in $\mathfrak{a}$, let $\mathscr{C}$ be the set of open connected components whose union is the complement of Φ in $\mathfrak{a}$. Then given $c \in \mathscr{C}$ and $X \in \mathfrak{a}$, we set

$$\omega(c, X) = \mathrm{vol}(c \cap B_X)\mathrm{vol}(B_X)^{-1}$$

where B_X is a small ball in $\mathfrak{a}$ centred at X. The function vanishes for any X outside the closure of c, and is locally constant on the strata of $\mathfrak{a}$ defined by the intersection of planes in Φ. We then define $\mathscr{H}^\Phi(G, V, \zeta)$ to be the space of functions f on G^Z_V such that

$$f(x) = \sum_{c\in\mathscr{C}} \omega(c, H_G(x)) f_c(x),$$

for $f_c \in \mathscr{H}(G, V, \zeta)$, and similarly $\mathscr{I}^\Phi(G, V, \zeta)$ to the space of functions g on $\Pi_{\mathrm{temp}}(G^Z_V, \zeta_V) \times \mathfrak{a}^Z_G$ of the form

$$g(\pi, X) = \sum_{c\in\mathscr{C}} \omega(c, X) g_c(\pi, X),$$

for $g_c \in \mathscr{I}(G, V, \zeta)$, and thirdly, $\mathscr{S}^\Phi(G, V, \zeta)$ to the space of functions h on $\Phi_{\mathrm{temp}}(G^Z_V, \zeta_V) \times \mathfrak{a}^Z_G$ of the form

$$h(\phi, X) = \sum_{c\in\mathscr{C}} \omega(c, X) h_c(\phi, X),$$

for $h_c \in \mathscr{S}(G, V, \zeta)$.

These spaces can be topologised in manner similar to the argument in [Art88a, §4] for the Hecke algebra. For example, we define for any positive integer N and finite subset Γ of $\Pi(K)$ the space $\mathscr{H}_N(G,V,\zeta)_\Gamma$ of smooth ζ-equivariant functions on $G(F_V)$ supported on the set

$$\{x \in G(F_V) : \log ||x|| \leq N\}$$

and which transform on each side under K according to representations in Γ. Its topology is given by the seminorms

$$||f||_D = \sup_{x\in G(F_V)} |Df(x)|, \qquad f \in \mathscr{H}_N(G,V,\zeta)_\Gamma,$$

where D is a differential operator on $G(F_{V\cap V_\infty})$, with V_∞ is the set of archimedean valuations of F. Then $\mathscr{H}^\Phi(G,V,\zeta)$ can be obtained as the topological direct limit

$$\varinjlim_{\Gamma} \varinjlim_{N} \mathscr{H}_N^\Phi(G,V,\zeta)_\Gamma.$$

Here $\mathscr{H}_N^\Phi(G,V,\zeta)_\Gamma$ denotes the space of functions f such that f_c belongs to $\mathscr{H}_N(G,V,\zeta)_\Gamma$. Its topology is defined by the seminorms

$$||f||_D = \sup_{c\in\mathscr{C}} \sup_{\{x\in G(F_V):H_G(x)\in c\}} |Df_c(x)|.$$

Now, the collection of Φ being a partially ordered set, we can then define

$$\widetilde{\mathscr{H}}(G,V,\zeta) = \varinjlim_{\Phi} \mathscr{H}^\Phi(G,V,\zeta),$$

$$\widetilde{\mathscr{I}}(G,V,\zeta) = \varinjlim_{\Phi} \mathscr{I}^\Phi(G,V,\zeta),$$

and

$$\widetilde{\mathscr{S}}(G,V,\zeta) = \varinjlim_{\Phi} \mathscr{S}^\Phi(G,V,\zeta).$$

If V contains no archimedean valuations, then $\mathfrak{a}_{G,V}$ is a lattice in $\mathfrak{a}$, and the spaces $\widetilde{\mathscr{H}}(G,V,\zeta)$, $\widetilde{\mathscr{I}}(G,V,\zeta)$, and $\widetilde{\mathscr{S}}(G,V,\zeta)$ are equal to $\mathscr{H}(G,V,\zeta)$, $\mathscr{I}(G,V,\zeta)$, and $\mathscr{S}(G,V,\zeta)$ respectively. The extensions $\widetilde{\mathscr{H}}_{\mathrm{ac}}(G,V,\zeta)$, $\widetilde{\mathscr{I}}_{\mathrm{ac}}(G,V,\zeta)$, and $\widetilde{\mathscr{S}}_{\mathrm{ac}}(G,V,\zeta)$ are defined similarly.

Specifically, we shall consider the set of hyperplanes in $\mathfrak{a}_M$ given by the collection

$$\Phi = \{\mathfrak{a}_L : L \in \mathscr{L}(M), \dim(A_M/A_L) = 1\}.$$

The associated components are the usual chambers $\{\mathfrak{a}_P^+ : P \in \mathscr{P}(M)\}$. For each $P \in \mathscr{P}(M)$, we shall set $\omega_P(X) = \omega(\mathfrak{a}_P^+, X)$ so that $\omega_P(0)$ is the number ω_P, and let ν_P be the point in the associated chamber $(\mathfrak{a}_P^*)^+$ in $\mathfrak{a}_M^*$ whose distance from the walls is very large, so that $\iota^{\mathscr{E}}_{M,\nu_P}(f,\pi,X)$ and $\tau_{M,\nu_P}(f,\pi,X)$ are independent of ν_P. We define

$${}^c\iota^{\mathscr{E}}_M(f,\pi,X) = \sum_{P\in\mathscr{P}(M)} \omega_P(X)\iota^{\mathscr{E}}_{M,\nu_P}(f,\pi,X)$$

and

$${}^c\tau_M(f,\phi,X) = \sum_{P\in\mathscr{P}(M)} \omega_P(X)\tau_{M,\nu_P}(f,\phi,X)$$

for $f \in \mathscr{H}_{\mathrm{ac}}(G,V,\zeta)$, $\pi \in \Pi_{\mathrm{temp}}(M_V^Z)$, $\phi \in \Phi_{\mathrm{temp}}(M_V^Z)$, and $X \in \mathfrak{a}_{M,V}$. One sees that $\iota^{\mathscr{E}}_{M,\nu_P}$ and τ_{M,ν_P} are continuous maps from $\widetilde{\mathscr{H}}(G,V,\zeta)$ to $\widetilde{\mathscr{I}}(G,V,\zeta)$ and $\widetilde{\mathscr{S}}(G,V,\zeta)$ respectively. It follows that ${}^c\iota^{\mathscr{E}}_M$ and ${}^c\tau_M$ also map $\widetilde{\mathscr{H}}(G,V,\zeta)$

continuously to $\widetilde{\mathscr{I}}(G,V,\zeta)$ and $\widetilde{\mathscr{S}}(G,V,\zeta)$ respectively. These definitions follow the supplementary map ${}^c\phi_M$, and accordingly send functions of compact support to functions of compact support.

Lemma 3.4. *The maps* ${}^c\iota_M^{\mathscr{E}}(f,\pi,X)$ *and* ${}^c\tau_M(f,\phi,X)$ *send* $\widetilde{\mathscr{H}}(G,V,\zeta)$ *continuously to* $\widetilde{\mathscr{I}}(M,V,\zeta)$ *and* $\widetilde{\mathscr{S}}(M,V,\zeta)$ *respectively.*

Proof. We would like to show that there is a positive integer N depending only on the support of $f \in \widetilde{\mathscr{H}}(G,V,\zeta)$ such that ${}^c\iota_M^{\mathscr{E}}(f,\pi,X)$ and ${}^c\tau_M(f,\phi,X)$ are supported on the ball in $\mathfrak{a}_{M,V}$ of radius N. From the definitions of ${}^c\iota_M^{\mathscr{E}}$ and ${}^c\tau_M(f,\pi,X)$, we see that it suffices to prove that for any $P \in \mathscr{P}(M)$ and X in the closure of $\mathfrak{a}_P^+ \cap \mathfrak{a}_{M,V}$, the functions ${}^c\iota_{M,\nu_P}^{\mathscr{E}}(f,\pi,X)$ and ${}^c\tau_{M,\nu_P}(f,\phi,X)$ are supported on the ball of radius N. Then from the decomposition

$$f(x) = \sum_{c\in\mathscr{C}} \omega(c,H_G(x))f_c(x),$$

we may assume that each f_c are each supported on a set which depends only on the support of f. We may therefore assume that f itself belongs to $\mathscr{H}(G,V,\zeta)$, and therefore

$${}^c\iota_M^{\mathscr{E}}(f,\pi,X) = \int_{\nu_P + i\mathfrak{a}_{M,V}^*} \mathrm{e}^{-\lambda(X)} I_M^{\mathscr{E}}(\pi_\lambda,f)d\lambda$$

and

$${}^c\tau_M(f,\phi,X) = \int_{\nu_P + i\mathfrak{a}_{M,V}^*} \mathrm{e}^{-\lambda(X)} S_M(\phi_\lambda,f)d\lambda.$$

