A WEIGHTED INVARIANT TRACE FORMULA

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ABSTRACT. We establish an invariant trace formula whose discrete spectral terms are weighted by automorphic L-functions. This involves extending the results of Finis, Lapid, and Müller on the continuity of Arthur's noninvariant trace formula to the stable and endoscopic trace formulas, while incorporating the use of basic functions at unramified places. Additionally, we extend the invariant and stable trace formulas of Arthur to noncompactly supported test functions that are equal to unit elements of the spherical Hecke algebra at unramified places.

CONTENTS

1. INTRODUCTION

The Arthur-Selberg trace formula is one of the major tools in harmonic analysis. Given a reductive group G over a number field F, there is a linear form J on G with parallel spectral and geometric expansions which constitute the trace formula. The monumental work of Arthur established the stabilization of the trace formula, which in turn has led to the endoscopic classification of automorphic representations of various classical groups. To do so one first makes the trace formula invariant, and its stabilization in turn depends crucially upon the Fundamental Lemma.

To gain deeper knowledge of Langlands' principle of functoriality, it is important to establish further refinements of the trace formula. The first refinement comes in the form of a trace formula whose spectral coefficients are weighted by factors related to automorphic L-functions, and that is valid for certain noncompactly supported test functions. This latter extension of the trace formula was

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established by Finis, Lapid, and Müller [FLM11, FL16] for the coarse expansion of the noninvariant form J, as a distribution on the group $G(\mathbf{A})^1$,

$$J(\dot{f}) = \sum_{\chi\in\mathfrak{X}} J_\chi(\dot{f}) = \sum_{\mathfrak{o}\in\mathscr{O}} J_{\mathfrak{o}}(\dot{f}).$$

Among such test functions is a distinguished test function, now referred to as the basic function, which can be used to weight the cuspidal spectral terms with the associated automorphic L-functions.

In this paper, we make use of basic functions to establish a weighted invariant trace formula, whose cuspidal spectral terms are weighted by automorphic L-functions. Fixing a central induced torus Z of G with an automorphic character ζ, we let V be a large finite set of valuations of F outside of which G and ζ are unramified. Let $G_V = \prod_{v\in V} G(F_v)$ and $G^V = \prod_{v\notin V} G(F_v)$. The main technical difficulty that we encounter is that Arthur's stabilization of the trace formula is valid only for test functions of the form

$$\dot{f} = f \times u^V$$

where f is a compactly-supported, ζ^{-1}-equivariant function on G_V and u^V is the unit element of the ζ^{-1}-equivariant Hecke algebra $\mathscr{H}(G,V,\zeta)$ of G^V. In order to properly weight the trace formula, we require instead test functions of the form

$$f^r_s = f \times b^V$$

where f is a non-compactly supported, ζ^{-1}-equivariant function on G_V, and b^V is the basic function which, as we recall in Section 2, does not have compact support. It depends on a complex finite-dimensional representation r of the L-group ${}^LG$ of G, and a complex number s with $\mathrm{Re}(s)$ large enough, which we shall assume to be fixed throughout this paper. Thus, we have to take the coarse expansion of the noninvariant linear form J as our starting point, and begin the process of refinement there. More precisely, we first establish a refined expansion for the noninvariant linear form J applied to f^r_s, namely

$$\begin{aligned} J(f^r_s) &= \sum_{M\in\mathscr{L}} |W^M_0||W^G_0|^{-1} \int_{\Pi(M,V,\zeta)} a^M_{r,s}(\pi) J_M(\pi,f)d\pi \\ &= \sum_{M\in\mathscr{L}} |W^M_0||W^G_0|^{-1} \sum_{\gamma\in\Gamma(M,V,\zeta)} a^M_{r,s}(\gamma) J_M(\gamma,f) \end{aligned}$$

which, taking b^V to be fixed, we may view as a linear form on f. The global coefficients $a^M_{r,s}(\pi)$ and $a^M_{r,s}(\gamma)$ that occur here are now weighted forms of the coefficients $a^M(\pi)$ and $a^M(\gamma)$ that occur in the usual trace formula. We note that the refined spectral expansion was already obtained in [FLM11], so the bulk of the work falls on refining the geometric expansion.

We then proceed to make this form invariant following the usual technique of Arthur. Indeed, our methods can be applied to obtain an invariant trace formula that is valid for any function $f\times g$ where g is a fixed function in $\mathscr{C}^\circ(G^V,\zeta^V)$, though we have no need of it here. The main result of this paper, then, is an invariant trace formula valid for the test functions $f \times b^V$.

Theorem 1. *There is an invariant linear form*

$$I^r_s(f) = I(\dot{f}^r_s), \qquad f \in \mathscr{C}^\circ(G,V,\zeta)$$

valid for $\mathrm{Re}(s)$ *large enough. It comes with the parallel expansions*

$$\begin{aligned}(1.1)\qquad &\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\int_{\Pi(M,V,\zeta)}a^M_{r,s}(\pi)I_M(\pi,f)d\pi\\ &=\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\sum_{\gamma\in\Gamma(M,V,\zeta)}a^M_{r,s}(\gamma)I_M(\gamma,f).\end{aligned}$$

The identity will follow from the geometric and spectral expansions established in Theorems 4.6 and 5.2, respectively.

To prove the theorem, we first extend the results of [FLM11, FL16] to the invariant linear form I, applied to test functions $f\times u^V$ in Section 3. This will take us part of the way in refining the coarse geometric expansion of $J(\dot{f})$. Then in Sections 4 and 5 we establish the geometric and spectral expansions of $I^r_s(f)$ respectively in (1.1). The key difference here is that the global coefficients are now weighted with the new coefficents that depend on the basic function b, while the local distributions remain unchanged. We again follow the general method of Arthur, and relying on the work Finis, Lapid, and Müller where the noncompact support is involved.

The next step, of course, is to stabilize the form $I^r_s(f)$. We take a step in this direction in Section 6. The second result of this paper is an extension of the stable and endoscopic trace formulas to the noncompactly supported test functions in Theorem 6.2.

Theorem 2. *The stable and endoscopic linear forms* S *and* $I^{\mathscr{E}}$ *extend continuously from* $\mathscr{H}(G,V,\zeta)$ *to* $\mathscr{C}^\circ(G,V,\zeta)$*.*

The proof of this result requires the extension of the Langlands-Shelstad transfer of smooth, compactly-supported test functions to our noncompactly supported ones. While it is the form $I^r_s(f)$ that needs to be stabilized rather than $I(f)$, in stabilizing the form $I(f)$ we show that the local distributions on either side of the stable trace formula hold for f in $\mathscr{C}^\circ(G,V,\zeta)$. As it happens, this is also the local statement required for the stabilization of $I^r_s(f)$, which we record as Corollaries 6.3 and 6.4. The global statement, on the other hand, regards the stabilization of the global coefficients is much more involved and will be studied in [Wonb]. In particular, this stabilization will require the transfer of weighted orbital integrals, which was not needed for the original stabilization of trace formula for functions $f\times u^V$. Such a transfer, we hope, should follow from the weighted fundamental lemma, analogous to the way that the transfer of unweighted orbital integrals follows from the fundamental lemma.

2. Basic notions

2.1. **Definitions.** Let G be a connected reductive group over a field F of characteristic zero. We denote by $\mathscr{L}(M)$ to be the collection of Levi subgroups of G containing M, $\mathscr{L}^0(M)$ the subset of proper Levi subgroups in $\mathscr{L}(M)$, and $\mathscr{P}(M)$ the collection of parabolic subgroups of G containing M. Let F be a global field, and V a finite set of places of F. We have the real vector space $\mathfrak{a}_M=\mathrm{Hom}(X(M)_F,\mathbf{R})$, and the set

$$\mathfrak{a}_{M,V}=\{H_M(m): m\in M(F_V)\}$$

is a subgroup of $\mathfrak{a}_M$, and $F_V=\prod_{v\in V}F_v$. It is equal to $\mathfrak{a}_M$ if V contains an archimedean place, and is a lattice in $\mathfrak{a}_M$ otherwise. The additive character group $\mathfrak{a}^*_{M,V}=\mathfrak{a}^*_M\backslash\mathfrak{a}^\vee_{M,V}$ equals $\mathfrak{a}^*_M$ in the first case, and is a compact quotient of $\mathfrak{a}^*_M$ in the second. Let A_M be the maximal split torus of a Levi subgroup M of G. We then identify the Weyl group of (G, A_M) with the quotient of the normaliser of M by M, thus

$$W^G(M) = \mathrm{Norm}_G(M)/M.$$

If M_0 is a minimal Levi subgroup of G, which we shall assume to be fixed, and denote $\mathscr{L} = \mathscr{L}(M_0), \mathscr{P} = \mathscr{P}(M_0), \mathscr{L}^0 = \mathscr{L}^0(M_0)$, and $W_0^G = W^G(M_0)$. Also write $P_0 = M_0N_0$ for the minimal parabolic subgroup containing M_0.

Let Z be a central induced torus of G over F. We define the pair (Z, ζ) where ζ is a character of $Z(F)$ if F is local, and an automorphic character of $Z(\mathbf{A})$ if F is global. Given a finite set of places V, we write $G_V = G(F_V)$ and write ζ_V for the restriction of ζ to the subgroup Z_V of $Z(\mathbf{A})$. We then write G_V^Z for the set of $x \in G_V$ such that $H_G(x)$ lies in the image of the canonical map from $\mathfrak{a}_Z$ to $\mathfrak{a}_G$. We shall assume that V contains the places over which G and ζ are ramified.

The stable trace formula requires that we work in fact with G a K-group as defined in [Art99, §1]. Thus

$$G = \coprod_\alpha G_\alpha \qquad \alpha \in \pi_0(G)$$

is a variety whose connected components G_α are reductive groups over F, equipped with an equivalence class of frames

$$(\psi, u) = \{(\psi_{\alpha\beta}, u_{\alpha\beta}) : \alpha, \beta \in \pi_0(G)\}$$

satisfying natural compatibility conditions. Here $\psi_{\alpha\beta} : G_\alpha \to G_\beta$ in an isomorphism over $\bar{F}$, and $u_{\alpha\beta}$ is a locally constant function from $\Gamma = \mathrm{Gal}(\bar{F}/F)$ to the simply connected cover $G_{\alpha,\mathrm{sc}}$ of the derived group of G_α. Any connected reductive group is a component of an K-group that is unique up to weak isomorphism. It comes with a local product structure

$$G_V = \prod_{v \in V} \coprod_{\alpha_v \in \pi_0(G_v)} G_{v,\alpha_v}.$$

The introduction of K-groups is to steamline certain aspects of the study of connected groups, and the definitions for connected groups will extend to K-groups in a natural way. For example, a central induced torus Z of a K-group G will have central embeddings $Z \xrightarrow{\sim} Z_\alpha \subset Z(G_\alpha)$ for each α, and ζ determines a character ζ_α for each α. We shall call a K-group G quasisplit if it has a connected component that is quasisplit over F.

Let $\mathscr{C}(G)$ be the space of Harish-Chandra Schwartz functions on $G(\mathbf{A})^1$, and $\mathscr{C}(G, \zeta)$ the ζ^{-1}-equivariant functions on $G(\mathbf{A})^Z$. We write $\mathscr{C}(G, V, \zeta) = \mathscr{C}(G_V^Z, \zeta_V)$ for the space of ζ^{-1}-equivariant Schwartz functions on G_V^Z, which contains the Hecke algebra $\mathscr{H}(G, V, \zeta) = \mathscr{H}(G_V^Z, \zeta_V)$ defined with respect to a choice of maximal compact subgroup K_∞ of G_{V_∞}, where V_∞ denotes the archimedean places in V. If F is a local field, we write $\mathscr{C}(G_v, \zeta_v)$ and $\mathscr{H}(G_v, \zeta_v)$ for the corresponding spaces. There are natural decompositions

$$\mathscr{C}(G_v, \zeta_v) = \bigoplus_{\alpha_v \in \pi_0(G_v)} \mathscr{C}(G_{\alpha_v}, \zeta_{\alpha_v})$$

and

$$\mathscr{C}(G_V, \zeta_V) = \bigotimes_{v \in V} \mathscr{C}(G_v, \zeta_v),$$

and similarly for the Hecke algebra. We shall write $\mathscr{C}(G_v, \zeta_v) = \mathscr{C}(G, \zeta)$ and $\mathscr{H}(G_v, \zeta_v) = \mathscr{H}(G, \zeta)$ when the context is clear. We will also denote by $I\mathscr{C}(G_V^Z, \zeta_V)$ and $S\mathscr{C}(G_V^Z, \zeta_V)$ the spaces of orbital integrals and stable orbital integrals of functions in $\mathscr{C}(G_V^Z, \zeta_V)$ respectively.

We also recall the space of functions constructed in [FLM11, §3] extending the usual space of test functions $C_c^\infty(G) = C_c^\infty(G(\mathbf{A})^1)$. For any compact open subgroup K of $G(\mathbf{A}_f)$ the space $G(\mathbf{A})^1/K$ is a differentiable manifold. Any element $X \in \mathcal{U}(\mathfrak{g}^1)$, the universal enveloping algebra of the Lie algebra $\mathfrak{g}^1$ of $G(\mathbf{R})^1 = G(\mathbf{R}) \cap G(\mathbf{A})^1$ defines a left-invariant differentiable operator $f * X$ on $G(\mathbf{A})^1/K$. Let $\mathscr{C}^\circ(G, K)$ be the space of smooth, right-K-invariant functions on $G(\mathbf{A})^1$ which belong to $L^1(G(\mathbf{A})^1)$ together with all their derivatives. It is a Fréchet space under the seminorms

$$||f * X||_{L^1(G(\mathbf{A})^1)}, \qquad X \in \mathcal{U}(\mathfrak{g}^1).$$

For any nonnegative integer k, we define the norms

$$||f||_{G,k} = \sum_i ||X_i * f||_{L^1(G(\mathbf{A})^1)}$$

where X_i ranges over a fixed basis of $\mathcal{U}(\mathfrak{g})_{\leq k}$ with respect to the standard filtration. Denote by $\mathscr{C}^\circ(G)$ the union of $\mathscr{C}^\circ(G, K)$ as K varies over open compact subgroups of $G(\mathbf{A}_f)^1$, and endow $\mathscr{C}^\circ(G)$ with the inductive limit topology.

As with the Hecke algebra, we shall also define the corresponding spaces $\mathscr{C}^\circ(G, \zeta)$ and $\mathscr{C}^\circ(G, V, \zeta)$ obtained from the spaces of ζ^{-1}-equivariant functions on $G(\mathbf{A})^Z$ and G_V^Z respectively, in a manner parallel to $\mathscr{C}^\circ(G)$. The resulting spaces are natural subspaces of the Schwartz spaces $\mathscr{C}(G), \mathscr{C}(G, \zeta)$, and $\mathscr{C}(G, V, \zeta)$ respectively. Moreover, we will again take G to be a K-group, so that

$$\mathscr{C}^\circ(G) = \bigoplus_{\alpha \in \pi_0(G)} \mathscr{C}^\circ(G_\alpha)$$

and similarly with the spaces $\mathscr{C}^\circ(G, \zeta)$ and $\mathscr{C}^\circ(G, V, \zeta)$.

2.2. **Basic functions and local L-factors.** For the moment let F be a nonarchimedean local field, and G a reductive group defined over F. Suppose moreover that G is unramified over F, meaning that G admits a reductive model over $\mathscr{O}_F$. Recall that we have an exact sequence

$$0 \to I_F \to \Gamma_F \to \Gamma_k \to 0$$

where I_F is the inertia group of Γ_F , and k is the residue field of F with Frobenius element denoted by σ_F. If G is quasisplit over F, it follows then that G is unramified if and only if the restriction of the homomorphism $\Gamma_F \to \mathrm{Out}(G \otimes_F \bar{F})$ to I_F is trivial. In particular, the action of Γ_F on $G^\vee$ factors through Γ_k, thus we may take ${}^LG$ to be $G^\vee \rtimes \langle \sigma_F \rangle$.

An irreducible smooth representation of $G(F)$ is unramified if it has a nonzero vector under $G(\mathscr{O}_F)$. Then the isomorphism classes of unramified representations π of $G(F)$ are in canonical bijection with the conjugacy classes α of $G^\vee$ in the connected component of ${}^LG$, $\sigma_F G^\vee \subset G^\vee \rtimes \langle \sigma_F \rangle$. Fix a maximal compact subgroup K of $G(F)$. There is a twisted form of the Satake isomorphism

$$\mathrm{Sat} : \mathscr{H}(G, K) \to \mathbf{C}[\sigma_F G^\vee]^{\mathrm{ad}(G^\vee)} \tag{2.1}$$

from the unramified Hecke algebra of G to the ring of regular functions on $\sigma_F G^\vee$ that are invariant under the adjoint action of $G^\vee$. The bijection $\pi \to \alpha_\pi$ is characterized by the requirement that $\mathrm{tr}(\pi(f)) = \mathrm{Sat}(f)(\alpha_\pi)$ for any unramified irreducible representation π.

Given a complex finite-dimensional representation $r : {}^L G \to \mathrm{GL}(V)$, we have the local L-factor of π given by

$$L(s,\pi,r) = \det(1 - r(\alpha_\pi)q^{-s})^{-1}$$

where q is the cardinality of the residue field of F. We may expand it as a formal series

$$\sum_{n=0}^{\infty} \mathrm{tr}((\mathrm{Sym}^n r)(\alpha_\pi))q^{-ns}$$

converging absolutely for $\mathrm{Re}(s)$ large enough, where the abscissa of convergence depends on the eigenvalues of α_π. Viewing $\det(1-r(\alpha)q^{-s})^{-1}$ as a rational function on $G^\vee$, we would like to invert the Satake isomorphism to obtain a function b^r_s such that

$$\mathrm{tr}(\pi(b^r_s)) = L(s,\pi,r).$$

Using the formal series expansion above, it is the same as asking for a family of functions b^r_n in $\mathscr{H}(G,K)$ such that

$$b^r_s = \sum_{n=0}^{\infty} b^r_n q^{-ns}, \tag{2.2}$$

and $\mathrm{tr}(\pi(b^r_n)) = \mathrm{tr}(\mathrm{Sym}^n(\alpha_\pi))$.

The basic function b^r_s is expected to be a distinguished vector in a certain Schwartz space of r-functions called the r-Schwartz space, defined as the global sections with compact support of a certain sheaf on a reductive monoid $M^r(F)$ containing $G(F)$ as an open subset [Ngô16]. Following Ngô, we may assume that G is equipped with a determinant homomorphism $\nu : G \to \mathbf{G}_m$ such that composition $r \circ \nu$ acts by scalar multiplication on the vector space V of r, giving an exact sequence

$$1 \to G_0 \to G \xrightarrow{\nu} \mathbf{G}_m \to 1$$

where G_0 is a semisimple group. This is not a restrictive condition, seeing as we may replace G by $G \times \mathbf{G}_m$ if necessary. Under this assumption, the sum (2.2) is locally finite, and its support can be described explicitly by the weights of r. Basic functions have been the subject of much study of late, but as we shall see, our interest will lie not in the functions themselves but their orbital integrals.

For our purposes, it will suffice to know that b^r_s belongs to the spherical subspace $\mathscr{H}_{\mathrm{ac}}(G,K)$ of the almost-compact unramified Hecke space $\mathscr{H}_{\mathrm{ac}}(G(F))$ [Art88a, §1]. Let T be a maximal split torus of G over F. Fix a Borel pair (B,T) of G and consider the Cartan decomposition $G(F) = KT(F)_+K$ using the anti-dominant Weyl chamber $X_*(T)_-$ in the cocharacter lattice $X_*(T)$, where $T(F)_+$ is the image of $X_*(T)_-$ under the map $\mu \mapsto \mu(\varpi)$ with ϖ a uniformizer of F. The homomorphism ν induces a map $X_*(T) \to \mathbf{Z}$. In this case, Li has given an explicit description of the basic function

$$b^r_s = \sum_{\mu \in X_*(T)_-} c_\mu(q)\delta^{\frac{1}{2}}_{B^-}(\mu(\varpi))\mathbf{1}_{K\mu(\varpi)K}q^{-\nu(\mu)s} \tag{2.3}$$

where $c(\mu)$ is polynomial in q^{-1}, and is a nonnegative integer given explicitly in terms of Kazhdan-Lusztig polynomials and symmetric powers of r.

Lemma 2.1. *The basic function b^r_s belongs to $\mathscr{H}_{\mathrm{ac}}(G,K)$ for* $\mathrm{Re}(s)$ *large enough.*

Proof. The case where G is split is due to [Li17, §3]. If G is quasisplit, we simply note that the Kato-Lusztig formula remains valid by [Hai18, Theorem 7.10] and [CCH19, Theorem 1.9.1], and applying the Satake inversion (2.1) for quasisplit G, it follows that the argument of [Li17, Proposition 3.4] and the preceding discussion can be applied. □

Remark 2.2. In particular, we note that there is a sequence of inclusions

$$\mathscr{H}_{\mathrm{ac}}(G,K) \subset \mathscr{H}_{\mathrm{ac}}(G) \subset \mathscr{C}(G) \subset \mathscr{C}^\circ(G),$$

so that in general the local theory applies to basic functions as well, though we shall need more precise information in this special case. We may also take b^r_s to be ζ^{-1}-equivariant by replacing the characteristic functions $\mathbf{1}_{K\mu(\varpi)K}$ with their ζ^{-1}-equivariant analogues. It is straightforward to extend the basic function to K-groups,

$$b^r_s = \bigoplus_{\alpha\in\pi_0(G)} b^r_{\alpha,s}$$

where $b^r_{\alpha,s}$ is the basic function defined by the component group G_α, thereby placing us in proper generality.

We now return to F being a global field. Enlarging V if necessary, we shall assume that G, ζ, and r are unramified outside of V. Recall the set of families $C(G^V,\zeta^V)$ of semisimple conjugacy classes in ${}^LG_v$, for $v \notin V$ in [Art02, p.202]. We shall in fact consider equivalence classes of families c^V, where two families c and c' in $C(G^V,\zeta^V)$ are identified if $c_v = c'_v$ for almost all $v \notin V$. Then given any c in $C(G^V,\zeta^V)$ and finite-dimensional representation r of ${}^LG$, the Euler product

$$L^V(s,c,r) = \prod_{v\notin V} \det(1 - r(c_v))q_v^{-s})^{-1}$$

converges to an analytic function in s in some right-half plane. The local components c_v determine unramified irreducible representations $\pi_v = \pi_v(c)$ of $G(F_v)$, and hence an unramified representation $\pi^V(c) = \otimes_{v\notin V}\pi_v(c)$ of G^V. Then if c is automorphic in the sense that there exists an irreducible representation π_V of G_V such that $\pi = \pi_V \otimes \pi^V(c)$ is an automorphic representation of $G(\mathbf{A})$, then a conjecture of Langlands asserts that $L^V(s,c,r)$ has meromorphic continuation [Lan70]. We can then identify the unramified automorphic L-function as

$$L^V(s,c,r) = L^V(s,\pi,r).$$

If π is tempered, the set of coefficients $\mathrm{tr}(r(c(\pi_v)))^k$ for $v \notin V$ and $k \geq 1$ is bounded; if moreover π is cuspidal, one expects the L-function to have meromorphic continuation to the complex plane, with at most a simple pole at $s = 1$.

We can now properly describe the test functions that we shall use. Given f in $\mathscr{C}^\circ(G,V,\zeta)$, we shall form the test function

$$\dot{f}^r_s = f \times b^V \tag{2.4}$$

in $\mathscr{C}^\circ(G,\zeta)$, where

$$b^V = b^V_{r,s} = \prod_{v\notin V} b^r_{v,s}.$$

so that
$$b_G^V(c) = b_G^V(\pi^V(c)) = L^V(s,c,r).$$
More generally, for nonarchimedean valuations v in V, we may choose f_v to be a ζ^{-1}-equivariant function in $L^1(G(F_v))$ and let f_∞ be a smooth ζ^{-1}-equivariant function on $G(F_\infty)$ where $F_\infty = \prod_{v|\infty} F_v$ such that $||f_\infty * X||_{L^1(G(F_\infty))}$ is finite for all $X \in \mathcal{U}(\mathfrak{g}^1)$. It follows then that $\dot{f}_s^r$ belongs to $\mathscr{C}^\circ(G,\zeta)$ for $\mathrm{Re}(s)$ large enough.

3. Continuity of the invariant trace formula

3.1. **The coarse expansion.** Let now F be a number field, and let $\mathscr{H}(G)$ be the Hecke algebra on $G(\mathbf{A})^1$. We first recall the noninvariant linear form $J(f)$ on $\mathscr{H}(G,V,\zeta)$ established in [Art02, §2] from the original linear form on $\mathscr{H}(G)$. It is a continuous, $Z(F)$-invariant linear form on $\mathscr{C}^\circ(G)$ consisting of two different expansions
$$J(\dot{f}^1) = \sum_{\mathfrak{o}\in\mathscr{O}} J_{\mathfrak{o}}(\dot{f}^1) = \sum_{\chi\in\mathfrak{X}} J_\chi(\dot{f}^1)$$
for any $\dot{f}^1 \in \mathscr{H}(G)$, with both sums converging absolutely. Here $\mathscr{O}$ is the set of $\mathscr{O}$-equivalence classes of element in $G(F)$, whereby two elements are equivalent if their semisimple parts are $G(F)$-conjugate, and $\mathfrak{X}$ is the set of equivalence classes of cuspidal automorphic data $\chi = \{(P,\sigma)\}$, where P is a standard parabolic subgroup of G with Levi subgroup M_P and σ is an irreducible representation of $M_P(\mathbf{A})^1$, up to a certain equivalence relation as described in [Art82]. There is a natural projection
$$\dot{f}^1 \to \dot{f}^\zeta$$
from $\mathscr{C}^\circ(G)$ onto the space $\mathscr{C}^\circ(G,\zeta) = \mathscr{C}^\circ(G(\mathbf{A})^Z,\zeta)$ given by
$$\dot{f}^\zeta(x) = \int_{Z(\mathbf{A})^x} \dot{f}^1(zx)\zeta(zx)dz \tag{3.1}$$
where $x \in G(\mathbf{A})^Z$ and $Z(\mathbf{A})^x$ is the set of $z \in Z(\mathbf{A})$ such that $H_G(zx) = 0$. We can then define a linear form on $\mathscr{C}^\circ(G,\zeta)$ by
$$J(\dot{f}^\zeta) = J^\zeta(\dot{f}^1) = \int_{Z(F)\backslash Z(\mathbf{A})^1} J(\dot{f}_z^1)\zeta(z)dz \tag{3.2}$$
where $\dot{f}_z^1$ denotes the translation of $\dot{f}^1$ by a point $z \in Z(\mathbf{A})^1$, and the integral depends only on the image $\dot{f}^\zeta$ of $\dot{f}^1$ in $\mathscr{C}^\circ(G,\zeta)$. Also, given a function $f \in \mathscr{C}^\circ(G,V,\zeta)$, we can also define a linear form on $\mathscr{C}^\circ(G,V,\zeta)$ by setting
$$J_s^r(f) = J(\dot{f}_s^r)$$
where $\dot{f}_s^r = f \times b^V$. We then have the noninvariant linear form on $\mathscr{C}^\circ(G,V,\zeta)$ given by
$$J_s^r(f) = J(\dot{f}_s^r) = J^\zeta(\dot{f}^1)$$
where $\dot{f}^1$ is any function in $\mathscr{C}^\circ(G)$ whose projection $\dot{f}^\zeta$ onto $\mathscr{C}^\circ(G,\zeta)$ equals $\dot{f}_s^r = f \times b^V$.

We next define an invariant linear form. I_s^r on $\mathscr{C}^\circ(G,V,\zeta)$ inductively by setting
$$I_s^r(f) = J_s^r(f) - \sum_{M\in\mathscr{L}^0} |W_0^M||W_0^G|^{-1}\hat{I}_s^{r,M}(\phi_M(f)) \tag{3.3}$$

for certain maps

$$\phi_M : \mathscr{H}_{\mathrm{ac}}(G, V, \zeta) \to \mathscr{I}_{\mathrm{ac}}(M, V, \zeta) \tag{3.4}$$

constructed from normalized weighted characters [Wonb, (2.2)] (see also [Art98]). To stabilize the invariant form I_s^r, we must first express the geometric and spectral expansions in terms of local distributions.

3.2. **The refined expansion.** For the remainder of this section, we shall work more generally with the noninvariant linear form on $\mathscr{H}(G, V, \zeta)$ given by

$$J(f) = J(\dot{f}) = J^\zeta(\dot{f}^1)$$

where $\dot{f}^1$ is any function in $\mathscr{H}(G)$ whose projection $\dot{f}^\zeta$ onto $\mathscr{H}(G, \zeta)$ equals $\dot{f} = f \times u^V$. It follows from the preceding discussion that $J(f)$ has the parallel expansions

$$J(f) = \sum_{\mathfrak{o} \in \mathscr{O}} J_{\mathfrak{o}}(f) = \sum_{\chi \in \mathfrak{X}} J_\chi(f)$$

which we would like extend to a larger family of noncompactly supported test functions. The following lemma extends the coarse expansion to a linear form on the space $\mathscr{C}^\circ(G, V, \zeta)$.

Lemma 3.1. *The linear form J on $\mathscr{H}(G, V, \zeta)$ extends to a continuous linear form on $\mathscr{C}^\circ(G, V, \zeta)$.*

Proof. We follow the passage of J from $\mathscr{H}(G)$ to $\mathscr{H}(G, V, \zeta)$. There is a natural projection

$$\dot{f}^1 \to \dot{f}^\zeta$$

from $\mathscr{C}^\circ(G)$ to $\mathscr{C}^\circ(G, \zeta)$ given by the formula (3.1). Given the linear form J on $\mathscr{C}^\circ(G)$, we define a linear form on $\mathscr{C}^\circ(G, \zeta)$ by

$$J(\dot{f}^\zeta) = J^\zeta(\dot{f}^1)$$

where the right-hand side is defined as in (3.2).

Now let f be a function in $\mathscr{C}^\circ(G, V, \zeta)$. Given any function $\dot{f}$ in $\mathscr{C}^\circ(G)$ whose projection $\dot{f}^\zeta$ onto $\mathscr{C}^\circ(G, \zeta)$ equals $\dot{f} = f \times u^V$, we have the noninvariant linear form on $\mathscr{C}^\circ(G, V, \zeta)$ given by

$$J(f) = J(\dot{f}) = J^\zeta(\dot{f}^1)$$

as before, with both spectral and geometric sides converging absolutely. By the construction of the linear forms on each space, it follows that the form $J(f)$ on $\mathscr{C}^\circ(G, V, \zeta)$ is the continuous extension of the corresponding linear form on $\mathscr{H}(G, V, \zeta)$. □

In order to pass to the invariant trace formula, we first have to refine the expansion of the noninvariant trace formula. In particular, we need to express both sides in terms of the basic distributions $J_M(\gamma, f)$ and $J_M(\pi, f)$. We first refine the geometric side. We refer to [Art02, (2.8)] for the construction of the global geometric coefficient $a^M(\gamma)$.

Proposition 3.2. *Let $f \in \mathscr{C}^\circ(G, V, \zeta)$. Then the linear form $J(f)$ has a geometric expansion*

$$\sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M, V, \zeta)} a^M(\gamma) J_M(\gamma, f). \tag{3.5}$$

Proof. The linear form $J(f)$ obtained in Lemma 3.1 has the coarse geometric expansion

$$J(f) = \sum_{\mathfrak{o}\in\mathscr{O}} J_{\mathfrak{o}}(f)$$

with the sums converging absolutely. Let G^0 be the connected component of the identity in G, and G_c the identity component of the centralizer of a semisimple element c in $G(F)$. Then the equivalence class $\mathfrak{o}$ consists of elements in $G(F)$ whose semisimple Jordan components belong in the same $G^0(F)$ orbit. There is another equivalence relation, which depends on a finite set of places S, which we shall assume contains V. The (G,S)-equivalence classes are defined to be the sets

$$G(F)\cap(\sigma U)^{G^0(F)} = \{g^{-1}\sigma ug : g\in G^0(F), u\in U\cap G^0(F)\}$$

where σ is a semisimple element of $G^0(F)$, and U is a unipotent conjugacy class in $G_\sigma(F)$. Any class $\mathfrak{o}\in\mathscr{O}$ breaks up into a finite set $(\mathfrak{o})_{G,S}$ of (G,S)-equivalence classes.

Let $\dot{f}^1$ be any function in $\mathscr{C}^\circ(G)$ whose projection $\dot{f}^\zeta$ onto $\mathscr{C}^\circ(G,\zeta)$ equals the function $\dot{f} = f\times u^V$. Suppose moreover that

$$\dot{f}^1 = \dot{f}^1_S\times u^{S,1}, \qquad \dot{f}^1_S\in\mathscr{C}^\circ(G(F_S)^1).$$

for $S\supset V$ large enough. The space $\mathscr{C}^\circ(G(F_S)^1)$ naturally embeds in $\mathscr{C}^\circ(G)$. It follows from [Art86, Theorem 8.1] that there is an expansion

$$J_{\mathfrak{o}}(\dot{f}^1) = \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\dot\gamma\in(M(F)\cap\mathfrak{o})_{M,S}} a^M(S,\dot\gamma)J_M(\dot\gamma,\dot{f}^1_S) \tag{3.6}$$

for any $\mathfrak{o}\in\mathscr{O}$, $\dot{f}^1_S\in C_c^\infty(G(F_S)^1)$, and S containing a finite set $S_{\mathfrak{o}}$ of valuations of F including the archimedean places. Here $J_M(\dot\gamma,\dot{f}^1_S)$ is the weighted orbital integral of $\dot{f}^1_S$ over the conjugacy class of $\dot\gamma$ in G_S, and is a tempered distribution by [Art94]. The derivation of this formula relies on a combinatorial argument and descent to unipotent weighted orbital integrals, and in particular remains valid so long as the distribution $J_{\mathfrak{o}}(\dot{f}^1)$ is absolutely convergent, and thus for $\dot{f}^1$ belonging to the larger space $\mathscr{C}^\circ(G)$. (We discuss the unipotent terms in greater detail in [Wonb, §2].)

In order to sum over the classes $\mathfrak{o}\in\mathscr{O}$, we have to modify the proof of [Art86, Theorem 9.2] and appeal to [FL16, Theorem 7.1] instead for the convergence of the sum since $\dot{f}^1$ no longer has compact support. Let

$$\mathrm{ad}(G^0(\mathbf{A}))_{\mathfrak{o}} = \{x^{-1}\gamma x : x\in G^0(\mathbf{A}), \gamma\in\mathfrak{o}\},$$

and write $\mathscr{O}_\Delta$ for the set of classes $\mathfrak{o}$ such that $\mathrm{ad}(G^0(\mathbf{A}))_{\mathfrak{o}}$ meets the support of $\dot{f}^1_S$. Since $J_{\mathfrak{o}}$ annihilates any function which vanishes on $\mathrm{ad}(G^0(\mathbf{A}))_{\mathfrak{o}}$, we obtain therefore

$$\sum_{\mathfrak{o}\in\mathscr{O}} J_{\mathfrak{o}}(\dot{f}^1) = \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\mathfrak{o}\in\mathscr{O}_\Delta}\sum_{\dot\gamma\in(M(F)\cap\mathfrak{o})_{M,S}} a^M(S,\dot\gamma)J_M(\dot\gamma,\dot{f}^1_S).$$

Now suppose that $\dot\gamma$ is any element of $(M(F))_{F,S}$. Then $\dot\gamma$ is contained in a unique class $\mathfrak{o}\in\mathscr{O}$, and it follows from [Art88c, Theorem 5.2] that $J_M(\dot\gamma,\dot{f}^1_S)$ vanishes if $\dot{f}^1_S$ vanishes on $\mathrm{ad}(G^0(\mathbf{A}))_{\mathfrak{o}}$, hence $J_M(\dot\gamma,\dot{f}^1_S)$ vanishes unless $\mathfrak{o}$ belongs to $\mathscr{O}_\Delta$. From this we have that

$$J(\dot{f}^1) = \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\dot\gamma\in(M(F_S))_{F,S}} a^M(S,\dot\gamma)J_M(\dot\gamma,\dot{f}^1_S).$$

The rest of the argument is similar to the proof of [Art02, Proposition 2.2], so we can be brief. For a fixed set of valuations S, the linear form $J(\dot{f}^1)$ is K^S-invariant, we may then write

$$J(f) = \int_{Z(F)Z(\mathfrak{o}^S)\backslash Z(\mathbf{A})^1} J(\dot{f}^1_z)\zeta(z)dz$$

as

$$\sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\dot\gamma\in(M(F))_{M,S}} a^M(S,\dot\gamma)\int_{Z_{S,\mathfrak{o}}\backslash Z_S^1} J_M(z\dot\gamma,\dot{f}^1_S)\zeta(z)dz$$

where $Z_{S,\mathfrak{o}} = Z(F)\cap Z_S Z(\mathfrak{o}^S)$ and $\mathfrak{o}^S = \prod_{v\notin S}\mathfrak{o}_v$, since $Z(\mathbf{A}) = Z(F)Z_S Z(\mathfrak{o}^S)$ and $J_M(\dot\gamma,\dot{f}^1_{S,z}) = J_M(z\dot\gamma,\dot{f}^1_S)$ for any $z\in Z_S$. Then using the definition of the coefficient $a^M(\gamma)$, it follows that the geometric expansion of $J(f)$ can be written as

$$\sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in\Gamma(M,V,\zeta)} a^M(\gamma)J_M(\gamma,f)$$

as required. □

Remark 3.3. We note that we have not obtained the absolute convergence of this refined geometric expansion. For semisimple elements γ, this follows from [FL11, Theorem 1], which proves the absolute convergence of the semisimple contribution to (3.6), and by the argument above one deduces the absolute convergence of the semisimple contribution to the refined geometric expansion (3.5). As the authors point out, the absolute convergence of the unipotent contribution would require a uniform bound on the global geometric coefficients, which at present are known only for $\mathrm{GL}(n)$ [Mat15, Theorem 1.1]. Fortunately, this is not needed for the applications that we are interested in, which is the comparison of trace formulae.

We next refine the spectral expansion.

Proposition 3.4. *Let $f\in\mathscr{C}^\circ(G,V,\zeta)$. Then the linear form $J(f)$ has a spectral expansion*

$$\sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1}\int_{\Pi(M,V,\zeta)} a^M(\pi)J_M(\pi,f)d\pi, \tag{3.7}$$

with the integrals converging absolutely.

Proof. The linear form $J(f)$ obtained in Lemma 3.1 has the fine spectral expansion

$$J(f) = \sum_{\chi\in\mathfrak{X}} J_\chi(f)$$

which converges absolutely, and where $J_\chi(f)$ is equal to the sum over $M\in\mathscr{L}$ of the product of

$$|W_0^M||W_0^G|^{-1}|\det(s-1)_{\mathfrak{a}_M^G}|^{-1}$$

with

$$\sum_{\pi\in\Pi_{\mathrm{unit}}(M,\zeta)}\sum_{L\in\mathscr{L}(M)}\sum_{s\in W^L(M)_{\mathrm{reg}}}\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \mathrm{tr}(\mathscr{J}_L(P,\lambda)J_P(s,0)\mathscr{I}_{P,\chi,\pi}(\lambda,f))d\lambda,$$

as stated in [Art82, Theorem 8.2]. Here

$$\mathscr{J}_L(P,\lambda) = \lim_{\Lambda\to0}\sum_{Q\in\mathscr{P}(M)}\mathscr{J}_Q(P,\lambda,\Lambda)\theta_Q(\Lambda)^{-1},$$

for $\Lambda \in i\mathfrak{a}_M^*$ near to 0, is the limit of (G,M)-families

$$\mathscr{J}_Q(P,\lambda,\Lambda) = J_{P|Q}(\lambda)^{-1} J_{Q|P}(\lambda+\Lambda)$$

and $J_{Q|P}(\lambda)$ is the global unnormalized operator intertwining the actions of the induced representations $\mathscr{I}_P(\pi_\lambda)$ and $\mathscr{I}_Q(\pi_\lambda)$. Also

$$J_P(s,0) = J_{P|P}(s,\pi_{\lambda+\Lambda}).$$

It is a consequence of [FLM11, Corollary 1] that the sums are finite and the integrals are absolutely convergent with respect to the trace norm, and define distributions on $\mathscr{C}^\circ(G)$. We note that the absolute convergence is proved for an expansion slightly different from the above, but is shown to be equivalent in [FLM11, §5.3]. Importantly, the sum over π does not occur in the latter, but the necessary estimate for this sum, which is not necessarily finite, is contained in [FLM11, §5.1]. (See also [Par19, Theorem 7.2] for the twisted case.)