We then have to show that as a function of $X \in \mathfrak{a}_P^+$, the integrals are supported on a ball which depends only on the support of f. The proof now follows inductively from the same property of the map ${}^c\phi_M$ studied in [Art88a, §4] and the supplementary map defined by

$${}^c\iota_M(f,\pi,X) = \sum_{P\in\mathscr{P}(M)} \omega_P(X)\iota_{M,\nu_P}(f,\pi,X)$$

using (2.6). $\square$

Our study of the maps $\iota_M^{\mathscr{E}}$ and τ_M leads us naturally to invariant maps $\theta_{\iota,M}^{\mathscr{E}}$ and ${}^c\theta_{\iota,M}^{\mathscr{E}}$ from $\widetilde{\mathscr{H}}_{\mathrm{ac}}(G,V,\zeta)$ to $\widetilde{\mathscr{I}}_{\mathrm{ac}}(M,V,\zeta)$ defined inductively by

$$\theta_{\iota,M}^{\mathscr{E}}(f) = {}^c\iota_M^{\mathscr{E}}(f) - \sum_{L\in\mathscr{L}^0(M)} \hat{\theta}_{\iota,M}^{L,\mathscr{E}}(\iota_L^{\mathscr{E}}(f)) \tag{3.4}$$

$${}^c\theta_{\iota,M}^{\mathscr{E}}(f) = \iota_M^{\mathscr{E}}(f) - \sum_{L\in\mathscr{L}^0(M)} {}^c\hat{\theta}_{\iota,M}^{L,\mathscr{E}}({}^c\iota_L^{\mathscr{E}}(f)) \tag{3.5}$$

for any $f \in \widetilde{\mathscr{H}}_{\mathrm{ac}}(G,V,\zeta)$, and also stably invariant maps $\theta_{\tau,M}$ and ${}^c\theta_{\tau,M}$ from $\widetilde{\mathscr{H}}_{\mathrm{ac}}(G,V,\zeta)$ to $\widetilde{\mathscr{S}}_{\mathrm{ac}}(M,V,\zeta)$ inductively by setting

$$\theta_{\tau,M}(f) = {}^c\tau_M(f) - \sum_{L\in\mathscr{L}^0(M)} \hat{\theta}_{\tau,M}^{L}(\tau_L(f)) \tag{3.6}$$

$${}^c\theta_{\tau,M}(f) = \tau_M(f) - \sum_{L\in\mathscr{L}^0(M)} {}^c\hat{\theta}_{\tau,M}^{L}({}^c\tau_L(f)). \tag{3.7}$$

We shall use these maps to study the modified distributions $\tilde{I}^{\mathscr{E}}_M(\pi, X, f)$ and $\tilde{S}_M(\phi, X, f)$ and their variants on $\tilde{\mathscr{H}}_{\mathrm{ac}}(G, V, \zeta)$ defined by

$$ {}^c\tilde{I}^{\mathscr{E}}_M(\pi, X, f) = \tilde{I}^{\mathscr{E}}_M(\pi, X, f) - \sum_{L \in \mathscr{L}^0(M)} {}^c\hat{\tilde{I}}^{L,\mathscr{E}}_M(\pi, X, {}^c\iota^{\mathscr{E}}_L(f)) \tag{3.8}$$

and

$$ {}^c\tilde{S}_M(\phi, X, f) = \tilde{S}_M(\phi, X, f) - \sum_{L \in \mathscr{L}^0(M)} {}^c\hat{\tilde{S}}^{L}_M(\phi, X, {}^c\tau_L(f)). \tag{3.9}$$

We then have the following basic relation between these maps and the modified characters.

Lemma 3.5. *Given $f \in \widetilde{\mathscr{H}}_{\mathrm{ac}}(G, V, \zeta)$, we have that*

$$\theta^{\mathscr{E}}_{\iota,M}(f, \pi, X) = \sum_{P \in \mathscr{P}(M)} \omega_P(X) \tilde{I}_{M,\nu_P}(\pi, X, f),$$

$${}^c\theta^{\mathscr{E}}_{\iota,M}(f, \pi, X) = {}^c\tilde{I}^{\mathscr{E}}_M(\pi, X, f)$$

for $\pi \in \Pi_{\mathrm{temp}}(M^Z_V, \zeta_V)$ and G non-quasisplit over F, and

$$\theta_{\tau,M}(f, \phi, X) = \sum_{P \in \mathscr{P}(M)} \omega_P(X) \tilde{S}_{M,\nu_P}(\phi, X, f),$$

$${}^c\theta_{\tau,M}(f, \phi, X) = {}^c\tilde{S}_M(\phi, X, f)$$

for $\phi \in \Phi_{\mathrm{temp}}(M^Z_V, \zeta_V)$ and G quasisplit over F.

Proof. The proof is a simple adaptation of the argument of [Art88a, Lemma 4.7], so we will be content with the general argument. By induction, we may assume that for any $L \in \mathscr{L}^0(M)$,

$$\hat{\theta}^{L,\mathscr{E}}_{\iota,M}(\iota^{\mathscr{E}}_L(f), \pi, X) = \sum_{R \in \mathscr{P}^L(M)} \omega_R(X) \hat{\tilde{I}}^{L,\mathscr{E}}_{M,\nu_R}(\pi, X, \iota^{\mathscr{E}}_L(f))$$

where $\mathscr{P}^L(M)$ denotes the set of parabolic subgroups of L containing M. The summands on the right are independent of the point ν_R as long as it remains highly regular in $(\mathfrak{a}^*_R)^+$. Now given any subset Φ' of hyperplanes Φ in $\mathfrak{a}$, with connected components $\mathscr{C}' \subset \mathscr{C}$, we have that

$$\omega(c', X) = \sum_{c \in \mathscr{C} : c \subset c'} \omega(c, X)$$

for any $c' \in \mathscr{C}'$. It follows then that

$$\hat{\theta}^{L,\mathscr{E}}_{\iota,M}(\iota^{\mathscr{E}}_L(f), \pi, X) = \sum_{P \in \mathscr{P}(M)} \omega_P(X) \hat{\tilde{I}}^{L,\mathscr{E}}_{M,\nu_P}(\pi, X, \iota^{\mathscr{E}}_L(f)).$$

The definition (3.4) is equal to subtracting the sum over $L \in \mathscr{L}^0(M)$ of this from the function

$${}^c\iota^{\mathscr{E}}_M(f, \pi, X) = \sum_{P \in \mathscr{P}(M)} \omega_P(X) \iota^{\mathscr{E}}_{M,\nu_P}(f, \pi, X).$$

The difference is equal to

$$\sum_{P \in \mathscr{P}(M)} \omega_P(X) \tilde{I}^{\mathscr{E}}_{M,\nu_P}(\pi, X, f),$$

since $\iota^{\mathscr{E}}_{M,\nu_P}(f,\pi,X) = \tilde{I}^{\mathscr{E}}_{M,\nu_P}(f,\pi,X)$, and the first identity follows. The second identity follows from a similar inductive argument using (3.5) and (3.8). The third and fourth identities are also proved in a similar fashion, using the parallel definitions (3.6), (3.7), and (3.9). □

3.3. **Weighted orbital integrals.** We would like to construct the linear forms that will occur in the modified geometric expansion, but first we shall have to recall the linear forms appearing in the geometric sides of the endoscopic and stable trace formulae. Let $\mathscr{D}(G^Z_V,\zeta_V)$ be the space of distributions on G^Z_V that are (i) invariant under G^Z_V-conjugation, (ii) ζ_V-equivariant under translation by Z_V, and (iii) supported on the preimage in G^Z_V of a finite union of conjugacy classes in $\bar{G}^Z_V = G^Z_V/Z_V$. It contains the invariant orbital integrals, and also derivatives of orbital integrals if V contain archimedean places. We shall write $\Gamma(G^Z_V,\zeta_V)$ for the fixed basis of this space as chosen in [Art02, §1].

Furthermore let $S\mathscr{D}(G^Z_V,\zeta_V)$ be the subspace of stable distributions in $\mathscr{D}(G^Z_V,\zeta)$. Recall that a distribution on G^Z_V is called stable if it lies in the closed linear span of strongly regular, stable orbital integrals. Following [Art02, §5], there is an endoscopic basis $\Delta^{\mathscr{E}}(G^Z_V,\zeta)$ of $\mathscr{D}(G^Z_V,\zeta_V)$ such that the intersection

$$\Delta(G^Z_V,\zeta_V) = \Delta^{\mathscr{E}}(G^Z_V,\zeta_V) \cap S\mathscr{D}(G^Z_V,\zeta_V)$$

forms a basis of $S\mathscr{D}(G^Z_V,\zeta_V)$. For any $f\in\mathscr{C}(G^Z_V,\zeta_V)$ and $\gamma\in\Gamma(M^Z_V,\zeta_V)$, we have the invariant linear form defined by the usual inductive formula

$$I_M(\gamma,f) = J_M(\gamma,f) - \sum_{L\in\mathscr{L}^0(M)} \hat{I}^L_M(\gamma,\phi_L(f)),$$

where $\phi_L(f)$ is the map (2.2), and $J_M(\gamma,f)$ is the noninvariant weighted orbital integral as in [Art02, §2].

We continue to assume that the endoscopic data M' and $\tilde{G}',\tilde{\xi}'$ are unramified away from the finite set V. Let $\delta'\in\Delta((\tilde{M}'_V)^{\tilde{Z}'},\tilde{\zeta}'_V)$. The corresponding endoscopic and stable linear forms on $\mathscr{S}(\tilde{G}'_V,\tilde{\zeta}'_V)$ are then defined inductively by the formula

$$I^{\mathscr{E}}_M(\delta',f) = \sum_{G\in\mathscr{E}^0_{M'}(G)} \iota_{M'}(G,G')\hat{S}^{\tilde{G}'}_{M'}(\delta',f') + \varepsilon(G)S^G_M(M',\delta',f)$$

together with the supplementary requirement that

$$I^{\mathscr{E}}_M(\delta',f) = I_M(\delta,f)$$

in the case that G is quasisplit and δ' maps to the element δ in $\Delta^{\mathscr{E}}(M^Z_V,\zeta_V)$. The coefficient $\iota_{M'}(G,G')$ vanishes unless G belongs to the finite set subset of elliptic elements in $\mathscr{E}_{M'}(G)$. The actual endoscopic distributions are in fact the family

$$I^{\mathscr{E}}_M(\gamma,f),\qquad \gamma\in\Gamma(M^Z_V,\zeta_V)$$

defined by the formula

$$I^{\mathscr{E}}_M(\delta',f) = \sum_{\gamma\in\Gamma(M^Z_V,\zeta_V)} \Delta_M(\delta',\gamma)I^{\mathscr{E}}_M(\gamma,f),$$

or equivalently,

$$I^{\mathscr{E}}_M(\gamma,f) = \sum_{\delta\in\Delta^{\mathscr{E}}(M^Z_V,\zeta_V)} \Delta_M(\gamma,\delta')I^{\mathscr{E}}_M(\delta,f),$$

which holds because $I_M^{\mathscr{E}}(\delta', f)$ depends only on the image δ of δ' in $\Delta^{\mathscr{E}}(M_V^Z, \zeta_V)$. Here $\Delta(\gamma, \delta)$ is the geometric transfer factor [Art02, §5], which satisfies the adjoint relations

$$\sum_{\delta \in \Delta^{\mathscr{E}}(G_V^Z, \zeta_V)} \Delta(\gamma, \delta)\Delta(\delta, \gamma_1) = \delta(\gamma, \gamma_1), \qquad \gamma, \gamma_1 \in \Gamma(G_V^Z, \zeta_V) \tag{3.10}$$

and

$$\sum_{\gamma \in \Gamma(G_V^Z, \zeta_V)} \Delta(\delta, \gamma)\Delta(\gamma, \delta_1) = \delta(\delta, \delta_1), \qquad \delta, \delta_1 \in \Delta^{\mathscr{E}}(G_V^Z, \zeta_V), \tag{3.11}$$

where $\delta(\gamma, \gamma_1) = 1$ if $\gamma = \gamma_1$ and 0 otherwise, and similarly for $\delta(\gamma, \gamma_1)$. These distributions are the key objects on the geometric side of the stable trace formula.