Beginning with

$$J(f) = J^\zeta(\dot{f}^1) = \int_{Z(F)\backslash Z(\mathbf{A})^1} J(\dot{f}^1_z)\zeta(z)dz,$$

where $\dot{f}^1$ is any function in $\mathscr{C}^\circ(G)$ whose projection onto $\mathscr{C}^\circ(G,\zeta)$ equals $\dot{f} = f \times u^V$, it follows from the argument of [Art88b, Theorem 4.4] and the definition of $a^G_{\mathrm{disc}}(\dot{\pi})$ that $J(f)$ has an expansion

$$\int_{Z(F)\backslash Z(\mathbf{A})^1} \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\dot{\pi}\in\Pi_{\mathrm{disc}}(M)} \int_{i\mathfrak{a}^*_{M,Z}\backslash i\mathfrak{a}^*_{G,Z}} a^M_{\mathrm{disc}}(\dot{\pi}_\lambda) J_M(\dot{\pi}_\lambda, \dot{f}^1_z)\zeta(z) d\lambda dz$$

where

$$J_M(\dot{\pi}_\lambda, \dot{f}^1_z) = \mathrm{tr}(\mathscr{J}_M(\dot{\pi}_\lambda,P)\mathscr{I}_P(\dot{\pi}_\lambda,\dot{f}^1_z))$$

is the global unnormalized weighted character on $\mathscr{C}^\circ(G)$. It is a consequence of [FLM11, §5.1] that the inner integral converges absolutely. On the other hand, the integral over $Z(F)\backslash Z(\mathbf{A})^1$ annihilates the contribution of $\dot{\pi}$ coming from the complement of $\Pi_{\mathrm{disc}}(M,\zeta)$ in $\Pi_{\mathrm{disc}}(M)$, hence $J(f)$ equals

$$\sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\dot{\pi}\in\Pi_{\mathrm{disc}}(M,\zeta)} \int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} a^M_{\mathrm{disc}}(\dot{\pi}_\lambda) J_M(\dot{\pi}_\lambda,\dot{f})d\lambda. \tag{3.8}$$

Then arguing as in [Art02, Proposition 3.3], it follows from the definition of $a^M(\pi)$ that the spectral expansion (3.8) equals

$$\sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(M,V,\zeta)} a^M(\pi) J_M(\pi,f)d\pi$$

where

$$J_M(\pi_\lambda,f) = \mathrm{tr}(\mathscr{M}_M(\pi_\lambda,P)\mathscr{I}_P(\pi_\lambda,f)), \qquad L\in\mathscr{L}(M), P\in\mathscr{P}(L)$$

is the local normalized weighted character. It is related to the global unnormalized character by the formula

$$J_M(\dot{\pi}_\lambda,\dot{f}) = \sum_{L\in\mathscr{L}(M)} r^L_M(c_\lambda) J_L(\pi^L_\lambda, f),$$

and hence is defined for f belonging to $\mathscr{C}^\circ(G,V,\zeta)$. Also, the operator $\mathscr{J}_Q(\Lambda,\dot{\pi}_\lambda,P)$ is a scalar multiple of $\mathscr{M}_Q(\Lambda,\pi_\lambda,P)$, that is,

$$\mathscr{J}_Q(\Lambda,\dot{\pi}_\lambda,P) = r_Q(\Lambda,c_\lambda,P)\mu_Q(\Lambda,c_\lambda,P)\mathscr{M}_Q(\Lambda,\pi_\lambda,P),$$

where the coefficient $r_Q(\Lambda, c_\lambda, P)$ is defined in [Art98, §2], and it follows then that the integral over $\Pi(M,V,\zeta)$ converges absolutely. □

3.3. **The invariant expansion.** Given the noninvariant linear form J on $\mathscr{H}(G,V,\zeta)$, we have already discussed the invariant linear form I also on $\mathscr{H}(G,V,\zeta)$ obtained by setting inductively

$$I(f) = J(f) - \sum_{M \neq G} |W_0^M||W_0^G|^{-1} \hat{I}_M(\phi_M(f)) \tag{3.9}$$

for the maps ϕ_M described in (3.4).

Proposition 3.5. *The invariant linear form I on $\mathscr{H}(G,V,\zeta)$ extends to a continuous linear form on $\mathscr{C}^\circ(G,V,\zeta)$. It has the spectral and geometric expansions given by*

$$\begin{aligned}&\sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(M,V,\zeta)} a^M(\pi) I_M(\pi,f)d\pi \\ &= \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in\Gamma(M,V,\zeta)} a^M(\gamma) I_M(\gamma,f).\end{aligned}$$

Proof. We recall that for any $\tilde{f} \in \mathscr{C}(G_V,\zeta_V)$, the function $\phi_M(\tilde{f})$ is defined to be the function on $\Pi_{\text{temp}}(M_V^Z,\zeta_V)$ whose value at $\tilde{\pi}$ is the tempered distribution $J_M(\tilde{\pi},\tilde{f})$ [Art94, §2], and

$$\phi_M(f,\pi) = \int_{i\mathfrak{a}^*_{M,Z}} \phi_M(\tilde{f},\tilde{\pi}) d\lambda$$

where f and π are the restrictions of $\tilde{f}$ and $\tilde{\pi}$ to G_V^Z and M_V^Z respectively. We also define

$$\phi_M(\tilde{f},\tilde{\pi},X) = J_M(\tilde{f},\tilde{\pi},X), \qquad X\in\mathfrak{a}_{M,V}$$

and $\phi_M(f,\pi,X)$ using

$$J_M(\pi,X,f) = \int_{i\mathfrak{a}^*_M} J_M(\pi_\lambda,f)\mathrm{e}^{-\lambda(X)}d\lambda$$

if $J_M(\pi_\lambda,f)$ is regular for $\lambda\in i\mathfrak{a}^*_M$. In this case, it follows from [Art98, Lemma 3.1] that ϕ_M maps $\mathscr{C}(G_V^Z,\zeta_V)$ continuously to $I\mathscr{C}(G_V^Z,\zeta_V)$. For general π in $\Pi(M,V,\zeta)$, it follows from the proof of Proposition 3.4 that $J_M(\pi,f)$ is well-defined for $f\in\mathscr{C}^\circ(G,V,\zeta)$, and moreover the integral

$$J_M(\tilde{f},\tilde{\pi},X) = \int_{i\mathfrak{a}^*_{M,V}/i\mathfrak{a}^*_{G,V}} J_M(\tilde{\pi}_\lambda,\tilde{f}^Z)e^{-\lambda(X)}d\lambda$$

converges absolutely. Here Z is the image in $\mathfrak{a}_{G,V}$ of X.

On the other hand, the weighted orbital integrals $J_M(\gamma,f)$ are tempered distributions on $\mathscr{C}(G,V,\zeta)$ as a consequence of [Art94, Theorem 4.1]. Altogether, it follows that the invariant distributions defined inductively by

$$I_M(\pi,f) = J_M(\pi,f) - \sum_{L\in\mathscr{L}^0(M)} \hat{I}^L_M(\pi,\phi_L(f))$$

and

$$I_M(\gamma,f) = J_M(\gamma,f) - \sum_{L\in\mathscr{L}^0(M)} \hat{I}^L_M(\gamma,\phi_L(f))$$

on either side of the invariant trace formula hold for functions f in $\mathscr{C}^\circ(G,V,\zeta)$.

Beginning with the linear form J on $\mathscr{C}^\circ(G,V,\zeta)$, we define the invariant linear form I as in (3.9). We can see that the absolute value of $I(f)$ extends to a continuous linear form on $\mathscr{C}^\circ(G,V,\zeta)$, by assuming inductively that the statement holds for $L\in\mathscr{L}^0$ then applying the continuity of the map ϕ_M on $\mathscr{C}(G_V^Z,\zeta_V)$ and the linear form J. But we shall also arrive at the same conclusion once we have obtained the desired expansions. Let us first show that $I(f)$ has the geometric expansion

$$I(f)=\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\sum_{\gamma\in\Gamma(M,V,\zeta)}a^M(\gamma)I_M(\gamma,f).$$

Assume inductively that the required expansion holds if G is replaced by any group $L\in\mathscr{L}^0$. Combining this with the geometric expansion (3.5) for J, we see then that $I(f)$ equals

$$\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\sum_{\gamma\in\Gamma(M,V,\zeta)}a^M(\gamma)\left(J_M(\gamma,f)-\sum_{L\in\mathscr{L}^0(M)}\hat{I}_M^L(\gamma,f)\right),$$

and by definition of $I_M(\gamma,f)$ this yields the required geometric expansion for $I(f)$. On the other hand, the spectral expansion

$$I(f)=\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\int_{\Pi(M,V,\zeta)}a^M(\pi)I_M(\pi,f)d\pi$$

follows in a similar manner. That is, assuming inductively that the required identity holds for $L\in\mathscr{L}^0$, and using the spectral expansion (3.7) for J it follows that $I(f)$ equals

$$\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\int_{\Pi(M,V,\zeta)}a^M(\gamma)\left(J_M(\pi,f)-\sum_{L\in\mathscr{L}^0(M)}\hat{I}_M^L(\pi,f)\right)d\pi.$$

Then by definition of $I_M(\pi,f)$ this yields the required spectral expansion for $I(f)$. □

As we have alluded to in the beginning, the extension of the linear form I to noncompactly-supported test functions in $\mathscr{C}^\circ(G,V,\zeta)$ does not allow for proper use of the basic function, which belongs to $\mathscr{C}^\circ(G)$ and is nontrivial at almost all places v. To correct for this, we have to reconsider the passage from $\mathscr{C}^\circ(G,\zeta)$ to $\mathscr{C}^\circ(G,V,\zeta)$, which requires, among other things, a reconsideration of the global geometric coefficients that depend on the finite set S in a complicated way.

4. Weighting the geometric side

The treatment of the geometric side is more involved. Let $\mathfrak{a}_0=\mathfrak{a}_{M_0}$, and let A_0 be the split component of the center of M_0. The summands in the geometric expansion

$$J(\dot{f}^1)=\sum_{\mathfrak{o}\in\mathscr{O}}J_{\mathfrak{o}}(\dot{f}^1) \tag{4.1}$$

are obtained by evaluating certain polynomials $J_{\mathfrak{o}}^T(\dot{f}^1)$ at a distinguished point $T=T_0$ in $\mathfrak{a}_0$. We agree to write $J_{\mathfrak{o}}(\dot{f}^1)=J_{\mathfrak{o}}^{T_0}(\dot{f}^1)$. More precisely, let $\dot{f}^1\in\mathscr{C}^\circ(G)$ and T be a point in the positive chamber $\mathfrak{a}_0^+$ in $\mathfrak{a}_0$ associated to P_0, suitably regular in the sense that its distance from the walls of $\mathfrak{a}_0^+$ is large. Then $J_{\mathfrak{o}}^T(\dot{f}^1)$ is the integral over $x \in G(F)\backslash G(\mathbf{A})^1$ of the function

$$\sum_{P \in \mathscr{P}} (-1)^{\dim \mathfrak{a}_P} \sum_{\delta \in P(F)\backslash G(F)} k_{\mathfrak{o},P}(\delta x)\hat{\tau}_P(H_P(\delta x) - T)$$

where $\hat{\tau}_P$ is the characteristic function of the set

$$\{H \in \mathfrak{a}_0 : \varpi(H) > 0, \varpi \in \hat{\Delta}_P\},$$

and

$$k_{\mathfrak{o},P}(\delta x) = \sum_{\substack{\gamma \in M_P(F) \\ I_P(\gamma)=\mathfrak{o}}} \int_{N_P(\mathbf{A})} \dot{f}^1(x^{-1}\gamma n x) dn.$$

Here $\hat{\Delta}_P$ is the basis of $\mathfrak{a}_P^*/\mathfrak{a}_G^*$ which is dual to the simple roots Δ_P of (P, A_P). As a function of T, $J_{\mathfrak{o}}^T(\dot{f}^1)$ is a polynomial of degree at most $d_0 = \dim \mathfrak{a}_0$, thus it can be extended to all $T \in \mathfrak{a}_0$. Our present goal is to provide an expansion for (4.1) as a distribution on $\mathscr{C}^\circ(G)$ in terms of local distributions. According to the proof of Proposition 3.2, we can express the geometric side as

$$J(\dot{f}^1) = \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\dot{\gamma} \in (M(F))_{M,S}} a^M(\dot{\gamma}) J_M(\dot{\gamma}, \dot{f}_S^1),$$

and from [Art05, §22], it follows that the limit

$$\lim_S J(\dot{f}^1)$$

taken over increasing sets S, stabilizes for large finite S. Thus, in principle it may be possible to make use of the basic function in the form of $\dot{f}_s^r$ in the above limit, but we would like to have a more explicit form. For this, we shall revisit the refinement of the coarse geometric expansion.

4.1. **Unipotent terms.** We first have to deal with the unipotent contribution, which is the most delicate. It corresponds to the term

$$J_{\mathrm{unip}}^T(\dot{f}^1) = J_{\mathfrak{o}}^T(\dot{f}^1)$$

where $\mathfrak{o} = \mathscr{U}_G(F)$, the Zariski closure in G of the unipotent set in $G(F)$. It is a closed algebraic subvariety of G defined over F, and is one of the classes in $\mathscr{O}$. We recall that the distribution $J_{\mathrm{unip}}^T(\dot{f})$ is obtained by integrating an alternating sum over standard parabolic subgroups, whose leading term is given by

$$K_{\mathrm{unip}}(x, x) = \sum_{\gamma \in \mathscr{U}_G(F)} \dot{f}^1(x^{-1}\gamma x).$$

Let $(\mathscr{U}_G)$ be the set of $\mathrm{Gal}(\bar{F}/F)$-orbits of $\mathscr{U}_G$. Then the previous expression can be rewritten as the sum over $U \in (\mathscr{U}_G)$ of

$$K_U(x, x) = \sum_{\gamma \in U(F)} \dot{f}^1(x^{-1}\gamma x).$$

In order to establish the refined geometric expansion for functions $\dot{f} = f \times u^V$, where $f \in \mathscr{C}^\circ(G, V, \zeta)$ and u^V is the ζ^{-1}-equivariant characteristic function of the maximal compact subgroup K^V, we requier the existence of a measure on the unipotent variety for functions in $\mathscr{C}^\circ(G)$. We provide the argument here for $\dot{f}_s^r$, which we shall also need to construct our new geometric coefficients.

Fix a Euclidean norm $||\cdot||$ on $\mathfrak{a}_0$, and set $d(T)=\min_{\alpha\in\Delta_{P_0}}\{\alpha(T)\}$. Let Λ_d^T be the Arthur's truncation operator applied to the diagonal [Art85, p.1242]

Lemma 4.1. *There exist distributions J_U^T for each $U\in\mathscr{U}_G$ which are polynomials in T of total degree at most d_0 and such that*

$$J^T_{\mathrm{unip}}(\dot f^1)=\sum_U J_U^T(\dot f^1)$$

for $\dot f^1\in\mathscr{C}^\circ(G)$. Moreover, there is a continuous seminorm μ on $\mathscr{C}^\circ(G)$ and constants $\epsilon,\epsilon_0>0$ such that

$$\left|J_U^T(\dot f^1)-\int_{G(F)\backslash G(\mathbf{A})^1}\Lambda_d^T K_U(x,x)dx\right|\le\mu(\dot f^1)e^{\epsilon d(T)} \tag{4.2}$$

for all $U\in(\mathscr{U}_G)$, $\dot f^1\in\mathscr{C}^\circ(G)$ and every suitably regular T with $d(T)\ge\epsilon_0||T||$.

Proof. The proof of this statement is a mild generalization of [Art85, Theorem 4.2], where instead of the convergence estimate [Art85, Theorem 3.1] for $\dot f^1\in C_c^\infty(G)$ we shall rely on [FL16, Theorem 7.1] for $\dot f^1\in\mathscr{C}^\circ(G)$.

Fix an orbit $U\in(\mathscr{U}_G)$. It is a locally closed subset of G, defined over F, and its Zariski closure $\bar U$ is a closed subvariety of G, again defined over F. The ideal of polynomial functions on G which vanish on U is of the form $(q_1,\dots,q_l)$, where $q_1,\dots,q_l$ are polynomials on G defined over F. If v is nonarchimedean valuation of F, we define ρ_v to be the characteristic function of $[-1,1]$, and if v is archimedean, define ρ_v to be any function such that $0\le\rho_v\le1$, equal to 1 on $[-\frac12,\frac12]$ and zero outside of $[-1,1]$.Then for any $\dot f^1\in\mathscr{C}^\circ(G)$ and $\epsilon>0$, we define the truncated function

$$\dot f^{1,\epsilon}_{U,v}(x)=\dot f^1(x)\rho_v(\epsilon^{-1}|q_1(x)|_v)\cdots\rho_v(\epsilon^{-1}|q_l(x)|_v)$$

where $x\in G(\mathbf{A})^1$. It again belongs to $\mathscr{C}^\circ(G)$, and equals f in a sufficiently small neighborhood of $\bar U(\mathbf{A})$.

Let v be any valuation of F. We shall construct J_U^T by examining the behavior of $J^T_{\mathrm{unip}}(\dot f^{1,\epsilon}_{U,v})$ as ϵ approaches zero. Let us write

$$K_{\bar U}(x,x)=\sum_{\{U'\in(\mathscr{U}_G):U'\subset\bar U\}}K_{U'}(x,x).$$

It will suffice to show there exists a continuous seminorm on $\mathscr{C}^\circ(G)$ such that for all $\dot f^1\in\mathscr{C}^\circ(G)$, the difference

$$\left|J^T_{\mathrm{unip}}(\dot f^{1,\epsilon}_{U,v})-\int_{G(F)\backslash G(\mathbf{A})^1}\Lambda_d^T K_{\bar U}(x,x)dx\right| \tag{4.3}$$

is bounded by

$$\mu(\dot f^1)\delta^{rm}(1+||T||)^{d_0}, \tag{4.4}$$

for some δ such that $0<\delta<1$, $r\ge0$, and m large enough. The desired result will then follow by the same argument as in the proof of [Art85, Theorem 4.2].