We now would like to define our modification of these distributions using the maps we have introduced earlier. Define inductively the modified linear forms

$$\tilde{I}_M^{\mathscr{E}}(\gamma, f), \qquad f \in \mathscr{H}_{\mathrm{ac}}(G, V, \zeta)$$

by setting

$$\tilde{I}_M^{\mathscr{E}}(\gamma, f) = I_M^{\mathscr{E}}(\gamma, f) - \sum_{L \in \mathscr{L}^0(M)} \hat{\tilde{I}}_M^{L,\mathscr{E}}(\gamma, \iota_L^{\mathscr{E}}(f)), \tag{3.12}$$

if G is arbitrary, and

$$\tilde{S}_M(\delta, f), \qquad f \in \mathscr{H}_{\mathrm{ac}}(G, V, \zeta)$$

by setting

$$\tilde{S}_M(\delta, f) = S_M(\delta, f) - \sum_{L \in \mathscr{L}^0(M)} \hat{\tilde{S}}_M^L(\delta, \tau_L(f)). \tag{3.13}$$

in the case that G is quasisplit. More generally, we define $\tilde{S}_M^G(M', \delta', f)$ analogously, whereby

$$\tilde{S}_M^G(\delta, f) = \tilde{S}_M^G(M^*, \delta^*, f)$$

and omitting the subscript G when the context is clear. The modified endoscopic distribution then satisfies the analogous inversion formula

$$\tilde{I}_M^{\mathscr{E}}(\delta', f) = \sum_{\gamma \in \Gamma(M_V^Z, \zeta_V)} \Delta_M(\delta', \gamma)\tilde{I}_M^{\mathscr{E}}(\gamma, f), \tag{3.14}$$

by the adjoint property of the transfer factors. We also define the variants

$${}^c\tilde{I}_M^{\mathscr{E}}(\gamma, f) = I_M^{\mathscr{E}}(\gamma, f) - \sum_{L \in \mathscr{L}^0(M)} {}^c\hat{\tilde{I}}_M^{L,\mathscr{E}}(\gamma, {}^c\iota_L^{\mathscr{E}}(f)), \tag{3.15}$$

and

$${}^c\tilde{S}_M(\delta, f) = S_M(\delta, f) - \sum_{L \in \mathscr{L}^0(M)} {}^c\hat{\tilde{S}}_M^L(\delta, {}^c\tau_L(f)). \tag{3.16}$$

The supplementary maps that we have defined earlier allow us to describe the asymptotic behaviour of these distributions.

Proposition 3.6. *Let $f \in \widetilde{\mathscr{H}}(G, V, \zeta)$. Then for $\gamma \in \Gamma(M_V^Z, \zeta_V)$, we have*

$$\tilde{I}_M^{\mathscr{E}}(\gamma, f) = {}^c\tilde{I}_M(\gamma, f) + \sum_{L \in \mathscr{L}^0(M)} {}^c\hat{\tilde{I}}_M^{L,\mathscr{E}}(\gamma, \theta_{\iota,L}^{\mathscr{E}}(f)) \tag{3.17}$$

$$ {}^c\tilde{I}^{\mathscr{E}}_M(\gamma, f) = \tilde{I}_M(\gamma, f) + \sum_{L \in \mathscr{L}^0(M)} \hat{\tilde{I}}^{L,\mathscr{E}}_M(\gamma, {}^c\theta^{\mathscr{E}}_{\iota,L}(f)) \tag{3.18} $$

if G is arbitrary, and for $\delta \in \Delta(M^Z_V, \zeta_V)$, we have

$$ \tilde{S}_M(\delta, f) = {}^c\tilde{I}_M(\delta, f) + \sum_{L \in \mathscr{L}^0(M)} {}^c\hat{\tilde{S}}_M(\delta, \theta_{\tau,L}(f)) \tag{3.19} $$

$$ {}^c\tilde{S}_M(\delta, f) = \tilde{I}_M(\delta, f) + \sum_{L \in \mathscr{L}^0(M)} \hat{\tilde{S}}_M(\delta, {}^c\theta_{\tau,L}(f)) \tag{3.20} $$

if G is quasisplit.

Proof. We assume inductively that each formula holds when G is replaced by a proper Levi subset. The formulas for G are then easily established from the definitions. We shall prove (3.17) here, and the rest will follow in the same way. Using the definitions (3.12) and (3.15) above, we have that

$$ \tilde{I}^{\mathscr{E}}_M(\gamma, f) - {}^c\tilde{I}^{\mathscr{E}}_M(\gamma, f) = \sum_{L \in \mathscr{L}^0(M)} {}^c\hat{\tilde{I}}^{L,\mathscr{E}}_M(\gamma, {}^c\iota^{\mathscr{E}}_L(f) - \sum_{L_1 \in \mathscr{L}^0(M)} \hat{\tilde{I}}^{L_1,\mathscr{E}}_M(\gamma, \iota^{\mathscr{E}}_{L_1}(f)). $$

By (3.5) the first sum is equal to

$$ \sum_{L \in \mathscr{L}^0(M)} {}^c\hat{\tilde{I}}^{L,\mathscr{E}}_M(\gamma, \theta^{\mathscr{E}}_{\iota,L}(f)) + \sum_{L_1 \in \mathscr{L}^0(M)} \sum_{L \in \mathscr{L}^{L_1}(M)} {}^c\hat{\tilde{I}}^{L,\mathscr{E}}_M(\gamma, \hat{\theta}^{L_1,\mathscr{E}}_{\iota,L}(\iota^{\mathscr{E}}_{L_1}(f))). $$

Applying (3.17) inductively to each $L_1 \in \mathscr{L}^0(M)$, we have that

$$ \sum_{L \in \mathscr{L}^{L_1}(M)} {}^c\hat{\tilde{I}}^{L,\mathscr{E}}_M(\gamma, \hat{\theta}^{L_1,\mathscr{E}}_{\iota,L}(\iota^{\mathscr{E}}_{L_1}(f))) = \hat{\tilde{I}}^{L_1,\mathscr{E}}_M(\gamma, \iota^{\mathscr{E}}_{L_1}(f)) $$

and the formula (3.17) then follows for G. □

Remark 3.7. By [Art88b, Corollary 5.3], we know that the distributions $I_M(\gamma, f)$ are supported on characters. Recall that a continuous linear map θ from $\mathscr{H}_{\text{ac}}(G, V, \zeta)$ to a topological vector space $\mathscr{V}$ is said to be supported on characters if it vanishes on the kernel of $\mathscr{I}(G, V, \zeta)$. That is, if $\theta(f) = 0$ for every function $f \in \mathscr{H}_{\text{ac}}(G, V, \zeta)$ such that $f_G = 0$. Using this and a slight variation of the proof of [Art88a, Theorem 6.1], it follows inductively that the new distributions we have introduced, and in particular those occurring in the preceding proposition are also supported on characters. Since we do not have need of this property explicitly, we neglect to provide the details here.

Let us also define inductively the modified invariant linear form

$$ \tilde{I}_M(\delta, f), \qquad f \in \mathscr{H}_{\text{ac}}(G, V, \zeta) $$

by setting

$$ \tilde{I}_M(\delta, f) = I_M(\delta, f) - \sum_{L \in \mathscr{L}^0(M)} \hat{\tilde{I}}^L_M(\delta, \iota_L(f)). $$

Then we have the following property as an immediate consequence of Local Theorem $1'$ of [Art02], which will also be needed for the global expansion.

Lemma 3.8. *Let V be a finite set of valuations such that G and ζ are unramified outside of V.*

(a) *if G is arbitrary,*

$$\tilde{I}^{\mathscr{E}}_M(\delta, f) = \tilde{I}_M(\delta, f), \qquad \pi \in \Gamma(M^Z_V, \zeta_V),\ f \in \mathscr{H}(G, V, \zeta)$$

(b)*Suppose that G is quasisplit, and that δ' belongs to $\Delta(\tilde{M}', \tilde{\zeta}')$ for some $M' \in \mathscr{E}_{\mathrm{ell}}(M)$. Then the linear form*

$$f \to \tilde{S}^G_M(M', \delta', X, f), \qquad f \in \mathscr{H}(G, V, \zeta)$$

vanishes unless $M' = M^*$, in which case it is stable.

4. Descent and splitting formulas

We now want to establish descent and splitting formulas for our modified distributions, which will reduce the study of the compound distributions on the geometric side to the local setting.

4.1. **Descent.** For the descent formula, we shall prove it for a finite set of places V, which includes the special case that V contains a single place v. We shall take f to be a fixed function in $\mathscr{C}(G^Z_V, \zeta_V)$. Let R be a Levi subgroup of M, so that $\mathfrak{a}_M$ is a subspace of $\mathfrak{a}_R$ and whose orthogonal complement we denote by $\mathfrak{a}^M_R$. If L belongs to $\mathscr{L}(R)$, we then have a map

$$\mathfrak{a}^M_R \oplus \mathfrak{a}^L_R \to \mathfrak{a}^G_R.$$

As in the special case of [Art88a, §7], we define the coefficient $d^G_R(M, L)$ by setting it to be zero if the map is not an isomorphism, and otherwise we define $d^G_R(M, L)$ to be the volume in $\mathfrak{a}^G_R$ of the image of a unit cube in $\mathfrak{a}^M_R \oplus \mathfrak{a}^L_R$.