Given standard parabolic subgroups $P_1\subset P_2$, we write $A^\infty_{P_1}$ for the identity component $A_{P_1}(\mathbf{R})^0$ of $A_{P_1}(\mathbf{R})$, and $A^\infty_{P_1,P_2}=A^\infty_{P_1}\cap M_{P_2}(\mathbf{A})^1$. Moreover, given $T_1,T_2\in\mathfrak{a}_0$, we denote by $A^\infty_{P_1,P_2}(T_1,T)$ the set

$$\{a\in A^\infty_{P_1,P_2}:\alpha(H_{P_1}(a)-T_1)>0,\alpha\in\Delta^{P_1\cap M_{P_2}};\varpi(H_{P_1}(a)-T)<0,\varpi\in\hat\Delta_{P_1\cap M_{P_2}}\}.$$

Let $T \in \mathfrak{a}_0$ be a suitably regular point. We define $F(x,T)$ to be the characteristic function of the compact subset of $G(F)\backslash G(\mathbf{A})^1$ obtained by the projection

$$N_0(\mathbf{A})M_0(\mathbf{A})A^{\infty}_{P_0,G}(T_1,T)K \to G(F)\backslash G(\mathbf{A})^1.$$

Using [Art85, Lemma 2.3], which states that

$$\int_{G(F)\backslash G(\mathbf{A})^1} \Lambda^T_d K_U(x,x)dx = \int_{G(F)\backslash G(\mathbf{A})^1} \Lambda^T_d F(x,T)\left(\sum_{\gamma\in U(F)} \dot{f}^1(x^{-1}\gamma x)\right)dx$$

and the property that

$$K_{\bar{U}}(x,x) = \sum_{\gamma\in\bar{U}(F)} f(x^{-1}\gamma x) = \sum_{\gamma\in\bar{U}(F)} \dot{f}^{1,\epsilon}_{U,v}(x^{-1}\gamma x),$$

we may bound the difference (4.3) by the sum of

$$\left| J^T_{\text{unip}}(\dot{f}^{1,\epsilon}_{U,v}) - \int_{G(F)\backslash G(\mathbf{A})^1} F(x,T)\left(\sum_{\gamma\in\mathscr{U}_G(F)} \dot{f}^{1,\epsilon}_{U,v}(x^{-1}\gamma x)\right)dx\right| \tag{4.5}$$

and

$$\int_{G(F)\backslash G(\mathbf{A})^1} F(x,T)\sum_{\gamma\in\mathscr{U}_G(F)\backslash\bar{U}(F)} |\dot{f}^1(x^{-1}\gamma x)|dx. \tag{4.6}$$

The first expression (4.5) is bounded by

$$\mu(\dot{f}^{1,\epsilon}_{U,v})(1+||T||)^{d_0}e^{-d(T)}$$

for some continuous seminorm μ_1 on $\mathscr{C}^\circ(G)$, as an application of [FL16, Theorem 7.1]. Replacing the seminorm $\mu(f)$ with $\mu(f)N(f)^n$ where the $N(f)$ is defined according to [Art85, p.1257] and for n large enough, we can apply [Art85, Corollary 3.3], which remains valid for functions in $\mathscr{C}^\circ(G)$, to conclude that the latter expression is bounded by

$$\epsilon^{-l}\mu(\dot{f}^1)(1+||T||)^{d_0}e^{-d(T)}.$$

On the other hand, the second expression (4.6) is bounded by

$$\mu(\dot{f}^1)(1+||T||)^{d_0}\epsilon^r$$

for some $r>0$, using [Art85, Lemma 4.1], which holds also in our case as the characteristic function $F(x,T)$ implies that the integral is taken over a compact set. Taking $\epsilon=\delta^m$ then, the required bound (4.4) follows. □

We shall apply the lemma to obtain the following expansion for the unipotent term.

Proposition 4.2. *Fix a representation r of ${}^LG$ and $s\in\mathbf{C}$ with $\mathrm{Re}(s)$ large enough. Then for any S, there are uniquely determined numbers*

$$a^M_{r,s}(S,u), \qquad M\in\mathscr{L},\ u\in(\mathscr{U}_M(F))_{M,S}$$

such that

$$J^L_{\text{unip}}(\dot{f}^1_{r,s}) = \sum_{M\in\mathscr{L}} |W^M_0||W^L_0|^{-1}\sum_{u\in(\mathscr{U}_M(F))_{M,S}} a^M_{r,s}(S,u)J_M(u,\dot{f}^1_S) \tag{4.7}$$

for any $L\in\mathscr{L}$ and $\dot{f}^1_{r,s}=\dot{f}^1_S\times b^{S,1}_{r,s}$ with $\dot{f}^1_S\in\mathscr{C}^\circ(G^1_S)$.

Proof. Assume inductively that the result holds for any Levi M properly containing L. Define $T^L(\dot{f}^1_{r,s})$ to be the difference

$$J^L_{\text{unip}}(\dot{f}^1_{r,s}) - \sum_{\substack{M\in\mathscr{L}^L\\ M\neq L}} |W_0^M||W_0^L|^{-1} \sum_{u\in(\mathscr{U}_M(F))_{M,S}} a^M_{r,s}(S,u)J^L_M(u,\dot{f}^1_S)$$

for $\dot{f}$ as above. We can thus view T^L as a distribution on L^1_S that annihilates any function which vanishes on $\mathscr{U}_L(F_S)$. It is an invariant distribution by the same argument in pp.1269–1270 of [Art85]. We need to show that there exist uniquely determined numbers $a^L_{r,s}(S,u)$ such that

$$T^L(\dot{f}^1_{r,s}) = \sum_{u\in(\mathscr{U}_L(F))_{L,S}} a^L_{r,s}(S,u)J^L_L(u,\dot{f}^1_S). \tag{4.8}$$

The uniqueness follows from the linear independence of the invariant orbital integrals $J^L_L(u)$, thus it remains to prove their existence.

For any integer d, let $\mathscr{U}_{L,d}$ be the union of orbits U in $(\mathscr{U}_L)$ of maximal dimension d. The set

$$\mathscr{U}_{L,d}(F_S) = \prod_{v\in S}\mathscr{U}_{L,d}(F_v)$$

of F_S-valued points is a closed subspace of L_S consisting of a finite union of L_S-conjugacy classes. Let $\mathscr{U}_{L,d}(F_S)'$ denote the union over orbits $U\in(\mathscr{U}_L)$ such that $\dim(U)\le d$ and such that $U(F)$ is nonempty, of the spaces $U(F_S)$. It is the union of L_S-conjugacy classes parametrized by elements $u\in(\mathscr{U}_L(F))_{L,S}$ with $\dim(U^L_u)\le d$. We see then that if there exist numbers

$$a^L_{r,s}(S,u), \qquad u\in(\mathscr{U}_L(F))_{L,S}$$

such that for any d the distribution

$$T^L_d(\dot{f}^1_{r,s}) = T^L(\dot{f}^1_{r,s}) - \sum_{\substack{u\in(\mathscr{U}_L(F))_{L,S}\\ \dim(U^L_u)>d}} a^L_{r,s}(S,u)J^L_L(u,\dot{f}^1_S)$$

annihilates any function $\dot{f}^1_S\in\mathscr{C}^\circ(G^1_S)$ which vanishes on $\mathscr{U}_{L,d}(F_S)$, the required expression (4.8) will follow.

If $d\ge\dim(\mathscr{U}_L)$, then $\mathscr{U}_{L,d}(F_S)'$ is the union of spaces U_S such that $U(F)$ is not empty, and $T^L_d(\dot{f}^1_{r,s}) = T^L(\dot{f}^1_{r,s})$. In this case, $T^L_d(\dot{f}^1_{r,s})$ is the difference between the distribution obtained in Lemma 4.1,

$$J^L_{\text{unip}}(\dot{f}^1_{r,s}) = \sum_{U\in(\mathscr{U}_L)} J^L_U(\dot{f}^1_{r,s}),$$

and a sum of integrals over $U(F_S)$ for which $U(F)$ is nonempty. Since J^L_U is zero when $U(F)$ is empty, it follows that T^L_d annihilates any function which vanishes on $\mathscr{U}_{L,d}(F_S)'$.

If $d>\dim(\mathscr{U}_L)$, assume inductively that $a^L_{r,s}(S,u)$ is defined for any u with $\dim(U^L_u)>d$ and $\frac{L}{d}$ annihilates any function which vanishes on $\mathscr{U}_{L,d}(F_S)'$. Let $\mathscr{U}^0_{L,d}$ be the union over orbits U in $(\mathscr{U}_L)$ with $\dim(U)=d$, and let C^d be the complement of $\mathscr{U}^0_{L,d}(F_S)$ in $\mathscr{U}_{L,d}(F_S)$. Thus C^d equals to union over $v\in S$ and

$U \in (\mathscr{U}_L)$ with $\dim(U) < d$ of the sets

$$C^d_{U,v} = U(F_v) \prod_{\substack{w \in S \\ w \neq v}} \mathscr{U}_{L,d}(F_w).$$

it is a closed subset of $L(F_S)^1$. We shall first consider the restriction of T^L_d to the complement of C^d in L^1_S. The space

$$\mathscr{U}_{L,d}(F_S)' \backslash C^d = \mathscr{U}_{L,d}(F_S)' \cap \mathscr{U}^0_{L,d}(F_S)$$

is a disjoint union of L_S-conjugacy classes which are closed in $L^1_S \backslash C^d$. The conjugacy classes are parametrized by $u \in (\mathscr{U}_L(F))_{L,S}$ such that $\dim(U^L_u) = d$. For each such u, let L_u be the centralizer in L of a fixed representative of u in $L(F)$. There is a surjective L^1_S-equivariant map

$$\mathscr{C}^\circ(L^1_S)\backslash C^d \to \bigoplus_u \mathscr{C}^\circ(L_S/L_{u,S}),$$

with kernel consisting of functions in $\mathscr{C}^\circ(L^1_S \backslash C^d)$ which vanish on $\mathscr{U}_{L,d}(F_S)'$. We may view any function that annihilates the kernel as the pullback of an L^1_S-equivariant distribution on the right-hand side. It follows then that we can choose constants $a^L_{r,s}(S,u)$ for each $u \in (\mathscr{U}_L(F))_{L,S}$ with $\dim(U^L_u) = d$ such that

$$T^L_d(\dot{f}^1_{r,s}) = \sum_{\substack{u \in (\mathscr{U}_L(F))_{L,S} \\ \dim(U^L_u) = d}} a^L_{r,s}(S,u) J^L_L(\dot{f}^1_S)$$

for any $\dot{f}^1_S \in \mathscr{C}^\circ(L^1_S \backslash C^d)$.

On the other hand, if f is any arbitrary function in $\mathscr{C}^\circ(L^1_S)$, we set

$$T^L_{d-1}(\dot{f}^1_{r,s}) = T^L_d(\dot{f}^1_{r,s}) - \sum_{\substack{u \in (\mathscr{U}_L(F))_{L,S} \\ \dim(U^L_u) = d}} a^L_{r,s}(S,u) J^L_L(\dot{f}^1_S).$$

Then $T^L_{d-1}(\dot{f}^1_{r,s})$ is an invariant distribution supported on C^d which annihilates any function that vanishes on $\mathscr{U}_{L,d}(F_S)'$. By the inductive assumption, it will suffice to show that

$$T^L_{d-1}(\dot{f}^1_{r,s}) = 0$$

for any function $\dot{f}^1_S$ that vanishes on $\mathscr{U}_{L,d-1}(F_S)'$. Consider then the sets $C^d_{U,v}$ for which $\dot{f}^1_S$ does not vanish on a neighborhood of the closures $\bar{C}^d_{U,v}$. If no such set exists, then $\dot{f}^1_S$ belongs to $\mathscr{C}^\circ(L^1_S \backslash C^d)$ and hence $T^L_{d-1}(\dot{f}^1_{r,s}) = 0$. Otherwise, let there be $(k+1)$ such sets with $k \geq 0$. Let $C^d_{U,v}$ be one such set. Then for any $\epsilon > 0$, we have that $f^\epsilon_{U,v}$ is equal to f in a neighborhood of $\bar{C}^d_{U,v}$, and $(f - f^\epsilon_{U,v})$ vanishes in a neighborhood of the closure of all but at most k sets. We may assume inductively then that

$$T^L_{d-1}(\dot{f}^1_{r,s} - (\dot{f}^1_{r,s})^\epsilon_{U,v}) = 0.$$

On the one hand, $T^L_{d-1}(\dot{f}^1_{r,s})$ is the difference between $J^L_{\mathrm{unip}}(\dot{f}^1_{r,s})$ and

$$\sum_{\substack{M \subset L \\ M \neq L}} |W^M_0||W^L_0|^{-1} \sum_{u \in (\mathscr{U}_M(F))_{M,S}} a^M_{r,s}(S,u) J^L_M(u, \dot{f}^1_S) + \sum_{\substack{u \in (\mathscr{U}_L(F)_{L,S} \\ \dim(U^L_u) \geq d}} a^L_{r,s}(S,u) J^L_L(u, \dot{f}^1_S).$$

On the other hand, using the property that

$$\lim_{\epsilon\to 0} J_M(u, (\dot f^1_{r,s})^\epsilon_{U,v}) = J_M(u, \dot f^1_{r,s}),$$

if $U^G_u \subset \bar U$, otherwise it is zero, and

$$\lim_{\epsilon\to 0} J^T_{\text{unip}}((\dot f^1_{r,s})^\epsilon_{U,v}) = \sum_{\{U'\in(\mathscr{U}_G): U'\subset \bar U\}} J^T_{U'}(\dot f^1_{r,s})$$

for any valuation v, which follows from Lemma 4.1 by the same argument as [Art85, Corollary 4.3], we then have that

$$\lim_{\epsilon\to 0} T^L_{d-1}((\dot f^1_{r,s})^\epsilon_{U,v})$$

is equal to

$$J^L_{\bar U}(\dot f^1_{r,s}) - \sum_{\substack{M\subset L\\ M\neq L}} |W^M_0||W^L_0|^{-1} \sum_{\substack{u\in(\mathscr{U}_M(F))_{M,S}\\ U^L_u\subset \bar U}} a^M_{r,s,}(S,u) J^L_M(u, \dot f^1_S)$$

for $\dim U < d$. Since $\dot f^1_{r,s}$ vanishes on $\mathscr{U}_{L,d-1}(F)'$, it follows from (4.2) that $J^L_{\bar U}(\dot f^1_{r,s}) = 0$. The other terms in the preceding expression also vanish, thus $T^L_{d-1}(\dot f^1_{r,s}) = 0$ as desired. □

Specializing to the case $L = G$, we have our desired expression for the unipotent contribution.

Corollary 4.3. *For any $\dot f^1_S \in \mathscr{C}^\circ(G^1_S)$, we have*

$$J_{\text{unip}}(\dot f^1_{r,s}) = \sum_{M\in\mathscr{L}} |W^M_0||W^G_0|^{-1} \sum_{u\in(\mathscr{U}_M(F))_{M,S}} a^M_{r,s}(S,\dot u) J_M(\dot u, \dot f^1_S).$$

In particular, the restriction of J_{unip} to G^1_S is a measure.

4.2. **Refinement.** Having treated the unipotent terms, the rest of the geometric expansion follows naturally. Let M be a Levi subgroup of G, and c a semisimple element of $M(F)$. For any $c \in G$, we denote by $G_{c,+}$ the centraliser of c in G, and G_c the connected component of the identity in $G_{c,+}$. We say a semisimple element c is F-elliptic in G if $A_{G_c} = A_G$. We recall that two elements $\dot\gamma$ and $\dot\gamma_1$ in $G(F_S)$ with standard Jordan decompositions $\dot\gamma = c\dot\alpha$ and $\dot\gamma_1 = c_1\dot\alpha_1$ are said to be (G,S)-equivalent if there is an element $\dot\delta \in G(F)$ such that $\dot\delta^{-1}c_1\delta = c$ and $\dot\delta^{-1}\alpha_1\delta$ is conjugate to α in $G_c(F_S)$. For a general element $\dot\gamma = c\dot\alpha$, we define the general coefficient by the descent formula

$$a^G_{r,s}(S,\dot\gamma) = i^G(S,c)|\text{Stab}(c,\dot\alpha)|^{-1} a^{G_c}_{r,s}(S,\dot\alpha) \tag{4.9}$$

where $\text{Stab}(c,\dot\alpha)$ denotes the stabilizer of $\dot\alpha$ in the finite group $G_{c,+}(F)/G_c(F)$, and $i^G(S,c)$ is equal to 1 if c is F-elliptic in G and the $G(\mathbf{A})^S$-conjugacy class of c meets K^S, and is zero otherwise.

Proposition 4.4. *Let $\dot f^1_S \in \mathscr{C}^\circ(G^1_S)$. We then have*

$$J(\dot f^1_{r,s}) = \sum_{M\in\mathscr{L}} |W^M_0||W^G_0|^{-1} \sum_{\dot\gamma\in(M(F))_{M,S}} a^M_{r,s}(S,\dot\gamma) J_M(\dot\gamma, \dot f^1_S).$$

Proof. The result will follow from the proof of [Art86, Theorem 9.2] if we can show that for each $\mathfrak{o} \in \mathscr{O}$, there is a finite set $S_{\mathfrak{o}}$ containing the archimedean places, such that for any finite set $S \supset S_{\mathfrak{o}}$ and $f \in \mathscr{C}^\circ(G_S^1)$,

$$J_{\mathfrak{o}}(f) = \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(F)\cap\mathfrak{o})_{M,S}} a_{r,s}^M(S,\dot{\gamma}) J_M(\dot{\gamma}, \dot{f}_S^1)).$$

This in turn follows from the proof of [Art86, Theorem 8.1], where we need only indicate the changes that must be made in our setting. To that end, fix a semisimple element $c \in \mathfrak{o}$ such that $c \in M_{P_1}(F)$ for a fixed standard parabolic P_1, and such that it does not belong to any proper parabolic subset of $M_1 = M_{P_1}$. Also let $\iota^G(c) = |G_{c,+}(F)/G_c(F)|$, and let T be a suitably regular point in $\mathfrak{a}_0$. Then following [Art86, Lemma 3.1], the distribution $J_{\mathfrak{o}}^T(f)$ can be expressed as the integral over x in $G(F)\backslash G(\mathbf{A})^1$, and the sums over standard parabolic subgroups R of G_c and elements $\xi \in R(F)\backslash G(F)$ of the product of

$$|\iota^G(c)|^{-1} \sum_{u \in M_R(F)} \int_{N_R(\mathbf{A})} f(x^{-1}\xi^{-1}cun\xi x)dn$$

with

$$\sum_{P \in \mathscr{F}_R(M_1)} (-1)^{\dim(A_P/A_G)} \hat{\tau}_P(H_P(\xi x) - Z_P(T-T_0) - T_0).$$

Here $\mathscr{F}_R(M_1)$ is the set of parabolic subsets P with Levi factor M_1 with centralizer $P_c = R$, and Z_P is defined in [Art86, (3.3)].

Let $P_{1c} = P_1 \cap G_c$, and let M_{1c} be its Levi factor. Let $\mathscr{F}^c$ be the set of parabolic subgroups of G_c containing M_{1c}. Let

$$\mathscr{Y}_R^T(\delta x, y) = \{Y_P^T(\delta x, y) : P \in \mathscr{F}_R(M_1)\},$$

where

$$Y_P^T(\delta x, y) = -H_P(K_{P_c}(\delta x)y) - Z_P(T-T_0) - T_c + T_0),$$

and $K_{P_c}(x)$ is the component of x in K_c relative to the decomposition $G_c(\mathbf{A}) = N_{P_c}(\mathbf{A})M_{P_c}(\mathbf{A})K_c$. We would like to rewrite $J_{\mathfrak{o}}^T(f)$ using a series of changes of variables as

$$|\iota^G(c)|^{-1} \int_{G_c(\mathbf{A})\backslash G(\mathbf{A})} \sum_{\{S \in \mathscr{F}^c : S \supset P_{1c}\}} \left(\int_{K_c} \int_{A_S^\infty \cap G(\mathbf{A})^1} J_{\text{unip}}^{M_S, T_c}(\Phi_{S,a,k,y}^T) da\ dk \right) dy$$

where $\Phi_{S,a,k,y}^T(m), m \in M_S(\mathbf{A})^1$ is given by

$$\Gamma_S^G(H_S(a) - T_c, \mathscr{Y}_S^T(k,y))\delta_S(m)^{\frac{1}{2}} \int_{N_S(\mathbf{A})} f(y^{-1}ck^{-1}mnky)dn$$

and $\Gamma_S^G(X, \mathscr{Y}_R)$ is a compactly supported function on $X \in \mathfrak{a}_R^G$, depending continuously on $\mathscr{Y}_R$ defined in [Art86, §4]. We can do so by applying the combinatorial arguments of [Art86, §6], noting that the integral over y remains absolutely integrable by the rapid decay of f, and that a weaker form [Art86, Lemma 6.1] holds by a similar argument (and simpler, as we do not require compactness). Namely, given a subset Δ of $G(\mathbf{A})^1$ we can choose a subset Σ of $G_c(\mathbf{A})\backslash G(\mathbf{A})$ such that $y^{-1}c\mathscr{U}_{G_c}(\mathbf{A})y \cap \Delta$ is empty unless y belongs to Σ. We can then choose $S_{\mathfrak{o}}$ to be the finite set of valuations as described in [Art86, p.203] (see also [Art02, p.193]). □

We have the following formula for the global geometric coefficient in the case of semisimple elements.

Corollary 4.5. *Let $\dot\gamma \in G(F)$ be a semisimple element. Then for any finite set $S \supset S_{\mathfrak{o}}$, we have*

$$a^G_{r,s}(S,\dot\gamma) = |G_{\dot\gamma,+}(F)/G_{\dot\gamma}(F)|^{-1}\mathrm{vol}(G_{\dot\gamma}\backslash G_{\dot\gamma}(\mathbf{A})^1)b^S_{r,s}(1)$$

if $\dot\gamma$ is F-elliptic, and zero otherwise.

Proof. Let us first show that

$$a^G_{r,s}(S,1) = a^G(S,1)b^S_{r,s}(1) = \mathrm{vol}(G(F)\backslash G(\mathbf{A})^1)b^S_{r,s}(1).$$

Notice that if U is the trivial unipotent class, then

$$\int_{G(F)\backslash G(\mathbf{A})} \Lambda^T_d K_U(x,x)dx = \dot f^1_{r,s}(1)\int_{G(F)\backslash G(\mathbf{A})^1} F(x,T)dx,$$

and by the dominated convergence theorem, we see that $J^T_{\{1\}}(\dot f^1_{r,s})$ is equal to the product of $\mathrm{vol}(G(F)\backslash G(\mathbf{A})^1)$ with $b^S_{r,s}(1)\dot f^1_S(1)$. From Proposition 4.2 we deduce the desired expression for trivial $\dot\gamma$, and the claim then follows from the descent formula (4.9). ☐

4.3. **The invariant geometric expansion.** Following [Art02, §1], we now want to reindex the geometric terms in a difference way. Let $\mathscr{D}(G^Z_V,\zeta_V)$ be the space of ζ_V-equivariant distributions that are invariant under G^Z_V-conjugation and supported on the preimage in G^Z_V of a finite union of conjugacy classes in $\bar G^Z_V = G^Z_V/Z_V$. Let $\mathscr{D}_{\mathrm{orb}}(G^Z_V,\zeta_V)$ be the subspace spanned by distributions

$$f \mapsto \int_{Z_V}\zeta_V(z)f_G(z\gamma_V)dz, \qquad \gamma_V \in G^Z_V$$

where

$$f_G(\gamma_V) = |D(\gamma_V)|^{1/2}\int_{G_{\gamma_V}\cap G^Z_V\backslash G^Z_V} f(x^{-1}\gamma_V x)dx,$$

for $G_{\gamma_V} = \prod_{v\in V} G_{\gamma_v}$ and $|D(\gamma_V)| = \prod_{v\in V}|D(\gamma_v)|_v$ is the usual discriminant. Let $\Gamma(G^Z_V,\zeta)$ be a fixed basis of $\mathscr{D}(G^Z_V,\zeta_V)$, and let $\Gamma_{\mathrm{orb}}(G^Z_V,\zeta_V) = \Gamma(G^Z_V,\zeta_V)\cap \mathscr{D}_{\mathrm{orb}}(G^Z_V,\zeta_V)$. Let $(\dot\gamma_S/\dot\gamma)$ be the ratio of the invariant measure on $\dot\gamma_S$ and the signed measure on $\dot\gamma_S$ that comes with $\dot\gamma$, so that $f_G(\dot\gamma_S) = (\dot\gamma_S/\dot\gamma)f_G(\dot\gamma)$ for any $f \in \mathscr{C}(G^Z_V,\zeta_V)$ and $\dot\gamma \in \Gamma_{\mathrm{orb}}(\bar G^Z_V,\zeta_V)$. For any $f \in \mathscr{C}(G,V,\zeta)$ and $\gamma \in \Gamma(M^Z_V,\zeta_V)$, we inductively define the linear forms

$$I_M(\gamma,f) = J_M(\gamma,f) - \sum_{L\in\mathscr{L}^0(M)}\hat I^L_M(\gamma,\phi_L(f))$$

where $J_M(\gamma,f)$ is the generalized weighted orbital integral defined in [Art02, §2] and ϕ_L is the map defined in (3.4).

We define the elliptic coefficients

$$a^G_{r,s,\mathrm{ell}}(\dot\gamma_S) = \sum_{\{\dot\gamma\}}|Z(F,\dot\gamma)|^{-1}a^G_{r,s}(S,\dot\gamma)(\dot\gamma/\dot\gamma_S) \tag{4.10}$$

where the sum over $\{\dot\gamma\}$ runs over a set of representatives of $Z_{S,\mathfrak{o}}$-orbits in $(G(F))_{G,S}$, and $Z(F,\dot\gamma)$ is the subset of $z \in Z_{S,\mathfrak{o}}$ such that $z\dot\gamma = \dot\gamma$. We note that $a^G_{r,s}(S,\dot\gamma)$ exists for any S-admissible element $\dot\gamma \in G(F)$, by [Art02, Lemma 2.1] and the analogue of [Art86, Lemma 7.1] for $f \in \mathscr{C}^\circ(G(F_S)^1)$ which follows from the proof of Proposition 4.4.

Let $V_{\mathrm{ram}}(G,\zeta)$ be the finite set of valuations of F outside of which G and ζ are unramified. We shall fix a subset V of S containing $V_{\mathrm{ram}}(G,\zeta)$, and that f is S-admissible in the sense of [Art02, §1]. We specialize the test function $\dot{f}_s^r = f \times b_S^V$, hence

$$f \to \dot{f}_s^r = f \times b_S^V$$

which gives a map from $\mathscr{C}^\circ(G,V,\zeta)$ to $\mathscr{C}^\circ(G,S,\zeta)$. We shall combine the elliptic coefficients with unramified weighted orbital integrals of basic functions at the places in $S-V$. Let $\mathcal{K}(\bar{G}_S^V)$ denote the set of conjugacy classes in $\bar{G}_S^V = G_S^V/Z_S^V$ that are bounded in the sense that for any v in $V-S$, the image of any representative lies in a compact subgroup of G_v. Any element $k \in \mathcal{K}(\bar{G}_S^V)$ induces a distribution $\gamma_S^V(k)$ in $\Gamma_{\mathrm{orb}}(G_S^V,\zeta_S^V)$. Given γ in $\Gamma(G_V^Z,\zeta_V)$, we write $\gamma \times k = \gamma \times \gamma_S^V(k)$ for the associated element in $\Gamma(G_S^Z,\zeta_S)$. Furthermore, let $\mathcal{K}_{\mathrm{ell}}^V(\bar{M},S)$ denote the elements in $\mathcal{K}(\bar{G}_S^V)$ such that $\gamma\times k$ belongs to $\Gamma_{\mathrm{ell}}(G,S,\zeta)$ for some γ. Here $\Gamma_{\mathrm{ell}}(G,S,\zeta)$ is the set of γ in $\Gamma_{\mathrm{orb}}(G_S^Z,\zeta_S)$ such that there is a $\dot{\gamma} \in G(F)$ such that (i) the semisimple part of $\dot{\gamma}$ is F-elliptic in G, (ii) the conjugacy class of $\dot{\gamma}$ in G_V maps to γ, and (iii) $\dot{\gamma}$ is bounded at each $v \notin S$. We can then define the unramified weighted orbital integrals

$$r_M^G(k,b) = J_M(\gamma_S^V(k), b_S^V), \qquad k \in \mathcal{K}(\bar{M}_S^V). \tag{4.11}$$

Now the set $\Gamma(G,V,\zeta)$ is given by the union of induced distributions μ^G where μ runs over elements in $\Gamma_{\mathrm{ell}}(M,V,\zeta)$ and M runs over Levis in $\mathscr{L}$. Recall that the induction is defined by the relation $f_G(\mu^G) = f_M(\mu)$ for any $f \in \mathscr{C}(G_V,\zeta_V)$, and $f \mapsto f_M$ is the canonical map from $\mathscr{C}(G_V,\zeta_V)$ to $I\mathscr{C}(G_V,\zeta_V)$ factoring through the map $f \mapsto f_G$. We also have the adjoint restriction map $\gamma \mapsto \gamma_M$ from $\mathscr{D}(G_V,\zeta_V)$ to $\mathscr{D}(M_V,\zeta_V)$, such that

$$\sum_{\gamma\in\Gamma(G_V,\zeta_V)} a_M(\gamma_M) b_G(\gamma) = \sum_{\mu\in\Gamma(M_V,\zeta_V)} a_M(\mu) b_G(\mu^G) \tag{4.12}$$

for any linear function a_M on $\mathscr{D}(M_V,\zeta_V)$ and b_G on $\mathscr{D}(G_V,\zeta_V)$. We then define for any $\gamma \in \Gamma(G_V^Z,\zeta_V)$, the geometric coefficient

$$a_{r,s}^G(\gamma) = \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{k\in\mathcal{K}_{\mathrm{ell}}^V(\bar{M},S)} a_{r,s,\mathrm{ell}}^M(\gamma_M \times k) r_M^G(k,b) \tag{4.13}$$

where S is any finite set of valuations of containing V such that $\gamma \times K^V$ is S-admissible. It follows from the definitions that $a_{r,s}^G(\gamma)$ is supported on the discrete subset $\Gamma(G,V,\zeta)$ of $\Gamma(G_V^Z,\zeta_V)$.

Theorem 4.6. *Let $f \in \mathscr{C}^\circ(G,V,\zeta)$. Then the invariant linear form $I_s^r(f)$ has the geometric expansion*

$$I_s^r(f) = \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in\Gamma(M,V,\zeta)} a_{r,s}^M(\gamma) I_M(\gamma,f) \tag{4.14}$$

Proof. Recall from Proposition 4.4 the expression

$$J(f_s^r) = J(\dot{f}_{r,s}^1) = \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\dot{\gamma}\in(M(F_S))_{F,S}} a_{r,s}^M(S,\dot{\gamma}) J_M(\dot{\gamma},\dot{f}_S^1).$$

For a fixed set of valuations S, the linear form $J(\dot{f}^1)$ is K^S-invariant, we may then write

$$J(f_s^r) = \int_{Z(F)Z(\mathfrak{o}^S)\backslash Z(\mathbf{A})^1} J(\dot{f}_{r,s,z}^1)\zeta(z)dz$$

as

$$\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\sum_{\dot\gamma\in(M(F))_{M,S}}a^M_{r,s}(S,\dot\gamma)\int_{Z_{S,\mathfrak{o}}\backslash Z_S^1}J_M(z\dot\gamma,\dot f_S^1)\zeta(z)dz$$

since $Z(\mathbf{A})=Z(F)Z_SZ(\mathfrak{o}^S)$ and $J_M(\dot\gamma,\dot f^1_{S,z})=J_M(z\dot\gamma,\dot f^1_S)$ for any $z\in Z_S$. The inner sum can be written as

$$\sum_{\{\dot\gamma\}}|Z(F,\dot\gamma)|^{-1}a^M_{r,s}(S,\dot\gamma)\int_{Z_S^1}J_M(z\dot\gamma,\dot f^1_S)\zeta(z)dz$$

Then applying the definition of (4.10), we have

$$J^r_s(f)=\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\sum_{\dot\gamma_S\in\Gamma_{\text{ell}}(M,V,\zeta)}a^M_{r,s,\text{ell}}(\dot\gamma_S)J_M(\dot\gamma,\dot f_S) \tag{4.15}$$

where $\dot f_S$ is the projection of $\dot f^1_S$ onto $\mathscr{H}(G,S,\zeta)$.

Since $\dot f_S$ is equal to $f\times b^V_S$, we claim that

$$J_M(\dot\gamma_S,\dot f_S)=\sum_{L\in\mathscr{L}(M)}J^L_M(\dot\gamma^V_S,(b^V_S)_L)J_L(\dot\gamma^L_V,f).$$

This follows from the descent and splitting properties of the weighted orbital integrals, stated in (18.7) and (18.8) of [Art05] or, more directly in [MW16b, VI.1.9(2)]). Here we use the fact that $(b^V_S)_L=(b^V_S)_Q$ is independent of $Q\in\mathscr{P}(L)$, where

$$f_Q(m)=\delta_Q(m)^{\frac12}\int_K\int_{N_Q(F_S)}f(k^{-1}mnk)dn\,dk,$$

since by (2.3) we see that the basic function depends only on the restriction to $T(F_v)_+$ for each $v\in V-S$. Moreover, $J^L_M(\dot\gamma^V_S,(b^V_S)_L)$ vanishes unless $\dot\gamma^V_S=\dot\gamma^V_S(k)$ for some $k\in\mathcal{K}(\bar M^V_S)$, in which case it equal $r^L_M(k,b)$ by definition. Hence $a^M_{r,s,\text{ell}}(\dot\gamma_S)$ vanishes unless $\mu=\dot\gamma_V$ lies in $\Gamma_{\text{ell}}(M,V,\zeta)$ and k lies in $\mathcal{K}^V_{\text{ell}}(\bar M,S)$. We can thus write (4.15) as

$$\sum_{L\in\mathscr{L}}\sum_{M\in\mathscr{L}^L}|W_0^M||W_0^G|^{-1}\sum_{\mu\in\Gamma_{\text{ell}}(M,V,\zeta)}\sum_{k\in\mathcal{K}^V_{\text{ell}}(\bar M,S)}a^M_{r,s,\text{ell}}(\mu\times k)r^L_M(k,b)J_L(\mu^L,f).$$

Using the property (4.12) and the definition (4.13) it follows that the inner sum can be expressed as

$$\begin{aligned}&\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\sum_{\gamma\in\Gamma(L,V,\zeta)}\sum_{k\in\mathcal{K}^V_{\text{ell}}(\bar M,S)}a^M_{r,s,\text{ell}}(\gamma_M\times k)r^L_M(k,b)J_L(\gamma,f)\\&=|W_0^M||W_0^G|^{-1}\sum_{\gamma\in\Gamma(L,V,\zeta)}a^L_{r,s}(\gamma)J_L(\gamma,f).\end{aligned}$$

Writing M for L in the preceding expression, the geometric expansion of $J^r_s(f)$ can thus be written as

$$J^r_s(f)=\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\sum_{\gamma\in\Gamma(M,V,\zeta)}a^M_{r,s}(\gamma)J_M(\gamma,f).$$

Converting this expansion for $J(f^r_s)$ into an expansion for $I(f^r_s)$ is standard. Recall from the definition in (3.3) that

$$I^r_s(f)=J^r_s(f)-\sum_{M\in\mathscr{L}^0}|W_0^M||W_0^G|^{-1}\hat I^{r,M}_s(\phi_M(f)).$$

Assume inductively that the expansion (4.14) holds if G is replaced by any proper Levi $L \in \mathscr{L}^0$. Then using the expansion we have just obtained for $J_s^r(f)$, we see that

$$I_s^r(f) = \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in\Gamma(M,V,\zeta)} a_{r,s}^M(\gamma) \left(J_M(\gamma,f) - \sum_{L\in\mathscr{L}(M)} \hat{I}_M^L(\gamma,\phi_L(f)) \right).$$

By the definition of $I_M(\gamma, f)$ then this is equal to

$$\sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in\Gamma(M,V,\zeta)} a_{r,s}^M(\gamma) I_M(\gamma,f)$$

as required. □

Corollary 4.7. *The coefficients $a_{r,s}^G(\gamma)$ are independent of S.*

Proof. The linear form $I_s^r(f)$ is constructed from the noninvariant form $J_s^r(f)$, which is independent of S. Assume inductively that for any proper Levi subgroup M of G, the coefficients $a_{r,s}^M(\gamma)$ are independent of S. Then the terms corresponding to the M in (4.14) are independent of S, thus so is the term corresponding to G, which is

$$\sum_{\gamma\in\Gamma(M,V,\zeta)} a_{r,s}^G(\gamma) f_G(\gamma).$$

It follows then that the $a_{r,s}^G(\gamma)$ are independent of S. □

5. Weighting the spectral side

On the spectral side, we have the sum

$$J(\dot{f}^1) = \sum_{\chi\in\mathfrak{X}} J_\chi(\dot{f}^1),$$

whose summands are also obtained by evaluating the polynomials $J_\chi^T(\dot{f}^1)$ at a distinguished point $T \in \mathfrak{a}_0$. They are simpler to treat than the geometric expansion, given the absolute convergence of the spectral side in hand, together with the results of [FLM11] which concern the refined noninvariant spectral expansion.

5.1. **Refinement.** Let $\Pi_{\mathrm{unit}}(M(\mathbf{A})^1)$ be the set of unitary representations of $M(\mathbf{A})^1$. Given an element $\dot\pi \in \Pi_{\mathrm{unit}}(M(\mathbf{A})^1)$ and $\lambda \in i\mathfrak{a}_M^*$, we form the representation π_λ by multiplying by $e^{\lambda(H_P(\cdot))}$ and the induced representation $\mathscr{I}_P(\dot\pi_\lambda)$ for any $P \in \mathscr{P}(M)$. Let $W^L(M)_{\mathrm{reg}}$ be the subset of regular elements in $W^L(M)$ with kernel equal to $\mathfrak{a}_L$. Also let $\mathfrak{a}_{G,Z}^*$ be the subspace of linear forms on $\mathfrak{a}_G$ that are trivial on the image of $\mathfrak{a}_Z$ in $\mathfrak{a}_G$. For any $Q \in \mathscr{P}(M)$ and $s \in W(M)$ there is a global (unnormalized) unitary intertwining operator $J_{Q|P}(s,\dot\pi_\lambda)$ from $\mathscr{H}_P(\dot\pi)$ to $\mathscr{H}_Q(\dot\pi)$. We set $J_{Q|P}(1,\dot\pi_\lambda) = J_{Q|P}(\dot\pi_\lambda)$ and $J_P(s,0) = J_{P|P}(s,\dot\pi_{\lambda+\Lambda})$. We recall that there is a discrete subset $\Pi_{\mathrm{disc}}(G)$ that supports a finite linear combination of characters

$$I_{\mathrm{disc}}(\dot{f}^1) = \sum_{\dot\pi\in\Pi_{\mathrm{disc}}(G)} a_{\mathrm{disc}}^G(\dot\pi) \dot{f}_G^1(\dot\pi),$$

with $\dot{f}^1 \in \mathscr{C}^\circ(G)$. The spectral coefficients $a_{\mathrm{disc}}^G(\dot\pi)$ are the multiplicities defined by the spectral expansion of the discrete part

$$I_{\mathrm{disc}}(\dot{f}^1) = \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{s\in W^L(M)_{\mathrm{reg}}} |\det(s-1)_{\mathfrak{a}_M^G}|^{-1} \mathrm{tr}(J_P(s,0)\mathscr{I}_P(\dot\pi,\dot{f}^1))$$

in [Art88b, §4], written as a linear combination of characters.

We begin by recording the following spectral expansion of $J(\dot{f}^1_{r,s})$.