Let R be a proper Levi subgroup of M. Any Levi subgroup $L \in \mathscr{L}(R)$ comes with a dual Levi subgroup $\hat{L}$ of $\hat{G}$ that contains $\hat{R}$, from which we form the coefficient

$$\mathrm{e}^G_R(M, L) = d^G_R(M, L)|Z(\hat{M})^\Gamma \cap Z(\hat{L})^\Gamma / Z(\hat{G})^\Gamma|^{-1}$$

for each $L \in \mathscr{L}(R)$.

Proposition 4.1. (a) *Suppose that G is not quasiplit over F, and $\gamma \in \Gamma(M^Z_V, \zeta_V)$. Then for any $f \in \mathscr{H}(G, V, \zeta)$, we have*

$$\tilde{I}^{\mathscr{E}}_M(\gamma, f) = \sum_{L \in \mathscr{L}(R)} d^G_R(M, L) \hat{\tilde{I}}^{L,\mathscr{E}}_R(\gamma, f_L),$$

(b) *Suppose that G is quasiplit over F, and $\delta \in \Delta(M^Z_V, \zeta_V)$. Then for any $f \in \mathscr{H}(G, V, \zeta)$, we have*

$$\tilde{S}_M(\delta, f) = \sum_{L \in \mathscr{L}(R)} \mathrm{e}^G_R(M, L) \hat{\tilde{S}}^L_R(\delta, f_L). \tag{4.1}$$

Proof. We begin with the first equation. By the inversion formula (3.14) it is equivalent to establishing the analogue

$$\tilde{I}^{\mathscr{E}}_M(\delta', f) = \sum_{L \in \mathscr{L}(R)} d^G_R(M, L) \hat{\tilde{I}}^{L,\mathscr{E}}_R(\delta', f_L). \tag{4.2}$$

Recall that the endoscopic distribution also satisfies a descent formula

$$I^{\mathscr{E}}_M(\delta', f) = \sum_{L \in \mathscr{L}(R)} d^G_R(M, L) \hat{I}^{L,\mathscr{E}}_R(\delta', f_L) \tag{4.3}$$

in [Art99, Theorem 7.1] for strongly G-regular elements and [Won19a, Proposition 3.3] for general δ' (see also [MW16b, VI.4] for an alternate formulation). Then it follows from the definition (3.12) that $\tilde{I}^{\mathscr{E}}_M(\delta', f)$ can be expressed as the difference of the right-hand side of (4.3) and

$$\sum_{L_1 \in \mathscr{L}^0(M)} \hat{\tilde{I}}^{L_1,\mathscr{E}}_M(\delta', \iota^{\mathscr{E}}_{L_1}(f)). \tag{4.4}$$

We can assume inductively that (4.2) holds for each of the distributions $\hat{\tilde{I}}^{L_1,\mathscr{E}}_M(\delta')$ for $L_1 \in \mathscr{L}^0(M)$. Then the latter sum can be written as

$$\sum_{L_1 \in \mathscr{L}^0(M)} \sum_{M_1 \in \mathscr{L}^{L_1}(R)} d^{L_1}_R(M, M_1) \hat{\tilde{I}}^{M_1,\mathscr{E}}_R(\delta', \iota^{\mathscr{E}}_{L_1}(f)_{M_1}). \tag{4.5}$$

Now $\iota^{\mathscr{E}}_{L_1}(f)_{M_1}$ is a function in $\mathscr{I}_{\mathrm{ac}}(M_1, V, \zeta)$. By specializing the descent formula for endoscopic weighted characters $I^{\mathscr{E}}_M(\pi, X, f)$ in [MW16b, X.4.4] for twisted endoscopy, it follows that

$$I^{\mathscr{E}}_{L_1}(\pi, X, f) = \sum_{L \in \mathscr{L}(M_1)} d^G_{M_1}(L_1, L) I^L_{M_1}(\pi, X, f_L)$$

and hence the mapping $\iota^{\mathscr{E}}_{L_1}$ satisfies the descent property

$$\iota^{\mathscr{E}}_{L_1}(f)_{M_1} = \sum_{L \in \mathscr{L}(M_1)} d^G_{M_1}(L_1, L) \iota^{L,\mathscr{E}}_{M_1}(f)_L. \tag{4.6}$$

Let us agree to set $d^{L_1}_R(M, M_1) = 0$ if L_1 does not contain both M and M_1. It follows from this that (4.5) is equal to

$$\sum_{L_1 \in \mathscr{L}^0(M)} \sum_{M_1 \in \mathscr{L}(R)} \sum_{L \in \mathscr{L}(M_1)} d^{L_1}_R(M, M_1) d^G_{M_1}(L_1, L) \hat{\tilde{I}}^{M_1,\mathscr{E}}_R(\delta', \iota^{L,\mathscr{E}}_{M_1}(f)_L). \tag{4.7}$$

We need only consider M_1 such that the coefficient $d^{L_1}_R(M, M_1) \neq 0$, so that $\mathfrak{a}^{L_1}_R = \mathfrak{a}^M_R \oplus \mathfrak{a}^{M_1}_R$. Also, if L is any element in $\mathscr{L}(M_1)$ such that $d^G_{M_1}(L_1, L) \neq 0$, then we have $\mathfrak{a}^G_{M_1} = \mathfrak{a}^{L_1}_{M_1} \oplus \mathfrak{a}^L_{M_1}$, and the Levi L in (4.3) and (4.7) can taken to be the same. The only part of the expression (4.7) depending on L_1 is

$$\sum_{L_1 \in \mathscr{L}^0(M)} d^{L_1}_R(M, M_1) d^G_{M_1}(L_1, L). \tag{4.8}$$

If $L \neq M_1$, we can replace the sum over $\mathscr{L}^0(M)$ with $\mathscr{L}(M)$ since the term corresponding to $L_1 = G$ vanishes. Then using [Art88a, (7.1)], or rather its extension to K-groups [Art99, Lemma 4.1], it follows that the sum is equal to $d^G_R(M, L)$. On the other hand, if $L = M_1$ then

$$d^G_{M_1}(L_1, L) = d^G_{M_1}(L_1, M_1) = 0$$

since $L_1 \neq G$, so in this case all the summands are zero. It follows then that (4.7) is equal to

$$\sum_{L \in \mathscr{L}(R)} \sum_{\substack{M_1 \in \mathscr{L}^L(M) \\ M_1 \neq L}} d^G_R(M, L) \hat{\tilde{I}}^{M_1,\mathscr{E}}_R(\delta', \iota^{L,\mathscr{E}}_{M_1}(f)_L).$$

We can now combine this with (4.3). From the inductive definition of $\tilde{I}_M^{L,\mathscr{E}}(\delta')$ we see that the difference between (4.3) and (4.4) is equal to

$$\sum_{L\in\mathscr{L}(R)} d_R^G(M,L)\hat{\tilde{I}}_R^{L,\mathscr{E}}(\delta',f_L)$$

as desired.

The second equation is established in a similar manner, so we can afford to be brief, indicating the points of departure from the latter proof. Once again the stable linear forms $S_M(\delta,f)$ satisfy the descent formula

$$S_M(\delta,f)=\sum_{L\in\mathscr{L}(R)} \mathrm{e}_R^G(M,L)\hat{S}_R^L(\delta,f_L), \tag{4.9}$$

and it follows from the definition (3.13) that $\tilde{S}_M(\delta,f)$ can be expressed as the difference of the right hand side of (4.9) and

$$\sum_{L_1\in\mathscr{L}^0(M)} \hat{\tilde{S}}_M^{L_1}(\delta,\tau_{L_1}(f)).$$

Inductively applying the formula (4.1) to the distributions $\hat{\tilde{S}}_M^{L_1}(\delta)$ for $L_1\in\mathscr{L}^0(M)$, the latter sum can be written as

$$\sum_{L_1\in\mathscr{L}^0(M)}\sum_{M_1\in\mathscr{L}^{L_1}(R)} \mathrm{e}_R^{L_1}(M,M_1)\hat{\tilde{S}}_R^{M_1}(\delta,\tau_{L_1}(f)_{M_1}). \tag{4.10}$$

Again using the descent formula for stable characters $S_M(\phi,X,f)$ in [MW16b, X.4.4],

$$S_{L_1}(\phi,X,f)=\sum_{L\in\mathscr{L}(M_1)} \mathrm{e}_{M_1}^G(L_1,L)S_{M_1}^L(\phi,X,f_L)$$

it follows that the mapping τ_{L_1} satisfies

$$\tau_{L_1}(f)_{M_1}=\sum_{L\in\mathscr{L}(M_1)} \mathrm{e}_{M_1}^G(L_1,L)\tau_{M_1}^L(f)_L. \tag{4.11}$$

It follows from this that (4.10) is equal to

$$\sum_{L_1\in\mathscr{L}^0(M)}\sum_{M_1\in\mathscr{L}(R)}\sum_{L\in\mathscr{L}(M_1)} \mathrm{e}_R^{L_1}(M,M_1)\mathrm{e}_{M_1}^G(L_1,L)\hat{\tilde{S}}_R^{M_1}(\delta',\tau_{M_1}^L(f)_L). \tag{4.12}$$

We focus our attention on the product of the two coefficients, appealing to the definition to write

$$d_R^{L_1}(M,M_1)d_{M_1}^G(L_1,L)|Z(\hat{M})^\Gamma\cap Z(\hat{M}_1)^\Gamma/Z(\hat{L}_1)^\Gamma|^{-1}|Z(\hat{L}_1)^\Gamma\cap Z(\hat{L})^\Gamma/Z(\hat{G})^\Gamma|^{-1}.$$