Proposition 5.1. *Let $\dot{f}^1_S \in \mathscr{C}^\circ(G^1_V)$. We then have*

$$J(\dot{f}^1_{r,s}) = \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\dot{\pi}\in\Pi_{\rm disc}(M)} a^M_{\rm disc}(\dot{\pi}_\lambda)\int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} J_M(\dot{\pi}_\lambda, \dot{f}^1_{r,s})d\lambda.$$

Proof. We recall that $J_\chi(\dot{f}^1)$ is equal to the sum over $M\in\mathscr{L}$ of the product of

$$|W_0^M||W_0^G|^{-1}|\det(s-1)_{\mathfrak{a}^G_M}|^{-1}$$

with

$$\sum_{\pi\in\Pi_{\rm unit}(M(\mathbf{A})^1)}\sum_{L\in\mathscr{L}(M)}\sum_{s\in W^L(M)_{\rm reg}}\int_{i\mathfrak{a}^*_L/i\mathfrak{a}^*_G} {\rm tr}(\mathscr{J}_L(\dot{\pi}_\lambda,P)J_P(s,0)\mathscr{I}_P(\dot{\pi}_\lambda,\dot{f}))d\lambda,$$

as in the proof of Proposition 3.4. Here

$$\mathscr{J}_L(\dot{\pi}_\lambda,P) = \lim_{\Lambda\to 0}\sum_{Q\in\mathscr{P}(M)}\mathscr{J}_Q(\Lambda,\dot{\pi}_\lambda,P)\theta_Q(\Lambda)^{-1},$$

for $\Lambda\in i\mathfrak{a}^*_M$ near to 0, is the limit of (G,M)-families

$$\mathscr{J}_Q(\Lambda,\dot{\pi}_\lambda,P) = J_{P|Q}(\dot{\pi}_\lambda)^{-1}J_{Q|P}(\dot{\pi}_{\lambda+\Lambda}).$$

Then the required formula follows by the same argument in the proof of [Art88b, Theorem 4.4]. □

5.2. **The invariant spectral expansion.** As before, we have to convert this expansion to a distribution on G^Z_V. We continue to assume that V contains $V_{\rm ram}(G,\zeta)$. Let $\mathscr{F}(G^Z_V,\zeta_V)$ be the space of finite complex linear combinations of irreducible characters on G^Z_V with Z_V-central character equal to ζ_V, with a canonical basis $\Pi(G^Z_V,\zeta_V)$ of irreducible characters. We identify elements $\pi\in\mathscr{F}(G^Z_V,\zeta_V)$ with the linear form

$$f\mapsto f_G(\pi) = {\rm tr}(\pi(f))$$

on $\mathscr{H}(G^Z_V,\zeta_V)$. We write $\Pi_{\rm unit}(G^Z_V,\zeta_V)$ for the subset of unitary characters. The orbits of the action $\pi\mapsto\pi_\lambda$ of $i\mathfrak{a}^*_{G,Z}$ on $\pi\in\Pi_{\rm unit}(G_V,\zeta_V)$ can be identified with the set $\Pi_{\rm unit}(G^Z_V,\zeta_V)$. We also define the induced characters $f_M(\pi) = f_G(\pi^G) = {\rm tr}(\mathscr{I}_P(\pi,f))$ for any $\pi\in\Pi_{\rm unit}(M_V,\zeta_V)$. Given any $f\in\mathscr{H}(G,V,\zeta)$ and $\pi\in\Pi_{\rm unit}$, we inductively define the linear form

$$I_M(\pi,f) = J_M(\pi,f) - \sum_{M\in\mathscr{L}^0(M)}\hat{I}^L_M(\pi,\phi_L(f))$$

where $J_M(\pi,f)$ is the weighted character defined in [Art02, §3] and ϕ_L is the map defined in (3.4).

We shall combine the spectral coefficients with unramified characters using basic functions at places outside of V. Let us define $\Pi_{\rm disc}(G,\zeta)$ to be the set of representations in $\Pi(G(\mathbf{A})^Z,\zeta)$ whose restrictions to $G(\mathbf{A})^1$ lie in $\Pi_{\rm disc}(G)$. It can be identified with the representations in $\Pi_{\rm disc}(G)$ whose central character on $Z(\mathbf{A})^1$ is equal to ζ. We then define $\Pi_{\rm disc}(G,V,\zeta)$ to be the subset of $\pi\in\Pi(G^Z_V,\zeta_V)$ such that $\pi\times c$ belongs to $\Pi_{\rm disc}(G,\zeta)$ for some $c\in\mathscr{C}(G^V,\zeta^V)$. Now given $c\in C(G^V,\zeta^V)$, there is a natural action of $\lambda\in\mathfrak{a}^*_{G,Z,\mathbf{C}}$ sending c to $c_\lambda = \{c_{v,\lambda} : v\notin V\}$. Let $\pi\times c = \pi\otimes\pi^V(c)$ denotes the associated representation in $\Pi(G(\mathbf{A}),\zeta)$, and let

$\mathscr{C}_{\text{disc}}(G,\zeta)$ denote the elements $c \in \mathscr{C}(G^V,\zeta^V)$ such that $\pi \times c$ belongs to $\Pi_{\text{disc}}(G,\zeta)$ for some $\pi \in \Pi_{\text{disc}}(G,V,\zeta)$. Following Arthur, we define for any $c \in \mathscr{C}^V_{\text{disc}}(G,\zeta)$ the unramified normalizing factors $r_{Q|P}(c_\lambda)$ as the quotient of completed automorphic L-functions

$$L(0,c_\lambda,\rho_{Q|P})L(1,c_\lambda,\rho_{Q|P})^{-1},$$

for $P,Q \in \mathscr{P}(M)$, where $\rho_{Q|P}$ is the adjoint representation of ${}^LM$ on the Lie algebra of the intersection of the unipotent radicals of $\hat{P}$ and $\hat{Q}$. We also form the corresponding (G,M)-family of functions

$$r_Q(\Lambda,c_\lambda) = r_{Q|\bar{Q}}(c_\lambda)^{-1}r_{Q|\bar{Q}}(c_{\lambda+\Lambda/2})$$

for $Q \in \mathscr{P}(M)$ and $\Lambda \in i\mathfrak{a}^*_M$. The limit

$$r^G_M(c_\lambda) = \lim_{\Lambda\to 0}\sum_{Q\in\mathscr{P}(M)} r_Q(\Lambda,c_\lambda)\theta_Q(\Lambda)^{-1}$$

is defined as a meromorphic function of λ. The global unnormalized weighted character, on the other hand,

$$J_M(\dot{\pi}_\lambda,\dot{f}^1_z) = \text{tr}(\mathscr{J}_M(\dot{\pi}_\lambda,P)\mathscr{I}_P(\dot{\pi}_\lambda,\dot{f}^1_z))$$

is to be expressed in terms of the local normalized weighted characters

$$J_L(\pi^L_\lambda,f) = \text{tr}(\mathscr{M}_L(\pi^L_\lambda,P_L)\mathscr{I}_{P_L}(\pi^L_\lambda,f)), \quad L \in \mathscr{L}(M), P_L \in \mathscr{P}(L).$$

Since $\dot{\pi}$ is unramified outside of V, $\mathscr{J}_Q(\Lambda,\dot{\pi}_\lambda,P)$ is a scalar multiple of $\mathscr{M}_Q(\Lambda,\pi,P)$, namely

$$\mathscr{J}_Q(\Lambda,\dot{\pi}_\lambda,P) = r_Q(\Lambda,c_\lambda,P)\mu_Q(\Lambda,c_\lambda,P)\mathscr{M}_Q(\Lambda,\pi_\lambda,P)$$

according to [Art02, p.207]. We can then define the unramified character

$$r^L_M(c_\lambda,b) = r^L_M(c_\lambda)b^V_M(c_\lambda), \qquad c \in \mathscr{C}^V_{\text{disc}}(M,\zeta). \tag{5.1}$$

It follows from [Art02, Lemma 3.2] and the absolute convergence of $b^V_M(c_\lambda)$ for $\text{Re}(s)$ large enough that for $c \in \mathscr{C}^V_{\text{disc}}(M,\zeta)$, the function $r^L_M(c_\lambda,b)$ is an analytic function of $\lambda \in i\mathfrak{a}^*_{M,Z}$ and

$$\int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} r^L_M(c_\lambda,b)(1+||\lambda||)^{-N}d\lambda$$

converges for N large enough.

Let $\tilde{\Pi}_{\text{disc}}(M,V,\zeta)$ be the preimage of $\Pi_{\text{disc}}(M,V,\zeta)$ in $\Pi_{\text{unit}}(M_V,\zeta_V)$, and let $\Pi^G_{\text{disc}}(M,V,\zeta)$ be the set of $i\mathfrak{a}^*_{G,Z}$-orbits in $\tilde{\Pi}_{\text{disc}}(M,V,\zeta)$. There is a free action $\rho \to \rho_\lambda$ of $i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}$ on $\Pi^G_{\text{disc}}(M,V,\zeta)$ whose orbits can be identified with $\Pi_{\text{disc}}(M,V,\zeta)$. Any element $\rho \in \Pi^G_{\text{disc}}(M,V,\zeta)$ is an irreducible representation of $M_V \cap G^Z_V$, from which one can form the parabolically induced representation ρ^G of G^Z_V. Recall that the induction is defined by the relation $f_G(\rho^G) = f_M(\rho)$ for any $f \in \mathscr{C}(G_V,\zeta_V)$, with the adjoint restriction map $\pi \mapsto \pi_M$ from $\mathscr{F}(G_V,\zeta_V)$ to $\mathscr{F}(M_V,\zeta_V)$, such that

$$\sum_{\pi\in\Pi(G_V,\zeta_V)} c_M(\pi_M)d_G(\pi) = \sum_{\rho\in\Pi(M_V,\zeta_V)} c_M(\rho)d_G(\rho^G) \tag{5.2}$$

for any linear function c_M on $\mathscr{F}(M_V,\zeta_V)$ and d_G on $\mathscr{F}(G_V,\zeta_V)$. We then define $\Pi(G,V,\zeta)$ to be the union over $M\in\mathscr{L}$ and $\rho\in\Pi^G_{\mathrm{disc}}(M,V,\zeta)$ of irreducible constituents of ρ^G. It has a Borel measure $d\pi$ given by

$$\int_{\Pi(G,V,\zeta)} h(\pi)d\pi = \sum_{M\in\mathscr{L}} |W_0^M|^{-1}|W_0^G|^{-1} \sum_{\rho\in\Pi_{\mathrm{disc}}(M,V,\zeta)} \int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} h(\rho^G_\lambda)d\lambda$$

for any $h\in C_c(\Pi(G,V,\zeta))$. We then define for any $\pi\in\Pi(G^Z_V,\zeta_V)$, the spectral coefficient

$$a^G_{r,s}(\pi) = \sum_{L\in\mathscr{L}} |W_0^L||W_0^G|^{-1} \sum_{c\in C^V_{\mathrm{disc}}(M,\zeta)} a^M_{\mathrm{disc}}(\pi_M\times c)r^G_M(c,b), \tag{5.3}$$

where $\pi_M\times c$ is a finite sum of representations $\dot\pi$ in $\Pi_{\mathrm{unit}}(M(\mathbf{A}),\zeta)$, and $a^M_{\mathrm{disc}}(\pi_M\times c)$ is the sum of corresponding values $a^G_{\mathrm{disc}}(\dot\pi)$. It follows from the definitions that $a^G_{r,s}(\pi)$ is supported on the subset $\Pi(G,V,\zeta)$ of $\Pi(G^Z_V,\zeta_V)$.

Theorem 5.2. *Let $f\in\mathscr{C}^\circ(G,V,\zeta)$. Then the invariant linear form $I^r_s(f)$ has the spectral expansion*

$$I^r_s(f) = \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(M,V,\zeta)} a^M_{r,s}(\pi)I_M(\pi,f)d\pi, \tag{5.4}$$

with the integrals converging absolutely.

Proof. As in (3.2), we would like a parallel expansion for the linear form

$$J^r_s(f) = J^\zeta(\dot f^1_{r,s}) = \int_{Z(F)\backslash Z(\mathbf{A})^1} J(\dot f^1_{r,s,z})\zeta(z)dz$$

where $\dot f^1_{r,s}$ is any function in in $\mathscr{C}^\circ(G)$ whose projection onto $\mathscr{C}^\circ(G,\zeta)$ equals $\dot f^r_s = f\times b^V$. It then follows from Proposition 5.1 that $J^r_s(f)$ has an expansion

$$\int_{Z(F)\backslash Z(\mathbf{A})^1} \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\dot\pi\in\Pi_{\mathrm{disc}}(M)} \int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} a^M_{\mathrm{disc}}(\dot\pi_\lambda)J_M(\dot\pi_\lambda,\dot f^1_{r,s,z})\zeta(z)d\lambda\, dz.$$

Now the outer integral annihilates the contribution of $\dot\pi$ in the complement of $\Pi_{\mathrm{disc}}(M,\zeta)$ in $\Pi_{\mathrm{disc}}(M)$, so the expression simplifies to

$$J^r_s(f) = \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\dot\pi\in\Pi_{\mathrm{disc}}(M,\zeta)} \int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} a^M_{\mathrm{disc}}(\dot\pi_\lambda)J_M(\dot\pi_\lambda,\dot f_{r,s})d\lambda.$$

We have to express $J_M(\dot\pi_\lambda,\dot f)$ in terms of local normalized weighted characters. Applying the splitting formula

$$\mathscr{J}_M(\dot\pi_\lambda,P) = \sum_{L\in\mathscr{L}(M)} r^L_M(c_\lambda)\mathscr{M}_L(\pi^L_\lambda,P)$$

in [Art02, p.208], we have for the choice of test function function $\dot f^r_s$ the relation

$$J_M(\dot\pi_\lambda,\dot f^r_s) = \sum_{L\in\mathscr{L}(M)} r^L_M(c_\lambda,b)J_M(\pi^L_\lambda,f),$$

which vanishes unless $\dot\pi$ is unramified outside of V. Writing $\dot\pi=\pi\times c$, we can replace the sum over $\dot\pi$ with a sum over π in $\Pi_{\mathrm{disc}}(M,V,\zeta)$ and $c\in C^V_{\mathrm{disc}}(M,\zeta)$.

From the definition of the spectral coefficients, we write

$$a^L_{r,s}(\pi^L_\lambda) = \sum_{c \in \mathscr{C}^V_{\mathrm{disc}}(M,\zeta)} a^M_{\mathrm{disc}}(\pi_\lambda \times c_\lambda) r^L_M(c_\lambda, b)$$

for any $\lambda \in i\mathfrak{a}^*_{M,Z}$ in general position, ignoring sets of measure zero. We can therefore rewrite our expansion as

$$\sum_{L \in \mathscr{L}} |W^L_0||W^G_0|^{-1} \sum_{M \in \mathscr{L}^L} |W^M_0||W^L_0|^{-1} \sum_{\pi \in \Pi_{\mathrm{disc}}(M,V,\zeta)} \int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} a^L_{r,s}(\pi^L_\lambda) J_L(\pi^L_\lambda, f) d\lambda.$$

The coefficient $a^L(\pi^L_\lambda)$ and the integral

$$\int_{i\mathfrak{a}^*_{L,Z}/i\mathfrak{a}^*_{G,Z}} J_L(\pi^L_{\lambda+\Lambda}, f) d\Lambda$$

depend only on the image of λ in $i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{L,Z}$, hence on the restriction of π^L_λ to L^V_Z. Writing π for this restriction, which runs over $\Pi(L,V,\zeta)$, we arrive at

$$J(\dot{f}^r_s) = \sum_{L \in \mathscr{L}} |W^L_0||W^G_0|^{-1} \int_{\Pi(L,V,\zeta)} a^L(\pi) J_L(\pi, f) d\pi.$$

Again, to convert the expansion for $J(f^r_s)$ into an expansion of $I(f^r_s)$, we assume inductively that the expansion (5.4) holds if G is replaced by any proper Levi $L \in \mathscr{L}^0$. Then using the expansion we have just obtained for $J^r_s(f)$, we see that $I^r_s(f)$ is equal to

$$\sum_{M \in \mathscr{L}} |W^M_0||W^G_0|^{-1} \int_{\Pi(M,V,\zeta)} a^M_{r,s}(\pi) \left(J_M(\pi, f) - \sum_{L \in \mathscr{L}(M)} \hat{I}^L_M(\pi, \phi_L(f)) \right) d\pi.$$

By the definition of $I_M(\pi, f)$ then this is equal to

$$\sum_{M \in \mathscr{L}} |W^M_0||W^G_0|^{-1} \int_{\Pi(M,V,\zeta)} a^M_{r,s}(\pi) I_M(\pi, f) d\pi$$

as required. □

Putting Theorems 4.6 and 5.2 together, we have an invariant trace formula that is valid for $\dot{f}^r_s = f \times b$.

6. Continuity of the stable trace formula

6.1. **Transfer.** We now turn to the stabilization, referring to [Art02] and the references therein for precise definitions of the objects that we consider here. In particular, we will introduce noncompactly supported variants of the many objects involved in the stabilization of the trace formula. Let $\mathscr{E}(G_v)$ be the set of endoscopic data $(G'_v, \mathcal{G}'_v, s'_v, \xi'_v)$ for G over F_v, represented by G'_v. We shall aslo assume that the auxiliary data $\tilde{G}' \to G'$ and $\tilde{\xi}' : \mathcal{G}' \to {}^L\tilde{G}'$ to be chosen according to [Art02, Lemma 7.1]. The Langlands-Shelstad transfer conjecture states that for any $G'_v \in \mathscr{E}(G_v)$, the map that sends $f \in \mathscr{H}(G_v, \zeta_v)$ to the function

$$f'(\delta') = f^{G'}(\delta') = \sum_{\gamma \in \Gamma_{\mathrm{reg}}(G_v,\zeta_v)} \Delta(\delta', \gamma) f_G(\gamma)$$

on $\Delta_{\mathrm{reg}}(\tilde{G}'_v, \tilde{\zeta}'_v)$ exists, and maps $\mathscr{H}(G_v, \zeta_v)$ continuously to the space $\mathscr{S}(\tilde{G}'_v, \tilde{\zeta}'_v)$. In the nonarchimedean case, the Langlands-Shelstad transfer was proved for smooth functions $f \in C_c^\infty(G_v)$ as a consequence of [Wal97] and the solution of the Fundamental Lemma [Ngô10], and thus holds also for ζ^{-1}-equivariant space $C_c^\infty(G_v, \zeta_v)$. Moreover, since the orbital integrals are tempered distributions, it makes sense to formulate the smooth transfer for the larger Schwartz space $\mathscr{C}(G_v, \zeta_v)$, in which case the transfer would lie in the corresponding space of stable orbital integrals $S\mathscr{C}(\tilde{G}'_v, \tilde{\zeta}'_v)$ of functions in $\mathscr{C}(\tilde{G}'_v, \tilde{\zeta}'_v)$ [Art99, §3]. Recall that we are taking G to be a K-group, so if f equals $\oplus_\alpha f_\alpha$, then

$$f' = \sum_{\alpha \in \pi_0(G)} f'_\alpha.$$

The Langlands-Shelstad transfer for Schwartz functions is then a simple consequence of the smooth transfer. We note that in the archimedean case, the result follows from work of Shelstad (c.f. [She08]).

Lemma 6.1. *Let F_v be a nonarchimedean local field. Then for $f \in \mathscr{C}(G_v, \zeta_v)$, the map from f to the function*

$$f'(\delta') = f^{G'}(\delta') = \sum_{\gamma \in \Gamma_{\rm reg}(G_v, \zeta_v)} \Delta(\delta', \gamma) f_G(\gamma)$$

on $\Delta_{reg}(\tilde{G}'_v, \tilde{\zeta}'_v)$ exists, and maps $\mathscr{C}(G_v, \zeta_v)$ continuously to $S\mathscr{C}(G_v, \zeta_v)$.

Proof. The proof relies on the fact that $C_c^\infty(G_v, \zeta_v)$ is a dense subspace of $\mathscr{C}(G_v, \zeta_v)$. Given f in $\mathscr{C}(G_v, \zeta_v)$, we may choose a sequence (f_n) in $C_c^\infty(G_v, \zeta_v)$ converging to f as n tends to infinity. Applying the Langlands-Shelstad transfer, it follows then that there is a family of transfers (f'_n) in $\mathscr{C}(\tilde{G}'_v, \tilde{\zeta}'_v)$ such that for any $\delta' \in \Delta_{\rm reg}(\tilde{G}'_v, \tilde{\zeta}'_v)$, we have

$$f'_n(\delta') = \sum_{\gamma \in \Gamma_{\rm reg}(G_v, \zeta_v)} \Delta(\delta', \gamma) f_{n,G}(\gamma)$$

in $\mathscr{S}(\tilde{G}'_v, \tilde{\zeta}'_v)$.