Since we are assuming $d_{M_1}^G(L_1,L)$ is nonzero, it follows that the connected component $(Z(\hat{M}_1)^\Gamma)^0$ is equal to the product of the subgroups $(Z(\hat{L}_1)^\Gamma)^0$ and $(Z(\hat{L})^\Gamma)^0$. Since $Z(\hat{M}_1)^\Gamma$ is equal to the product of $(Z(\hat{M}_1)^\Gamma)^0$ and $Z(\hat{G})^\Gamma$, by [Art99, Lemma 1.1], we see then that

$$Z(\hat{M}_1)^\Gamma=Z(\hat{L}_1)^\Gamma Z(\hat{L})^\Gamma.$$

It follows then that the product of

$$\begin{aligned}|Z(\hat{L}_1)^\Gamma\cap Z(\hat{L})^\Gamma/Z(\hat{G})^\Gamma|^{-1}&=|Z(\hat{L}_1)^\Gamma/Z(\hat{L})^\Gamma\cap Z(\hat{L}_1)^\Gamma||Z(\hat{L}_1)^\Gamma/Z(\hat{G})^\Gamma|^{-1}\\&=|Z(\hat{L}_1)^\Gamma Z(\hat{L})^\Gamma/Z(\hat{L})^\Gamma||Z(\hat{L}_1)^\Gamma/Z(\hat{G})^\Gamma|^{-1},\end{aligned}$$

with the quantity $|Z(\hat{M})^\Gamma \cap Z(\hat{L}_1)^\Gamma Z(\hat{L})^\Gamma / Z(\hat{L}_1)^\Gamma|^{-1}$ is equal to

$$|Z(\hat{M})^\Gamma \cap Z(\hat{L})^\Gamma / Z(\hat{G})^\Gamma|^{-1}.$$

In particular, it is independent of L_1. Using the same identity for (4.8) to sum the product of $d_R^{L_1}(M, M_1)$ and $d_{M_1}^G(L_1, L)$, we have that

$$\sum_{L_1 \in \mathscr{L}^0(M)} \mathrm{e}_R^{L_1}(M, M_1)\mathrm{e}_{M_1}^G(L_1, L) = \mathrm{e}_R^G(M, L).$$

And again taking into account the vanishing conditions for the coefficients, it follows that (4.12) is equal to

$$\sum_{L_1 \in \mathscr{L}(R)} \sum_{\substack{M_1 \in \mathscr{L}^L(M) \\ M_1 \neq L}} \mathrm{e}_R^G(M, L)\hat{\tilde{S}}_R^{M_1}(\delta', \tau_{M_1}^L(f)_L). \tag{4.13}$$

Finally, using the inductive definition of $\tilde{S}_M^L(\delta)$ we see that the difference between (4.9) and (4.13) is equal to

$$\sum_{L \in \mathscr{L}(R)} \mathrm{e}_R^G(M, L)\hat{\tilde{S}}_R^L(\delta, f_L)$$

as desired. □

4.2. **Splitting.** For the splitting formula, we shall assume that V is a disjoint union of nonempty sets V_1 and V_2, and that the image of F_{V_i} in $\mathbf{R}$ under the absolute value is closed for each $i = 1, 2$. We shall also fix a function

$$f_V = f_{V_1} \times f_{V_2}, \quad f_{V_i} \in \mathscr{C}(G_{V_i}, \zeta_{V_i})$$

The splitting formulas will be expressed in terms of pairs of Levi subgroups $L_1, L_2 \in \mathscr{L}(M)$. For any such pair, we define the coefficient

$$\mathrm{e}_M^G(L_1, L_2) = d_M^G(L_1, L_2)|Z(\hat{L}_1)^\Gamma \cap Z(\hat{L}_2)^\Gamma / Z(\hat{G})^\Gamma|^{-1}.$$

Note that if $d_M^G(L_1, L_2)$ is nonzero, then $\mathfrak{a}_{L_1}^* \cap \mathfrak{a}_{L_2}^* = \mathfrak{a}_G^*$ and the identity component of $Z(\hat{L}_1)^\Gamma \cap Z(\hat{L}_2)^\Gamma$ is the same as that of $Z(\hat{G})^\Gamma$. Therefore $\mathrm{e}_M^G(L_1, L_2)$ is also nonzero. We shall generally write L_i for the image of L_{i,V_i}.

Proposition 4.2. (a) *Suppose that G is arbitrary, and $\gamma_V = (\gamma_{V_1}, \gamma_{V_2}) \in \Gamma(G_V^Z, \zeta_V)$. Then for any $f \in \mathscr{H}(G, V, \zeta)$, we have*

$$\tilde{I}_M^{\mathscr{E}}(\gamma_V, f_V) = \sum_{L_1, L_2 \in \mathscr{L}(M)} d_M^G(L_1, L_2)\hat{\tilde{I}}_M^{L_1,\mathscr{E}}(\gamma_{V_1}, f_{V_1,L_1})\hat{\tilde{I}}_M^{L_2,\mathscr{E}}(\gamma_{V_2}, f_{V_2,L_2}).$$

(b) *Suppose that G is quasisplit over F, and $\delta_V = (\delta_{V_1}, \delta_{V_2}) \in \Delta(G_V^Z, \zeta_V)$. Then for any $f \in \mathscr{H}(G, V, \zeta)$, we have*

$$\tilde{S}_M(\delta_V, f_V) = \sum_{L_1, L_2 \in \mathscr{L}(M)} \mathrm{e}_M^G(L_1, L_2)\hat{\tilde{S}}_M^{L_1}(\delta_{V_1}, f_{V_1,L_1})\hat{\tilde{S}}_M^{L_2}(\delta_{V_2}, f_{V_2,L_2}).$$

Proof. The required formula in (a) has the analogue

$$\tilde{I}_M^{\mathscr{E}}(\delta'_V, f_V) = \sum_{L_1, L_2 \in \mathscr{L}(M)} d_M^G(L_1, L_2)\hat{\tilde{I}}_M^{L_1,\mathscr{E}}(\delta'_{V_1}, f_{V_1,L_1})\hat{\tilde{I}}_M^{L_2,\mathscr{E}}(\delta'_{V_2}, f_{V_2,L_2}).$$

for $\delta'_V \in \Delta^{\mathscr{E}}((\tilde{M}')^{\tilde{Z}'}_V, \tilde{\zeta}'_V)$. According to (3.14), the two formulas are equivalent, so it will be sufficient to establish the latter. The original endoscopic distribution satisfies the splitting formula

$$I^{\mathscr{E}}_M(\delta'_V, f_V) = \sum_{L_1, L_2 \in \mathscr{L}(M)} d^G_M(L_1, L_2) \hat{I}^{L_1, \mathscr{E}}_M(\delta'_{V_1}, f_{V_1, L_1}) \hat{I}^{L_2, \mathscr{E}}_M(\delta'_{V_2}, f_{V_2, L_2}). \tag{4.14}$$

as in [Won19a, Proposition 3.4] for general δ' and [Art99, Theorem 6.1] for strongly G-regular elements (see also [MW16b, VII.2] for an alternate formulation). Then it follows from the definition (3.12) that $\tilde{I}^{\mathscr{E}}_M(\delta'_V, f)$ can be expressed as the difference of the right-hand side of (4.14) and

$$\sum_{L \in \mathscr{L}^0(M)} \hat{\tilde{I}}^{L, \mathscr{E}}_M(\delta', \iota^{\mathscr{E}}_L(f)). \tag{4.15}$$

We can assume inductively that (4.14) holds for each of the distributions $\hat{\tilde{I}}^{L, \mathscr{E}}_M(\delta')$ for $L \in \mathscr{L}^0(M)$. Then the latter sum can be written as

$$\sum_{L \in \mathscr{L}^0(M)} \sum_{M_1, M_2 \in \mathscr{L}^L(M)} d^L_M(M_1, M_2) \hat{\tilde{I}}^{M_1, \mathscr{E}}_M(\delta'_{V_1}, \iota^{\mathscr{E}}_L(f_{V_1})_{M_1}) \hat{\tilde{I}}^{M_2, \mathscr{E}}_M(\delta'_{V_2}, \iota^{\mathscr{E}}_L(f_{V_2})_{M_2}). \tag{4.16}$$

We shall apply the descent formula (4.6) to each $\iota^{\mathscr{E}}_L(f_{V_i})_{M_i}$, that is,

$$\iota^{\mathscr{E}}_L(f_{V_i})_{M_i} = \sum_{L_i \in \mathscr{L}(M_i)} d^G_{M_i}(L, L_i) \iota^{L_i, \mathscr{E}}_{M_i}(f_{V_i})_{L_i}.$$

for $i = 1.2$. It follows then that (4.16) can be written as the sum over $L \in \mathscr{L}^0(M)$, $M_1, M_2 \in \mathscr{L}(M)$, $L_1 \in \mathscr{L}(M_1)$, and $L_2 \in \mathscr{L}(M_2)$ of

$$d^L_M(M_1, M_2) d^G_{M_1}(L, L_1) d^G_{M_2}(L, L_2) \hat{\tilde{I}}^{M_1, \mathscr{E}}_M(\delta'_{V_1}, \iota^{L_1, \mathscr{E}}_{M_1}(f_{V_1})_{L_1}) \hat{\tilde{I}}^{M_2, \mathscr{E}}_M(\delta'_{V_2}, \iota^{L_2, \mathscr{E}}_{M_2}(f_{V_2})_{L_2}).$$

The only part that depends on L is again the sum

$$\sum_{L \in \mathscr{L}^0(M)} d^L_M(M_1, M_2) d^G_{M_1}(L, L_1) d^G_{M_2}(L, L_2).$$

We may assume that $d^L_M(M_1, M_2)$ and $d^G_{M_1}(L, L_1)$ are nonzero. This implies that $\mathfrak{a}^L_M = \mathfrak{a}^{M_1}_M \oplus \mathfrak{a}^{M_2}_M$ and $\mathfrak{a}^G_{M_1} = \mathfrak{a}^L_{M_1} \oplus \mathfrak{a}^{L_1}_{M_1}$, and it follows from this that $\mathfrak{a}^G_M$ is equal to

$$\mathfrak{a}^{M_1}_M \oplus \mathfrak{a}^G_{M_1} = \mathfrak{a}^{M_1}_M \oplus \mathfrak{a}^L_{M_1} \oplus \mathfrak{a}^{L_1}_{M_1} = \mathfrak{a}^{M_1}_M \oplus \mathfrak{a}^{M_2}_M \oplus \mathfrak{a}^{L_1}_{M_1} = \mathfrak{a}^{L_1}_M \oplus \mathfrak{a}^{M_2}_M,$$

hence $d^G_M(L_1, M_2) \neq 0$. We are also assuming that $d^G_{M_2}(L, L_2)$ is nonzero, and therefore $\mathfrak{a}^G_M = \mathfrak{a}^{M_1}_M \oplus \mathfrak{a}^{L_2}_M$, then by the same argument we have $d^G_M(M_1, L_2) \neq 0$. If $L_i \neq M_i$ and $L = G$ then $d^G_{M_i}(L, L_i) = 0$ for each $i = 1, 2$, so in this case we may replace the sum over $\mathscr{L}^0(M)$ with $\mathscr{L}(M)$. It follows then from [MW16b, VI.4.2] that the sum is equal to $d^G_M(L_1, L_2)$.

On the other hand, the summands corresponding to $L_i = M_i$ and $L \neq G$ all vanish, so we may write

$$\sum_{L_1, L_2 \in \mathscr{L}(M)} \sum_{\substack{M_i \in \mathscr{L}^{L_i}(M) \\ M_i \neq L_i, i=1,2}} d^G_M(L_1, L_2) \hat{\tilde{I}}^{M_1, \mathscr{E}}_M(\delta'_{V_1}, \iota^{L_1, \mathscr{E}}_{M_1}(f_{V_1})_{L_1}) \hat{\tilde{I}}^{M_2, \mathscr{E}}_M(\delta'_{V_2}, \iota^{L_2, \mathscr{E}}_{M_2}(f_{V_2})_{L_2})$$

and then applying inductive definition to the form

$$\tilde{I}^{L_1 \times L_2, \mathscr{E}}_{M \times M}(\delta'_V, f_{V, L_1 \times L_2}),$$

which applies in particular to products of groups in a manner similar to [Art88a, §9], we arrive at

$$\sum_{L_1,L_2\in\mathscr{L}(M)} d_M^G(L_1,L_2)\hat{\tilde{I}}_M^{L_1,\mathscr{E}}(\delta'_{V_1},f_{V_1,L_1})\hat{\tilde{I}}_M^{L_2,\mathscr{E}}(\delta'_{V_2},f_{V_2,L_2})$$

as required.

The second required formula proceeds in a similar manner. The stable linear form also satisfies the splitting formula

$$S_M^G(\delta_V,f_V)=\sum_{L_1,L_2\in\mathscr{L}(M)} \mathrm{e}_M^G(L_1,L_2)\hat{S}_M^{L_1}(\delta_{V_1},f_{V_1,L_1})\hat{S}_M^{L_2}(\delta_{V_2},f_{V_2,L_2}) \tag{4.17}$$

as in [Won19a, Proposition 3.4] for general δ' and [Art99, Theorem 6.1] for strongly G-regular elements (see also [MW16b, VII.2] for an alternate formulation). Then it follows from the definition (3.13) that $\tilde{I}_M^{\mathscr{E}}(\delta'_V,f)$ can be expressed as the difference of the right-hand side of (4.17) and

$$\sum_{L\in\mathscr{L}^0(M)} \hat{\tilde{S}}_M^L(\delta,\tau_L(f)). \tag{4.18}$$

We can assume inductively that (4.17) holds for each of the distributions $\hat{\tilde{S}}_M^L(\delta)$ for $L\in\mathscr{L}^0(M)$. Then the latter sum can be written as

$$\sum_{L\in\mathscr{L}^0(M)}\sum_{M_1,M_2\in\mathscr{L}^L(M)} \mathrm{e}_M^L(M_1,M_2)\hat{\tilde{S}}_M^{M_1}(\delta_{V_1},\tau_L(f_{V_1})_{M_1})\hat{\tilde{S}}_M^{M_2}(\delta_{V_2},\tau_L(f_{V_2})_{M_2}). \tag{4.19}$$

We shall apply the descent formula (4.11) to each $\tau_L^{\mathscr{E}}(f_{V_i})_{M_i}$, that is,

$$\tau_L(f_{V_i})_{M_i}=\sum_{L_i\in\mathscr{L}(M_i)} \mathrm{e}_{M_i}^G(L,L_i)\tau_{M_i}^{L_i}(f_{V_i})_{L_i}.$$

for $i=1.2$. It follows then that (4.19) can be written as the sum over $L\in\mathscr{L}^0(M)$, $M_1,M_2\in\mathscr{L}(M)$, $L_1\in\mathscr{L}(M_1)$, and $L_2\in\mathscr{L}(M_2)$ of

$$\mathrm{e}_M^L(M_1,M_2)\mathrm{e}_{M_1}^G(L,L_1)\mathrm{e}_{M_2}^G(L,L_2)\hat{\tilde{S}}_M^{M_1}(\delta_{V_1},\tau_{M_1}^{L_1}(f_{V_1})_{L_1})\hat{\tilde{S}}_M^{M_2}(\delta_{V_2},\tau_{M_2}^{L_2}(f_{V_2})_{L_2}).$$

The only part that depends on L is

$$\sum_{L\in\mathscr{L}^0(M)} \mathrm{e}_M^L(M_1,M_2)\mathrm{e}_{M_1}^G(L,L_1)\mathrm{e}_{M_2}^G(L,L_2). \tag{4.20}$$

It involves the product of the cardinalities

$$|Z(\hat{M}_1)^\Gamma\cap Z(\hat{M}_2)^\Gamma/Z(\hat{L})^\Gamma||Z(\hat{L})^\Gamma\cap Z(\hat{L}_1)^\Gamma/Z(\hat{G})^\Gamma||Z(\hat{L})^\Gamma\cap Z(\hat{L}_2)^\Gamma/Z(\hat{G})^\Gamma|.$$

It follows from [MW16b, VI.4.2] that this is equal to

$$|Z(\hat{L}_1)^\Gamma\cap Z(\hat{L}_2)^\Gamma/Z(\hat{G})^\Gamma|,$$

and hence (4.20) equals $\mathrm{e}_M^G(L_1,L_2)$, so that we may write (4.19) as

$$\sum_{L_1,L_2\in\mathscr{L}(M)}\sum_{\substack{M_i\in\mathscr{L}^{L_i}(M)\\ M_i\neq L_i, i=1,2}} \mathrm{e}_M^G(L_1,L_2)\hat{\tilde{S}}_M^{M_1}(\delta_{V_1},\tau_{M_1}^{L_1}(f_{V_1})_{L_1})\hat{\tilde{S}}_M^{M_2}(\delta_{V_2},\tau_{M_2}^{L_2}(f_{V_2})_{L_2}).$$

Then applying the inductive definition as before, we obtain

$$\sum_{L_1, L_2 \in \mathscr{L}(M)} e_M^G(L_1, L_2) \hat{\hat{S}}_M^{L_1}(\delta_{V_1}, f_{V_1, L_1}) \hat{\hat{S}}_M^{L_2}(\delta_{V_2}, f_{V_2, L_2}).$$

as required. □

5. The unitary part of the trace formula

We are now ready to apply the local study to the global endoscopic and stable trace formulas. We are assuming that G is a K-group over a global field F. Given $f \in \mathscr{H}(G, V, \zeta)$, define inductively

$$I^{\mathscr{E}}(f) = \sum_{G' \in \mathscr{E}^0(G,V)} \iota(G, G') \hat{S}'(f') + \varepsilon(G) S^G(f)$$

for linear forms $\hat{S}' = \hat{S}^{\tilde{G}'}$ on the spaces $\mathscr{S}(\tilde{G}', V, \tilde{\zeta}')$ defined inductively with the supplementary requirement that

$$I^{\mathscr{E}}(f) = I(f)$$

in the case that G is quasisplit. The distribution $S^G(f)$ is to be regarded as the stable part of $I(f)$, while the other terms can be considered as error terms. If G is quasisplit, in which case we take $G = G^*$, we write

$$S(f) = S^G(f) = I(f) - \sum_{G' \in \mathscr{E}^0(G,V)} \iota(G, G') \hat{S}'(f'),$$

which represents the stable trace formula.

We then define the modified linear forms on $\mathscr{H}(G, V, \zeta)$,

$$\tilde{I}^{\mathscr{E}}(f) = I^{\mathscr{E}}(f) - \sum_{L \in \mathscr{L}^0} |W_0^L||W_0^G|^{-1} \hat{\tilde{I}}^{L,\mathscr{E}}(\tau_L(f))$$

if G is arbitrary, and

$$\tilde{S}(f) = S(f) - \sum_{L \in \mathscr{L}^0} |W_0^L||W_0^G|^{-1} \hat{\tilde{S}}^L(\tau_L(f))$$

if G is quasisplit over F. Our present goal is to examine the spectral and geometric expansions of these new distributions.

5.1. **The spectral side.** To describe the global expansions further, we first have to recall the global coefficients. We shall describe them in some detail here, as we will also require them again later. We first consider families

$$c^V = \{c_v : v \notin V\}$$

of semisimple conjugacy classes c_v in the local L-group ${}^LG_v = G^\vee \rtimes W_{F_v}$, whose image in the local Weil group W_{F_v} is a Frobenius element. We let $C(G^V, \zeta^V)$ be the set of families c^V satisfying the requirement that each c_v is compatible with ζ_v in the sense that the image of c_v under the projection ${}^LG_v \to {}^LZ_v$ gives the unramified Langlands parameter of ζ_v. Moreover, we require that for any $G^\vee$ invariant polynomial A on ${}^LG$, we have that c satisfies the estimate

$$|A(c_v)| \leq q_v^{r_A}$$

for some $r_A > 0$ and for any $v \notin V$. By the Satake transform, any element $c \in \mathscr{C}(G^V, \zeta^V)$ can be identified with a K^V-unramified representation $\pi^V(c)$ in $\Pi(G^V, \zeta^V)$. Given $c \in \mathscr{C}(G^V, \zeta^V)$ and $\pi \in \Pi(G_V, \zeta_V)$, we write

$$\pi \times c = \pi \otimes \pi^V(c)$$

for the associated representation in $\Pi(G(\mathbf{A}), \zeta)$. If π belongs to $\Pi(G_V^Z, \zeta)$, then $\pi \times c$ belongs to the quotient $\Pi(G(\mathbf{A})^1, \zeta)$. We also define $C^V_{\mathrm{disc}}(G, \zeta)$ for the set of $c \in C(G^V, \zeta^V)$ such that $\pi \times c$ belongs to $\Pi_{\mathrm{disc}}(G, \zeta)$ for some $\pi \in \Pi_{\mathrm{disc}}(G, V, \zeta)$.