Estimating then the difference

$$|f'_n(\delta') - f'_{n+1}(\delta')| \leq \sum_{\gamma \in \Gamma_{\rm reg}(G_v, \zeta_v)} |\Delta(\delta', \gamma)||f_{n,G}(\gamma) - f_{n+1,G}(\gamma)|$$

for any fixed δ', where we note that the sums are finite since the orbital integral of f_n is compactly supported on the regular set for any n, we see that the difference

$$|f_{n,G}(\gamma) - f_{n+1,G}(\gamma)|$$

converges in the space of orbital integrals of functions in $C_c^\infty(G_v, \zeta_v)$. It follows that $f_n(\delta')$ converges in $\mathscr{S}(\tilde{G}'_v, \tilde{\zeta}'_v)$, and by continuity in $S\mathscr{C}(\tilde{G}'_v, \tilde{\zeta}'_v)$. By completeness, we denote by f' the function in $\mathscr{C}(\tilde{G}'_v, \tilde{\zeta}'_v)$ such that f'_n converges to f'. We note that the choice of f' is unique only up to stable conjugacy, and satisfies the identity

$$f'(\delta') = \sum_{\gamma \in \Gamma_{\rm reg}(G_v, \zeta_v)} \Delta(\delta', \gamma) f_G(\gamma).$$

as required. □

The stabilization of the trace formula relies on the local results of Arthur on orbital integrals such as in [Art96, Art99, Art06, Art08, Art16]. In order to stabilize the invariant linear form $I(f)$ for $f \in \mathscr{C}^\circ(G, V, \zeta)$, we note that these results hold for general Schwartz functions $f \in \mathscr{C}(G, V, \zeta)$ either as explicitly stated, or otherwise can be shown using the fact that the linear forms $I_M(\gamma, f)$ extend to tempered distributions on G. These local transfer mappings are required to construct the stable basis $\Delta(G_V^Z, \zeta_V)$ that is used to index the geometric side of the stable trace formula.

We have to show that this construction holds in our case also. While Arthur's stabilization is carried out for functions f belonging to $\mathscr{H}(G, V, \zeta)$, his construction of these spaces holds generally for functions in $\mathscr{C}(G, V, \zeta)$, as long as the transfer mappings exist. Note also that the twisted stabilization of Moeglin and Waldspurger is proved for functions in $C_c^\infty(G)$, given the existence of smooth transfer. We shall summarize Athur's construction here, extended to the slightly more general setting of $\mathscr{C}^\circ(G, V, \zeta)$.

6.2. **Geometric transfer factors.** As usual, if S' is a stable, tempered $\tilde{\zeta}'$-equivariant distribution on $\tilde{G}'(F_v)$, then we write $\hat{S}'$ for the corresponding continuous linear form on $S\mathscr{C}(\tilde{G}'_v, \tilde{\zeta}'_v)$. Applying the transfer to each of the components G_{α_v} of G_v, we have a mapping

$$f_v \to f'_v = f_v^{\tilde{G}'}$$

from $\mathscr{C}(G_v, \zeta_v)$ to $S\mathscr{C}(\tilde{G}'_v, \tilde{\zeta}'_v)$, which can be identified with a mapping

$$a_v \to a'_v$$

from $I\mathscr{C}(G_v, \zeta_v)$ to $S\mathscr{C}(\tilde{G}'_v, \tilde{\zeta}'_v)$. It follows that the product mapping from $\prod_v a_v$ to $\prod_v a'_v$ gives a linear mapping from $I\mathscr{C}(G_V, \zeta_V)$ to $S\mathscr{C}(\tilde{G}'_V, \tilde{\zeta}'_V)$. This mapping is attached to the product G'_V of the data G'_v, which we can think of as the endoscopic data of G over F_V. Letting G'_V vary, we obtain a mapping

$$I\mathscr{C}(G_V, \zeta_V) \to \prod_{G'_V} S\mathscr{C}(\tilde{G}'_V, \tilde{\zeta}'_V)$$

by putting together the individual images of a'. The image $I\mathscr{C}^{\mathscr{E}}(G_V, \zeta_V)$ of $I\mathscr{C}(G_V, \zeta_V)$ fits into a sequence of inclusions

$$I\mathscr{C}^{\mathscr{E}}(G_V, \zeta_V) \subset \bigoplus_{\{G'_V\}} I\mathscr{C}^{\mathscr{E}}(G'_V, G_V, \zeta_V) \subset \prod_{\Delta_V} S\mathscr{C}(\tilde{G}'_V, \tilde{\zeta}'_V) \tag{6.1}$$

in which the summand $I\mathscr{C}^{\mathscr{E}}(G'_V, G_V, \zeta_V)$ is a vector space of families of functions on $\tilde{G}'$ parametrized by transfer factors for G and $\tilde{G}'$, depending only on the F_V-isomorphism class of G'_V.

The mappings of functions have dual analogues for distributions. Given G'_V with auxiliary data $\tilde{G}'_V$ and $\tilde{\xi}'_V$, assume that δ' belongs to the space of stable distributions $S\mathscr{D}((\tilde{G}'_V)^{\tilde{Z}'_V}, \tilde{\zeta}'_V)$. By Lemma 6.1, we may evaluate the transfer f' of any function f in $\mathscr{C}^\circ(G, V, \zeta)$ at δ'. Since the distribution $f \to f'(\delta')$ belongs to $\mathscr{D}(G_V^Z, \zeta_V)$, we can construct the extended geometric transfer factors at each local place

$$\Delta(\delta', \gamma), \qquad G' \in \mathscr{E}(G), \delta' \in \Delta(\tilde{G}', \tilde{\zeta}'), \gamma \in \Gamma(G, \zeta).$$

defined for fixed bases $\Delta(\tilde{G}', \tilde{\zeta}')$ of the spaces $S\mathscr{D}(\tilde{G}', \tilde{\zeta}')$ such that

$$f'(\delta') = \sum_{\gamma \in \Gamma(G, \zeta)} \Delta(\delta', \gamma) f_G(\gamma) \tag{6.2}$$

holds for $\delta' \in \Delta(\tilde{G}', \tilde{\zeta}')$ and $f \in \mathscr{C}(G, \zeta)$. We can see that the extended local transfer factor, as a function on $\Delta(\tilde{G}', \tilde{\zeta}') \times \Gamma(G, \zeta)$ is defined in the exact same manner as [Art02, §4], and depends linearly on δ'. We can then define the global transfer factor as the corresponding product

$$\Delta(\delta,\gamma)=\prod_{v\in V}\Delta(\delta_v,\gamma_v)$$

for $\delta\in\Delta^{\mathscr{E}}(G_V,\zeta_V)$ and $\gamma\in\Gamma(G_V,\zeta_V)$. The sequence of inclusions (6.1) is dual to a sequence of surjective linear mappings

$$\prod_{G'_V} S\mathscr{D}((\tilde{G}'_V)^{\tilde{Z}'},\tilde{\zeta}'_V)\to\bigoplus_{\{G'_V\}}\mathscr{D}^{\mathscr{E}}(G'_V,G^Z_V,\zeta_V)\to\mathscr{D}^{\mathscr{E}}(G^Z_V,\zeta_V) \tag{6.3}$$

between spaces of distributions. Since f' is the image of the function f_G in $I\mathscr{C}(G,V,\zeta)$, it follows that $f'(\delta')$ depends only on the image δ of δ' in $\mathscr{D}^{\mathscr{E}}(G^Z_V,\zeta_V)$. In other words,

$$f'(\delta')=f^{\mathscr{E}}_G(\delta),$$

where $f^{\mathscr{E}}_G$ is the image of f_G in $I\mathscr{C}^{\mathscr{E}}(G,V,\zeta)$, so that by the adjoint relations satisfied by the geometric transfer factor [Art02, §5] the map $f_G\to f^{\mathscr{E}}_G$ is an isomorphism. The same is true therefore of the coefficients $\Delta_G(\delta',\gamma)$, so we may write

$$\Delta_G(\delta,\gamma)=\Delta_G(\delta',\gamma)$$

for $\gamma\in\Gamma(G^Z_V,\zeta_V)$ and complex numbers $\Delta_G(\delta,\gamma)$ that depend linearly on $\delta\in\mathscr{D}^{\mathscr{E}}(G^Z_V,\zeta_V)$. The image in $\Delta^{\mathscr{E}}(G^Z_V,\zeta_V)$ of the subspace

$$S\mathscr{D}((G^*_V)^{Z^*_V},\zeta^*_V)\simeq S\mathscr{D}(G^*_V,G^Z_V,\zeta_V)$$

can be identified with the subspace $S\mathscr{D}(G^Z_V,\zeta_V)$ of stable distributions in $\mathscr{D}(G^Z_V,\zeta_V)$.

The coefficients in the geometric expansion should really be regarded as elements in the appropriate completion of $\mathscr{D}(M^Z_V,\zeta_V)$ and $S\mathscr{D}(M^Z_V,\zeta_V)$, which we shall identify with the dual space of $\mathscr{D}(M^Z_V,\zeta_V)$ by fixing suitable bases $\Gamma(M^Z_V,\zeta_V)$ and $\Delta(M^Z_V,\zeta_V)$ of the relevant spaces of distributions. In particular, we shall fix a basis $\Delta((\tilde{G}'_V)^{\tilde{Z}'_V},\tilde{\zeta}'_V)$ of $S\mathscr{D}((\tilde{G}'_V)^{\tilde{Z}'_V},\tilde{\zeta}'_V)$ for any F_V-endoscopic datum G'_V with auxiliary data $\tilde{G}'_V$ and $\tilde{\xi}'_V$. We also fix a basis $\Delta^{\mathscr{E}}(G^Z_V,\zeta_V)$ of $\mathscr{D}^{\mathscr{E}}(G^Z_V,\zeta_V)$ such that

$$\Delta(G^Z_V,\zeta_V)=\Delta^{\mathscr{E}}(G^Z_V,\zeta_V)\cap S\mathscr{D}(G^Z_V,\zeta_V)$$

forms a basis of $S\mathscr{D}(G^Z_V,\zeta_V)$, and in the case that G is quasisplit, that $\Delta(G^Z_V,\zeta_V)$ is isomorphic to the image of the basis $\Delta((G^*_V)^{Z^*},\zeta_V)$.

6.3. **Spectral transfer factors.** The construction on the spectral side is parallel. In place of the spaces of distributions described by (6.3), we have the spectral analogue $\mathscr{F}(G^Z_V,\zeta_V)$ of $\mathscr{D}(G^Z_V,\zeta_V)$, and the sequence of maps

$$\prod_{G'_V} S\mathscr{F}((\tilde{G}'_V)^{\tilde{Z}'},\tilde{\zeta}'_V)\to\bigoplus_{\{G'_V\}}\mathscr{F}^{\mathscr{E}}(G'_V,G^Z_V,\zeta_V)\to\mathscr{F}^{\mathscr{E}}(G^Z_V,\zeta_V).$$

In place of the basis $\Gamma(G^Z_V,\zeta_V)$ of $\mathscr{D}(G^Z_V,\zeta_V)$, we have the basis

$$\Pi(G^Z_V,\zeta_V)=\coprod_{t\geq 0}\Pi_t(G^Z_V,\zeta_V)$$

of $\mathscr{F}(G^Z_V,\zeta_V)$ consisting of irreducible characters. If ϕ' belongs to $S\mathscr{F}((\tilde{G}'_V)^{\tilde{Z}'},\tilde{\zeta}'_V)$, then the distribution $f\to f'(\phi')$ belongs to $\mathscr{F}(G^Z_V,\zeta_V)$, we can construct the spectral transfer factors at each local place

$$\Delta(\phi',\pi),\qquad G'\in\mathscr{E}(G),\phi'\in\Phi(\tilde{G}',\tilde{\zeta}'),\pi\in\Pi(G,\zeta)$$

defined for fixed bases $\Phi(\tilde{G}', \tilde{\zeta}')$ of the spaces $S\mathscr{F}(\tilde{G}', \tilde{\zeta}')$ such that

$$f'(\phi') = \sum_{\pi \in \Pi(G_v, \zeta_v)} \Delta(\phi', \pi) f_G(\pi)$$

holds for $\phi' \in \Phi(\tilde{G}', \tilde{\zeta}')$ and $f \in \mathscr{C}(G, \zeta)$, parallel to (6.2). We also define the corresponding product

$$\Delta(\phi, \pi) = \prod_{v \in V} \Delta(\phi_v, \pi_v)$$

for $\phi \in \Phi^{\mathscr{E}}(G_V, \zeta_V)$ and $\pi \in \Pi(G_V, \zeta_V)$. Given an element ϕ' in $S\mathscr{F}((\tilde{G}'_V)^{\tilde{Z}'}, \tilde{\zeta}'_V)$, We have that $f'(\phi')$ depends only on the image ϕ of ϕ' in $\mathscr{F}^{\mathscr{E}}(G_V^Z, \zeta_V)$, that is,

$$f'(\phi') = f_G^{\mathscr{E}}(\phi),$$

and the spectral coefficients satisfy the relation

$$\Delta_G(\phi, \pi) = \Delta_G(\phi', \pi)$$

for $\pi \in \Pi(G_V^Z, \zeta_V)$ and complex numbers $\Delta_G(\phi, \pi)$ that depend linearly on $\phi \in \mathscr{F}^{\mathscr{E}}(G_V^Z, \zeta_V)$. They satisfy adjoint relations parallel to the geometric transfer factors. Here as in [Art02, §5] we shall fix an endoscopic basis $\Phi^{\mathscr{E}}(G_V^Z, \zeta_V)$ of $\mathscr{F}(G_V^Z, \zeta_V)$, and a subset

$$\Phi(G_V^Z, \zeta_V) = \Phi^{\mathscr{E}}(G_V^Z, \zeta_V) \cap S\mathscr{F}(G_V^Z, \zeta_V)$$

that forms a basis of $S\mathscr{F}(G_V^Z, \zeta_V)$, and in the case that G is quasisplit, such that $\Phi(G_V^Z, \zeta_V)$ is isomorphic to the image of the basis $\Phi((G_V^*)^{Z^*}, \zeta_V)$. If v is archimedean, we can identify $\Phi(\tilde{G}'_v, \tilde{\zeta}'_v)$ with the relevant set of Langlands parameters. If v is nonarchimedean, we construct $\Phi(\tilde{G}'_v, \tilde{\zeta}'_v)$ in terms of abstract bases $\Phi_{\mathrm{ell}}(G_v, \zeta_v)$ of the cuspidal subspaces $\mathscr{S}_{\mathrm{cusp}}(M_v, \zeta_v)$, and similar objects for endoscopic groups M' of M, where we observe that the relevant constructions of [Art96] extend readily to $\mathscr{C}(G_v, \zeta_v)$ (see [Art03, p.825]).

6.4. **The stable and endoscopic expansions.** Having defined the relevant objects, we now turn to the continuity of the stable trace formula. As before, our attention will be on extending the arguments in [Art02, Art01, Art03], which will essentially follow from properly constructing the natural generalizations of the required objects. As the stabilization of the trace formula involves a much more intricate argument than that needed for the invariant trace formula, we are forced to follow the same path here. We note that a similar argument is provided in [MW16a, MW16b] for the stabilization of the twisted trace formula.

Theorem 6.2. *The linear forms $I^{\mathscr{E}}$ and S extend continuously from $\mathscr{H}(G, V, \zeta)$ to $\mathscr{C}^\circ(G, V, \zeta)$.*

Proof. We first observe that Global Theorem 1′ in [Art02, §7] states that the global geometric coefficients satisfy

$$a^{G,\mathscr{E}}(\gamma) = a^G(\gamma), \qquad \gamma \in \Gamma^{\mathscr{E}}(G, V, \zeta)$$

for any G, and that

$$b^G(\delta), \qquad \delta \in \Delta^{\mathscr{E}}(G, V, \zeta)$$

vanishes on the complement of $\Delta(G, V, \zeta)$ if G is quasisplit. Notice that $\Gamma^{\mathscr{E}}(G, V, \zeta)$ and $\Delta(G, V, \zeta)$ are constructed as subsets of bases $\Gamma(G_V^Z, \zeta_V)$ and $\Delta(G_V^Z, \zeta_V)$ of the spaces $\mathscr{D}(G_V^Z, \zeta_V)$ and $S\mathscr{D}(G_V^Z, \zeta_V)$ respectively. In particular, we see that this space contains the orbital integrals, and also derivatives of orbital integrals in the archimedean cases, of functions f in $\mathscr{C}^\circ(G,V,\zeta)$. Similarly, Global Theorem $2'$ states that the global geometric coefficients satisfy

$$a^{G,\mathscr{E}}(\pi) = a^G(\pi), \qquad \pi \in \Pi_t^{\mathscr{E}}(G,V,\zeta)$$

for any G, and that

$$b^G(\phi), \qquad \delta \in \Phi_t^{\mathscr{E}}(G,V,\zeta)$$

vanishes on the complement of $\Phi_t(G,V,\zeta)$ if G is quasisplit. Here the spaces

$$\Pi_t^{\mathscr{E}}(G,V,\zeta), \quad \Phi_t^{\mathscr{E}}(G,V,\zeta), \quad \Phi_t(G,V,\zeta)$$

are the subset of elements in

$$\Pi^{\mathscr{E}}(G,V,\zeta), \quad \Phi^{\mathscr{E}}(G,V,\zeta), \quad \Phi(G,V,\zeta)$$

respectively whose archimedean infinitesimal characters ν have norms $t = ||\mathrm{Im}(\nu)||$. Notice that $\Phi_t(G,V,\zeta)$ and $\Pi_t^{\mathscr{E}}(G,V,\zeta)$ are constructed as discrete subsets of the bases $\Pi_t^{\mathscr{E}}(G_V^Z,\zeta_V)$ and $\Phi_t(G_V^Z,\zeta_V)$ of the spaces $\mathscr{F}(G_V^Z,\zeta_V)$ and $S\mathscr{F}(G_V^Z,\zeta_V)$ respectively. As we have indicated above, in both the geometric and spectral cases, the construction of the endoscopic spaces implicitly rely on the Langlands-Shelstad transfer, hence by Lemma 6.1 these spaces exist unconditionally.

Let S be a finite set of valuations containing V. There is a natural map

$$f \mapsto \dot{f}_S = f \times u_S^V$$

from $\mathscr{C}^\circ(G,V,\zeta)$ to $\mathscr{C}^\circ(G,S,\zeta)$. We shall define an admissible subspace $\mathscr{C}^\circ_{\mathrm{adm}}(G,S,\zeta)$ of $\mathscr{C}^\circ(G,S,\zeta)$, using the same notion of admissibility in [Art02, §1]. The polynomial $\det(1+t-\mathrm{Ad}(x)) = \sum_k D_k(x)t^k$ for $x \in G$ defines a morphism

$$\mathcal{D} = (D_0,\dots,D_d) : G \to \mathbf{G}_a^{d+1}$$

where $d = \dim G$. If X is a nonzero point in $\mathbf{G}_a^{d+1}$, we shall denote $X_{\min} = X_k$ where k is the smallest integer such that X_k is nonzero. Let $\mathscr{O}^S$ be product of all v not in S of $\mathscr{O}_v$, the ring of integers of F_v. We call a subset C_S of $F_S^{d+1}\backslash\{0\}$ admissible if any point X in the intersection

$$F^{d+1} \cap (C_S \times (\mathscr{O}^S)^{d+1})$$

satisfies $|X_{\min}|_v = 1$ for all $v \notin S$. Assume moreover that S contains the places over which G and ζ are ramified, and that $Z(A) = Z(F)Z_SZ(\mathscr{O}^S)$. Then we call a subset Δ_S of G_S admissible if $\mathcal{D}(\Delta_S)$ is admissible in F_S^{d+1}. This implies that

$$|D(\dot{\gamma})|_v = 1$$

for all $\gamma \in G(F) \cap (\Delta_S \times K^S)$ and $v \notin S$. Also, Δ_S is admissible if and only if its projection onto $\bar{G}_S = G_S/Z_S$ is admissible. Finally, we define $\mathscr{C}^\circ_{\mathrm{adm}}(G,S,\zeta)$ to be the subspace of functions in $\mathscr{C}^\circ(G,S,\zeta)$ whose support is admissible. Also, we shall say a subset Δ of $G(\mathbf{A})$ is S-admissible if for some finite set S there is an admissible subset C_S of F_S^{d+1} such that $\mathcal{D}(\Delta)$ is contained in $C_S \times (\mathscr{O}^S)^{d+1}$. We note that it is this condition of admissibility and S-admissibility that the reductions of [Art02, Art01, Art03] are based upon, rather than the compact support of the test functions f.