For $c \in C(G^V, \zeta^V)$ and a finite-dimensional representation ρ of ${}^LG$, we can form the Euler product

$$L(s, c, \rho) = \prod_{v \notin V} \det(1 - \rho(c_v) q_v^{-s})^{-1},$$

where $s \in \mathscr{C}$ and q_v is the cardinality of the residue field of F_v. There is a natural action of $\lambda \in \mathfrak{a}^*_{G,Z}$ on $c \to c_\lambda$, which we write as $c \mapsto c_\lambda$. Let $M \in \mathscr{L}$ and $M^\vee \subset G^\vee$ be a dual Levi subgroup. Then there is a bijection $P \mapsto P^\vee$ from $\mathscr{P}(M)$ to $\mathscr{P}(M^\vee)$, the set of Γ-stable parabolic subgroups of $G^\vee$ with Levi component $M^\vee$. Given $P, Q \in \mathscr{P}(M)$, let $\rho_{Q|P}$ denote the adjoint representation of ${}^LM$ the Lie algebra of the intersection of the unipotent radicals of $P^\vee$ and opposite $\bar{Q}^\vee$. We can then define the normalizing factors

$$r_{Q|P}(c_\lambda) = L(0, c_\lambda, \rho_{Q|P}) L(1, c_\lambda, \rho_{Q|P})^{-1}$$

and form the (G, M)-family

$$r_Q(\Lambda, c_\lambda) = r_{Q|\bar{Q}}(c_\lambda)^{-1} r_{Q|\bar{Q}}(c_{\lambda + \Lambda/2})$$

for $Q \in \mathscr{P}(M)$ and $\Lambda \in i\mathfrak{a}^*_M$.

To define the global spectral coefficient $a^G(\pi)$ on $\Pi(G_V^Z, \zeta_V)$, we set

$$a^G(\pi) = \sum_{L \in \mathscr{L}} |W_0^L||W_0^G|^{-1} \sum_{c \in C^V_{\mathrm{disc}}(M,\zeta)} a^M_{\mathrm{disc}}(\pi_M \times c) r^G_M(c)$$

where for each c, the product $\pi_M \times c$ represents a finite sum of representations $\dot{\pi}$ in $\Pi_{\mathrm{unit}}(M(\mathbf{A}), \zeta)$ and $a^M_{\mathrm{disc}}(\pi_M \times c)$ is the corresponding sum of spectral coefficients $a^M_{\mathrm{disc}}(\dot{\pi})$ defined in [Art88b, §4]. It follows from the definition that $a^G(\pi)$ is supported on the subset $\Pi(G, V, \zeta)$ of $\Pi(G_V^Z, \zeta)$. We fix a Borel measure $d\pi$ on $\Pi(G, V, \zeta)$ by requiring that

$$\int_{\Pi(G,V,\zeta)} h(\pi) d\pi = \sum_{M \in \mathscr{L}} |W_0^L||W_0^G|^{-1} \sum_{\rho \in \Pi_{\mathrm{disc}}(M, V\zeta)} \int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} h(\rho^G_\lambda) d\lambda$$

for any $h \in C_c(\Pi(G, V, \zeta)$. If π lies in $\Pi^{\mathscr{E}}(G_V^Z, \zeta_V)$, we set

$$a^{G,\mathscr{E}}(\pi) = \sum_{G'} \sum_{\phi'} \iota(G, G') b^{\tilde{G}'}(\phi') \Delta_G(\phi', \pi) + \varepsilon(G) \sum_{\phi} b^G(\phi) \Delta_G(\phi, \pi),$$

with G', ϕ', and ϕ summed over $\mathscr{E}^0_{\mathrm{ell}}(G, V)$, $\Phi((\tilde{G}'_V, \tilde{\zeta}'_V)$ and $\Phi^{\mathscr{E}}(G_V^Z, \zeta_V)$ respectively, and the coefficients $b^{\tilde{G}'}(\phi)$ defined inductively with the requirement that

$$a^{G,\mathscr{E}}(\pi) = a^G(\pi)$$

in the case that G is quasisplit. The coefficients $a^{G,\mathscr{E}}(\pi)$ and $b^G(\phi)$ are in fact supported on the discrete subsets $\Pi^{\mathscr{E}}(G, V, \zeta)$ and $\Phi(G, V, \zeta)$ respectively. These spaces come with corresponding Borel measures $d\pi$ and $d\phi$, given by

$$\int_{\Pi^{\mathscr{E}}(G,V,\zeta)} h(\pi)d\pi = \sum_{M\in\mathscr{L}} |W_0^L||W_0^G|^{-1} \sum_{\rho\in\Pi^{\mathscr{E}}_{\mathrm{disc}}(M,V\zeta)} \int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} h(\rho^G_\lambda)d\lambda$$

for any $h \in C_c(\Pi^{\mathscr{E}}(G,V,\zeta)$, and

$$\int_{\Phi(G,V,\zeta)} h(\phi)d\phi = \sum_{M\in\mathscr{L}} |W_0^L||W_0^G|^{-1} \sum_{\chi\in\Phi_{\mathrm{disc}}(M,V\zeta)} \int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} h(\chi^G_\lambda)d\lambda$$

for any $h \in C_c(\Phi(G,V,\zeta)$ respectively.

We then have for any f in $\mathscr{H}(G,V,\zeta)$ the spectral expansions of the endoscopic and stable linear forms

$$I^{\mathscr{E}}(f) = \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(M,V,\zeta)} a^{M,\mathscr{E}}(\pi) I^{\mathscr{E}}_M(\pi,f)d\pi \tag{5.1}$$

and

$$S(f) = \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Phi(M,V,\zeta)} b^M(\phi)S_M(\phi,f)d\phi. \tag{5.2}$$

Due to problems of absolute convergence, in [Art02] these forms are further decomposed into a sum of linear forms

$$I^{\mathscr{E}}(f) = \sum_{t\geq 0} I^{\mathscr{E}}_t(f)$$

and similarly $S_t(f)$, where $t \geq 0$ is the norm of the imaginary part of the archimedean infinitesimal parameter associated to a representation. This is no longer strictly necessary due to the results of [FLM11].

Theorem 5.1. (a) *If G is arbitrary, then $\tilde{I}^{\mathscr{E}}(f)$ has a spectral expansion*

$$\tilde{I}^{\mathscr{E}}(f) = \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi^{\mathscr{E}}(M,V,\zeta)} a^{M,\mathscr{E}}(\pi) \tilde{I}^{\mathscr{E}}_M(\pi,f)d\pi. \tag{5.3}$$

(b) *If G is quasisplit, then $\tilde{S}(f)$ has a spectral expansion*

$$\tilde{S}(f) = \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Phi(M,V,\zeta)} b^M(\phi)\tilde{S}_M(\phi,f)d\phi \tag{5.4}$$

for any $f \in \mathscr{H}(G,V,\zeta)$.

Proof. The proofs of the two statements are the same, so we will be content with proving (a). Using the spectral expansion (5.1) of $I^{\mathscr{E}}(f)$, it follows from the definition that $\tilde{I}^{\mathscr{E}}(f)$ is equal to the difference of

$$I^{\mathscr{E}}(f) = \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(M,V,\zeta)} a^{M,\mathscr{E}}(\pi) I^{\mathscr{E}}_M(\pi,f)d\pi$$

and

$$\sum_{L\in\mathscr{L}} |W_0^L||W_0^G|^{-1} \hat{\tilde{I}}^{L,\mathscr{E}}(\iota^{\mathscr{E}}_L(f)).$$

Now assume inductively that (5.3) holds for $L \in \mathscr{L}^0(M)$, so that we have

$$\tilde{I}^{L,\mathscr{E}}(g) = \sum_{M\in\mathscr{L}^L} |W_0^M||W_0^L|^{-1} \int_{\Pi^{\mathscr{E}}(M,V,\zeta)} a^{M,\mathscr{E}}(\pi) \tilde{I}^{L,\mathscr{E}}_M(\pi,g)d\pi$$

for any $g \in \mathscr{H}(L, V, \zeta_L)$ and ζ_L the restriction of ζ to L. The sums in the expression

$$\sum_{L \in \mathscr{L}} |W_0^L||W_0^G|^{-1} \sum_{M \in \mathscr{L}^L} |W_0^M||W_0^L|^{-1} \int_{\Pi^{\mathscr{E}}(M,V,\zeta)} a^{M,\mathscr{E}}(\pi) \tilde{I}_M^{L,\mathscr{E}}(\pi, \iota_L^{\mathscr{E}}(f)) d\pi$$

can be rewritten as

$$\sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{L \in \mathscr{L}^0(M)} \int_{\Pi^{\mathscr{E}}(M,V,\zeta)} a^{M,\mathscr{E}}(\pi) \tilde{I}_M^{L,\mathscr{E}}(\pi, \iota_L^{\mathscr{E}}(f)) d\pi.$$

It follows that $\tilde{I}^{\mathscr{E}}(f)$ can be expressed as

$$\sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi^{\mathscr{E}}(M,V,\zeta)} a^{M,\mathscr{E}}(\pi) \left(I^{\mathscr{E}}(\pi, f) - \sum_{L \in \mathscr{L}^0(M)} \hat{\tilde{I}}_M^{L,\mathscr{E}}(\pi, \iota_L^{\mathscr{E}}(f)) \right) d\pi,$$

which by (3.1) is equal to

$$\sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi^{\mathscr{E}}(M,V,\zeta)} a^{M,\mathscr{E}}(\pi) \tilde{I}_M(\pi, f) d\pi,$$

which proves (5.3).

The proof of (b) follows from the spectral expansion (5.2) of $S(f)$, assuming (5.4) inductively and the definition (3.2). □

5.2. **Geometric side.** We now turn to the geometric expansion. Recall that two elements $\dot\gamma$ and $\dot\gamma_1$ in $G(F_S)$ with standard Jordan decompositions $\dot\gamma = c\dot\alpha$ and $\dot\gamma_1 = c_1\dot\alpha_1$ are said to be (G, S)-equivalent if there is an element $\dot\delta \in G(F)$ such that $\dot\delta^{-1} c_1 \delta = c$ and $\dot\delta^{-1}\alpha_1\delta$ is conjugate to α in $G_c(F_S)$. Beginning with the global geometric coefficient $a^G(S, \dot\gamma)$ for the (G, S)-equivalence class $\dot\gamma \in (G(F))_{G,S}$ in [Art86, (8.1)], we define the geometric coefficient

$$a^G_{\mathrm{ell}}(\dot\gamma) = \sum_{\{\dot\gamma\}} |Z(F, \dot\gamma)|^{-1} a^G(S, \dot\gamma)(\dot\gamma/\dot\gamma_S)$$

for any admissible element $\dot\gamma \in \Gamma(G_S, \zeta_S)$ in the sense of [Art02, §1]. Here $\{\dot\gamma\}$ runs over $Z(F) \cap Z_S Z(\mathfrak{o}^S)$-orbits in $(G(F))_{G,S}$ that map to $\dot\gamma_S$, and such that the $G(\mathbf{A}^S)$-conjugacy class of $\dot\gamma$ in $G(\mathbf{A}^S)$ meets K^S,

$$Z(F, \dot\gamma) = \{z \in Z(F) : z\dot\gamma = \dot\gamma\},$$

and $(\dot\gamma/\dot\gamma_S)$ is the ratio of the invariant measure on $\dot\gamma$ and the signed measure on $\dot\gamma$ that comes with $\dot\gamma_S$. The coefficient vanishes on the complement of the subset of orbital integrals $\Gamma_{\mathrm{orb}}(G_S^Z, \zeta_S)$ in $\Gamma(G_S, \zeta_S)$.

Now let $\mathcal{K}(\bar G_S^V)$ be the set of conjugacy classes in $\bar G_S^V = G_S^V / Z_S^V$ that are bounded. Any element $k \in \mathcal{K}(\bar G_S^V)$ induces a distribution $\gamma_S^V(k)$ in the subset $\Gamma_{\mathrm{orb}}(G_S^V, \zeta_S^V)$. Given $k \in \mathcal{K}(\bar G_S^V)$ and $\gamma \in \Gamma(G_V^Z, \zeta_V)$, we write

$$\gamma \times k = \gamma \times \gamma_S^V(k)$$

for the associated element in $\Gamma(G_S^Z, \zeta_S)$. We then define the unramified weighted orbital integrals

$$r_M^G(k) = J_M(\gamma_S^V(k), u_S^V)$$

as functions on $\mathcal{K}(\bar{G}_S^V)$. To define the global geometric coefficient $a^G(\gamma)$ on $\Gamma(G_V^Z, \zeta_V)$, we set

$$a^G(\gamma) = \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{k \in \mathcal{K}_{\mathrm{ell}}^V(\bar{M},S)} a_{\mathrm{ell}}^M(\gamma_M \times k) r_M^G(k) \tag{5.5}$$

where S is is any finite set of valuations containing V and such that the set $\gamma \times K^V$ is S-admissible. Here $\gamma \times k$ is viewed as a finite linear combination of elements $\dot{\gamma} \in \Gamma(M_S^Z, \zeta_S)$ and $a_{\mathrm{ell}}^M(\gamma_M \times k)$ is the corresponding finite linear combination of values $a_{\mathrm{ell}}^M(\dot{\gamma}_S)$.

If γ lies in $\Gamma(G_V^Z, \zeta_V)$, we set

$$a^{G,\mathscr{E}}(\gamma) = \sum_{G'} \sum_{\delta'} \iota(G,G') b^{\tilde{G}'}(\delta') \Delta_G(\delta', \gamma) + \varepsilon(G) \sum_{\delta} b^G(\delta) \Delta_G(\delta, \gamma)$$

with G', δ', and δ summed over $\mathscr{E}_{\mathrm{ell}}^0(G,V), \Delta((\tilde{G}'_V, \tilde{\zeta}'_V)$ and $\Delta^{\mathscr{E}}(G_V^Z, \zeta_V)$ respectively, and the coefficients $b^{\tilde{G}'}(\delta)$ defined inductively with the requirement that

$$a^{G,\mathscr{E}}(\gamma) = a^G(\gamma)$$

in the case that G is quasisplit. The coefficients $a^{G,\mathscr{E}}(\gamma)$ and $b^G(\delta)$ are in fact supported on the discrete subsets $\Gamma^{\mathscr{E}}(G,V,\zeta)$ and $\Delta^{\mathscr{E}}(G,V,\zeta)$ respectively.

We have for any f in $\mathscr{H}(G,V,\zeta)$ the geometric expansions of the endoscopic and stable linear forms

$$I^{\mathscr{E}}(f) = \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma^{\mathscr{E}}(M,V,\zeta)} a^{M,\mathscr{E}}(\gamma) I_M^{\mathscr{E}}(\gamma, f) \tag{5.6}$$

in the case that G is arbitrary, and

$$S(f) = \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\delta \in \Delta(M,V,\zeta)} b^M(\delta) S_M(\delta, f) \tag{5.7}$$

in the case that G is quasisplit.

Theorem 5.2. (a) *If G is arbitrary, then $\tilde{I}^{\mathscr{E}}(f)$ has a geometric expansion*

$$\tilde{I}^{\mathscr{E}}(f) = \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma^{\mathscr{E}}(M,V,\zeta)} a^{M,\mathscr{E}}(\gamma) \tilde{I}_M^{\mathscr{E}}(\gamma, f). \tag{5.8}$$

(b) *If G is quasisplit, then $\tilde{S}(f)$ has a geometric expansion*

$$\tilde{S}(f) = \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\delta \in \Delta(M,V,\zeta)} b^M(\delta) \tilde{S}_M(\delta, f) \tag{5.9}$$

for any $f \in \mathscr{H}(G,V,\zeta)$.

Proof. The proofs of the two statements are the same and also parallel to Theorem 5.1, so we will be content with proving (a). Assume inductively that (5.8) holds for $L \in \mathscr{L}^0(M)$, so that we have

$$\tilde{I}^{L,\mathscr{E}}(f) = \sum_{M \in \mathscr{L}^L} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma^{\mathscr{E}}(M,V,\zeta)} a^{M,\mathscr{E}}(\gamma) \tilde{I}_M^{\mathscr{E}}(\gamma, f)$$

for any $g \in \mathscr{H}(L,V,\zeta_L)$ and ζ_L the restriction of ζ to L. Using the geometric expansion (5.6) of $I^{\mathscr{E}}(f)$, it follows from the definition that $\tilde{I}^{\mathscr{E}}(f)$ is equal to the difference between

$$\sum_{M \in \mathscr{L}} |W_0^L||W_0^G|^{-1} \sum_{\gamma \in \Gamma^{\mathscr{E}}(M,V,\zeta)} a^{M,\mathscr{E}}(\gamma) I_M^{\mathscr{E}}(\gamma, f)$$

and

$$\sum_{L\in\mathscr{L}^0}|W_0^L||W_0^G|^{-1}\sum_{M\in\mathscr{L}^L}|W_0^M||W_0^L|^{-1}\sum_{\gamma\in\Gamma^{\mathscr{E}}(M,V,\zeta)}a^{M,\mathscr{E}}(\gamma)\hat{\tilde{I}}_M^{L,\mathscr{E}}(\gamma,\iota_L^{\mathscr{E}}(f)).$$

The second expression can be written as

$$\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\sum_{\gamma\in\Gamma^{\mathscr{E}}(M,V,\zeta)}a^{M,\mathscr{E}}(\gamma)\sum_{L\in\mathscr{L}^0(M)}\hat{\tilde{I}}_M^{L,\mathscr{E}}(\gamma,\iota_L^{\mathscr{E}}(f)),$$

and it follows that $\tilde{I}(f)$ equals

$$\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\sum_{\gamma\in\Gamma^{\mathscr{E}}(M,V,\zeta)}a^{M,\mathscr{E}}(\gamma)\left(I_M(\gamma,f)-\sum_{L\in\mathscr{L}^0(M)}\hat{\tilde{I}}_M^{L,\mathscr{E}}(\gamma,\iota_L^{\mathscr{E}}(f))\right),$$

which by (3.12) is equal to

$$\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\sum_{\gamma\in\Gamma^{\mathscr{E}}(M,V,\zeta)}a^{M,\mathscr{E}}(\gamma)\tilde{I}_M(\gamma,f),$$

thus proving (5.8).

The proof of (b) follows from the geometric expansion (5.7) of $S(f)$, assuming (5.9) inductively and the definition (3.13). ☐

We can now put these together to obtain our first main result.

Corollary 5.3. *The linear forms $I^{\mathscr{E}}_{\text{unit}}(f)$ and $S_{\text{unit}}(f)$ have geometric expansions given by the geometric expansions of $\tilde{I}^{\mathscr{E}}(f)$ and $\tilde{S}(f)$ in (5.8) and (5.9) respectively.*

Proof. We first observe that the modified distributions $\tilde{I}^{\mathscr{E}}(f)$ and $\tilde{S}(f)$ differ from the original distributions $I^{\mathscr{E}}(f)$ and $S(f)$ in the contribution of the proper Levi subgroups $L\in\mathscr{L}^0$. On the spectral side, it follows from Lemma 3.1 that the latter contribution vanishes, so we have only the terms corresponding the $M=G$, namely

$$\tilde{I}^{\mathscr{E}}(f)=I^{\mathscr{E}}_{\text{unit}}(f)$$

and

$$\tilde{S}(f)=S_{\text{unit}}(f)$$

on the one hand as a result of Theorem 5.1, and on the hand the geometric expansions given in Theorem 5.2(a) and 5.2(b) respectively. ☐

We observe that the proofs of the global expansions are more or less formal, following the inductive definitions of the modified linear forms. In particular, once we have established the validity of the forms $I^{\mathscr{E}}$ and S for a larger space of non-compactly supported test functions, we shall also have the modified global expansion as above.

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