Having made these preliminary remarks, we now proceed as follows. Let I be the invariant linear form on $\mathscr{C}^\circ(G,V,\zeta)$ obtained in Proposition 3.5. If G is arbitrary, we define an endoscopic linear form inductively by setting

$$I^{\mathscr{E}}(f) = \sum_{G' \in \mathscr{E}_{\mathrm{ell}}(G,V)} \iota(G,G')\hat{S}'(f')$$

for stable linear forms $\hat{S}' = \hat{S}^{\tilde{G}'}$ on $S\mathscr{C}^\circ(G,V,\zeta)$. In the case that G is quasisplit, we define a linear form

$$S^G(f) = I(f) - \sum_{G' \in \mathscr{E}^0_{\mathrm{ell}}(G,V)} \iota(G,G')\hat{S}'(f')$$

and also the endoscopic linear form by the trivial relation

$$I^{\mathscr{E}}(f) = I(f).$$

We assume inductively that if G is replaced by a quasisplit inner K-form of $\tilde{G}'$, the corresponding analogue of S^G is defined and stable. At this stage, the reductions of [Art02, Art01] can now be applied without difficulty. In particular, if on the geometric side, we define $I^{\mathscr{E}}_{\mathrm{orb}}(f)$ and $S^G_{\mathrm{orb}}(f)$ to be the summands corresponding to $M = G$ in $I^{\mathscr{E}}(f)$ and $S^G(f)$ respectively, we see from the proof of [Art02, Theorem 10.1] that if G is arbitrary,

$$I^{\mathscr{E}}(f) - I^{\mathscr{E}}_{\mathrm{orb}}(f) = \sum_{M \in \mathscr{L}^0} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M,V,\zeta)} a^{M,\mathscr{E}}(\gamma) I^{\mathscr{E}}_M(\gamma, f)$$

and if G is quasisplit, we have that $S^G(f) - S^G_{\mathrm{orb}}(f)$ is equal to

$$\sum_{M \in \mathscr{L}^0} |W_0^M||W_0^G|^{-1} \sum_{M' \in \mathscr{E}'_{\mathrm{ell}}(M,V)} \iota(M,M') \sum_{\delta' \in \Delta(\tilde{M}',V,\tilde{\zeta}')} b^{\tilde{M}'}(\delta') S^G_M(M',\delta',f).$$

Here $I^{\mathscr{E}}_M(\gamma,f)$ and $S^G_M(M',\delta',f)$ are the local geometric distributions defined in [Art02, §6]. While on the spectral side, we define $I^{\mathscr{E}}_{t,\mathrm{unit}}(f)$ and $S^G_{t,\mathrm{unit}}$ using the decomposition according to the norm of the archimedean infinitesimal character,

$$I^{\mathscr{E}}(f) = \sum_{t \geq 0} I^{\mathscr{E}}_t(f)$$

and

$$S^G(f) = \sum_{t \geq 0} S^G_t(f).$$

It follows then from the proof of [Art02, Theorem 10.6] that if G is arbitrary,

$$I^{\mathscr{E}}_t(f) - I^{\mathscr{E}}_{t,\mathrm{unit}}(f) = \sum_{M \in \mathscr{L}^0} |W_0^M||W_0^G|^{-1} \int_{\Pi^{\mathscr{E}}_t(M,V,\zeta)} a^{M,\mathscr{E}}(\pi) I^{\mathscr{E}}_M(\pi,f)d\pi$$

and if G is quasisplit, we have that $S^G_t(f) - S^G_{t,\mathrm{unit}}(f)$ is equal to

$$\sum_{M \in \mathscr{L}^0} |W_0^M||W_0^G|^{-1} \sum_{M' \in \mathscr{E}'_{\mathrm{ell}}(M,V)} \iota(M,M') \int_{\Phi_{t'}(\tilde{M}',V,\tilde{\zeta}')} b^{\tilde{M}'}(\phi') S^G_M(M',\phi',f)d\phi'.$$

Here again $I^{\mathscr{E}}_M(\pi,f)$ and $S^G_M(M',\phi',f)$ are the local spectral distributions defined in [Art02, §6]. These identities reduce the study of the global geometric coefficients $a^{G,\mathscr{E}}(\gamma), b^G(\delta)$ and global spectral coefficients $a^{G,\mathscr{E}}(\pi), b^G(\phi)$ to the terms $M = G$ in their expansion, namely $a^{G,\mathscr{E}}_{\mathrm{ell}}(\gamma), b^G_{\mathrm{ell}}(\delta)$ and $a^{G,\mathscr{E}}_{\mathrm{disc}}(\pi), b^G_{\mathrm{disc}}(\phi)$ respectively by the arguments of Propositions 10.3 and 10.7 of [Art02]. Moreover, the global descent formula of [Art01, Corollary 2.2] further reduces the study of the global geometric coefficients to unipotent elements. (Our extension of the notion of admissibility is crucial here for the extension of this result, which is a long but straightforward verification.) More precisely, given an admissible element $\dot\gamma_S$ in $\Gamma^{\mathscr{E}}_{\text{ell}}(G,S,\zeta)$ with Jordan decomposition $\dot\gamma_S = c_S\dot\alpha_S$, we have

$$a^{G,\mathscr{E}}_{\text{ell}}(\dot\gamma_S) = \sum_c \sum_{\dot\alpha} i^{\bar G}(S,c)|\bar G_{c,+}(F)/\bar G_c(F)|^{-1} a^{G_c,\mathscr{E}}_{\text{ell}}(\dot\alpha)$$

and if G is quasisplit, given an admissible element $\dot\delta_S$ in $\Delta_{\text{ell}}(G,S,\zeta)$ with Jordan decomposition $\dot\delta_S = d_S\dot\beta_S$, we have

$$b^G_{\text{ell}}(\dot\delta_S) = \sum_d \sum_{\dot\beta} j^{\bar G^*}(S,d)|\bar G^*_{d,+}(F)/\bar G^*_d(F)|^{-1} b^{G^*_d}_{\text{ell}}(\dot\beta),$$

where $G_{c,+}$ denotes the centralizer of c in G, and G_c is the identity component of $G_{c,+}$. We refer the reader to [Art01] for complete definitions of these expressions.

Turning to the local setting, the analogues of Local Theorems 1 and 2 of [Art02, §6] for $f \in \mathscr{C}^\circ(G,V,\zeta)$ follow from the analogues of Local Theorems $1'$ and $2'$, which concern the compound linear forms $I^{\mathscr{E}}_M(\gamma,f)$, $S^G_M(M',\delta',f)$ and $I^{\mathscr{E}}_M(\pi,f)$, $S^G_M(M',\phi',f)$ as a consequence of the geometric and spectral splitting and descent formulae respectively [Art02, Propositions 6.1 and 6.3]. We recall that the required geometric formulae are given in [Wona, §3], whereas the spectral formulae can be deduced from [MW16b, X.4].

To apply the arguments of [Art03], we require analogous constructions of various subspaces of the Hecke space $\mathscr{H}(G,V,\zeta)$ used therein. If G is quasisplit, we define the unstable subspace

$$\mathscr{C}^{\circ,\text{uns}}(G,V,\zeta)$$

of functions $f \in \mathscr{C}^\circ(G,V,\zeta)$ such that $f^G = 0$. It is spanned by functions $f = \prod_{v\in V} f_v$ such that for some $v \in V$, f_v satisfies the property that $f^G_v = 0$. We shall also define the subspace

$$\mathscr{C}^\circ_M(G,V,\zeta)$$

of functions $f \in \mathscr{C}^\circ(G,V,\zeta)$ such that f_v is M-cuspidal at two places $v \in V$. Recall that $f_v \in \mathscr{C}(G_v,\zeta_v)$ is said to be M-cuspidal if $f_{v,L_v} = 0$ for any element $L_v \in \mathscr{L}_v$ that does not contain a G_v-conjugate of M_v. If v is a nonarchimedean place, we define

$$\mathscr{C}^\circ(G_v,\zeta_v)^{00}$$

to be the subspace of functions $f \in \mathscr{C}^\circ(G_v,\zeta_v)$ such that $f_{v,G}(z_v\alpha_v) = 0$ for any z_v in the center of $\bar G_v = G_v/Z_v$ and α_v in the basis $R_{\text{unip}}(G_v,\zeta_v)$ of unipotent orbital integrals in [Art03, §3]. We lastly define

$$\mathscr{C}^\circ(G_v,\zeta_v)^0$$

analogously, with α_v ranging over the parabolic subset $R_{\text{unip,par}}(G_v,\zeta_v)$. We also write $\mathscr{C}^\circ(G,V,\zeta)^0$ for the product of functions $f_v \in \mathscr{C}^\circ(G_v,\zeta_v)^0$ for $v \in V$, and similarly for $\mathscr{C}^\circ(G,V,\zeta)^{00}$. We shall denote by the intersections of these various spaces by using overlapping notation, for example, we write $\mathscr{C}^\circ_M(G_v,\zeta_v)^0 = \mathscr{C}^\circ_M(G_v,\zeta_v) \cap \mathscr{C}^\circ(G_v,\zeta)^0$.

The remainder of the proof proceeds by a double induction on integers r_{der} and d_{der} such that

$$0 < r_{\text{der}} < d_{\text{der}}.$$

Namely, we assume inductively that Local Theorem 1 holds if $\dim(G_{\mathrm{der}}) < d_{\mathrm{der}}$ and if

$$\dim(G_{\mathrm{der}}) = d_{\mathrm{der}}, \quad \dim(A_M \cap G_{\mathrm{der}}) < d_{\mathrm{der}}$$

for a local non-archimedean field; the archimedean transfer for $f \in \mathscr{C}(G,\zeta)$ follows from [Art08, Theorem 1.1]. We assume inductively that Global Theorems 1 and 2 hold if $\dim(G_{\mathrm{der}}) < r_{\mathrm{der}}$. In both local and global cases, we assume that if G is not quasisplit and $\dim(G_{\mathrm{der}}) = d_{\mathrm{der}}$, the relevant theorems hold for the quasisplit inner K-form of G.

We will be content with recapitulating the broad strokes of the arguments in [Art03], indicating the points in which we use the new spaces that we have defined above instead of the compactly supported ones. We shall use the subscripts 'unip' to denote the unipotent variant of objects with subscript 'ell,' and 'par' with the objects corresponding to terms $M \neq G$. For example, we write

$$I_{\mathrm{unip}}(f,S) = \sum_{\alpha \in \Gamma_{\mathrm{unip}}(G,V,\zeta)} a^G_{\mathrm{unip}}(\alpha,S) f_G(\alpha)$$

where

$$a^G_{\mathrm{unip}}(\alpha,S) = \sum_{k \in \mathcal{K}^V_{\mathrm{unip}}((\bar{G},S)} a^G_{\mathrm{ell}}(\alpha \times k) r_G(k)$$

for $\alpha \in \Gamma_{\mathrm{unip}}(G,V,\zeta)$. By the inductive definitions we obtain $I^{\mathscr{E}}_{\mathrm{unip}}$ and S^G_{unip} analogously. The global induction hypothesis then implies that

$$I^{\mathscr{E}}_{\mathrm{par}}(f) - I_{\mathrm{par}}(f) = \sum_t \left(I^{\mathscr{E}}_{t,\mathrm{disc}}(f) - I_{t,\mathrm{disc}}(f)\right) - \sum_z \left(I^{\mathscr{E}}_{t,\mathrm{unip}}(f,S) - I_{t,\mathrm{unip}}(f,S)\right)$$

and if G is quasisplit and f belongs to $\mathscr{C}^{\circ,\mathrm{uns}}(G,V,\zeta)$, then

$$S^G_{\mathrm{par}}(f) = \sum_t S^G_{t,\mathrm{disc}}(f) - \sum_z S^G_{z,\mathrm{unip}}(f,S),$$

where z belongs to the quotient $Z(G)_{V,\mathfrak{o}} Z_V / Z_V$, and $Z(G)_{V,\mathfrak{o}}$ is the subgroup of elements in $(Z(G))(F)$ such that for every $v \notin V$, the element z_v is bounded in $(Z(G))(F_v)$. The induction hypotheses further lead to a cancellation of p-adic singularities, allowing us to express

$$I^{\mathscr{E}}_{\mathrm{par}}(f) - I_{\mathrm{par}}(f) = |W(M)|^{-1} \hat{I}^M(\varepsilon_M(f))$$

for f in the intersection $\mathscr{C}^\circ_M(G,V,\zeta)^0$. and if G is quasisplit,

$$S^G_{\mathrm{par}}(f) = |W(M)|^{-1} \sum_{M' \in \mathscr{E}_{\mathrm{ell}}(M,v)} \iota(M,M') \hat{S}^{\tilde{M}'}(\varepsilon^{M'}(f))$$

for f in the intersection $\mathscr{C}^{\circ,\mathrm{uns}}_M(G,V,\zeta)^0$ as in [Art03, Corollary 3.3]. Here ε_M is a map from $\mathscr{C}^\circ(G_v,\zeta_v)^0$ to the subspace of cuspidal functions in $\mathscr{I}_{\mathrm{ac}}(M_v,\zeta_v)$ such that

$$\varepsilon_M(f_v,\gamma_v) = I^{\mathscr{E}}_M(\gamma_v,f_v) - I_M(\gamma_v,f_v)$$

for any $\gamma_v \in \Gamma(M_v,\zeta_v)$, and in the case that G_v is quasisplit, ε^M is a map from $\mathscr{C}^{\circ,\mathrm{uns}}(G_v,\zeta_v)^0$ to $\mathscr{S}_{\mathrm{ac}}(M_v,\zeta_v)$ such that

$$\varepsilon^M(f_v,\delta_v) = S^G_M(\delta_v,f_v)$$

for any $\delta_v \in \Delta(M_v,\zeta_v)$. These maps are given in [Art03, Proposition 3.1], also studied in Chapters VIII and IX of [MW16b], and can be seen as generalizing the mapping ϕ_M in a direction different from the maps $\iota_M, \iota_M^{\mathscr{E}}$, and τ_M that we have constructed earlier.

The separation of the spectral sides according to infinitesimal character follows from [Art03, §4–5] and the properties of the function spaces we have defined, but is not strictly necessary given the absolute convergence of the spectral side. On the other hand, the stabilization of the invariant local trace formula in [Art02, §10] and [Art03, §6] extends to our setting following Lemma 6.1, and together with the global results above lead to the proof of Local Theorem 1 in the nonarchimedean case, again using the local and global induction hypotheses. We note that this implies Local Theorem 2 according to an unpublished work of Arthur, and we may also refer to sections X.5 and X.7 of [MW16b] for a variant argument.

To complete the global theorems, we apply the local theorems to conclude that

$$I^{\mathscr{E}}_{\mathrm{par}}(f) - I_{\mathrm{par}}(f) = \sum_{M \in \mathscr{L}^0} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M,V,\zeta)} a^M(\gamma)(I^{\mathscr{E}}_M(\gamma, f) - I_M(\gamma, f))$$

vanishes for $\mathscr{C}^\circ(G, V, \zeta)$, and if G is quasisplit, that

$$S^G_{\mathrm{par}}(f) = \sum_{M \in \mathscr{L}^0} |W_0^M||W_0^G|^{-1} \sum_{\delta^* \in \Delta(M^*,V,\zeta^*)} b^{M^*}(\delta^*) S^G_M(M^*, \delta^*, f)$$

vanishes for $f \in \mathscr{C}^{\circ,\mathrm{uns}}(G, V, \zeta)$. The induction argument on r_{der} implies that the terms $I^{\mathscr{E}}_{t,\mathrm{disc}}(f) - I_{t,\mathrm{disc}}(f)$ and $S^G_{t,\mathrm{disc}}(f)$ vanish for f in $\mathscr{C}^\circ(G, V, \zeta)$ and $\mathscr{C}^{\circ,\mathrm{uns}}(G, V, \zeta)$ respectively, so that

$$\sum_z \left(I^{\mathscr{E}}_{t,\mathrm{unip}}(f, S) - I_{t,\mathrm{unip}}(f, S)\right) = 0, \qquad f \in \mathscr{C}^\circ(G, V, \zeta)$$

and

$$\sum_z S^G_{z,\mathrm{unip}}(f, S) = 0, \qquad f \in \mathscr{C}^{\circ,\mathrm{uns}}(G, V, \zeta)$$

in the case that G is quasisplit. Choosing $V = S$, and using the property that the linear forms

$$\dot{f}_S \to \dot{f}_{S,G}(z\dot{\alpha}_S), \qquad z \in Z(\bar{G})_{S,\mathfrak{o}},\ \dot{\alpha}_S \in \Gamma^{\mathscr{E}}_{\mathrm{unip}}(G, S, \zeta)$$

on the subspace of admissible functions in $\mathscr{C}^\circ(G, S, \zeta)$ are linearly independent, we conclude from the definitions of $I_{t,\mathrm{unip}}$ and $I^{\mathscr{E}}_{t,\mathrm{unip}}$ that

$$a^{G,\mathscr{E}}_{\mathrm{ell}}(\dot{\alpha}_S) - a^G_{\mathrm{ell}}(\dot{\alpha}_S) = 0$$

for $\dot{\alpha}_S \in \Gamma^{\mathscr{E}}_{\mathrm{unip}}(G, S, \zeta)$, and similarly

$$\dot{f}_S \to \dot{f}^{\mathscr{E}}_{S,G}(z\dot{\beta}_S), \qquad z \in Z(\bar{G})_{S,\mathfrak{o}},\ \dot{\beta}_S \in \Delta^{\mathscr{E}}_{\mathrm{unip}}(G, S, \zeta) \backslash \Delta_{\mathrm{unip}}(G, S, \zeta)$$

on the subspace of admissible functions in $\mathscr{C}^{\circ,\mathrm{uns}}(G, S, \zeta)$ are linearly independent, whence we conclude that

$$b^G_{\mathrm{ell}}(\dot{\beta}_S) = 0$$

for $\dot{\alpha}_S$ in the complement of $\Delta_{\mathrm{unip}}(G, S, \zeta)$ in $\Delta^{\mathscr{E}}_{\mathrm{unip}}(G, S, \zeta)$. Applying the global descent formula to the coefficients then yields the geometric Global Theorem 1. The spectral Global Theorem 2 follows similarly, using the vanishing of

$$\sum_t \left(I^{\mathscr{E}}_{t,\mathrm{disc}}(f) - I_{t,\mathrm{disc}}(f)\right) = 0, \qquad \dot{f} \in \mathscr{C}^\circ(G, \zeta)$$

and

$$\sum_t S^G_{t,\mathrm{disc}}(f) = 0, \qquad \dot{f} \in \mathscr{C}^{\circ,\mathrm{uns}}(G,\zeta).$$

Arguing as in the geometric case we conclude that

$$a^{G,\mathscr{E}}_{\mathrm{disc}}(\dot\pi) - a^{G,\mathscr{E}}_{\mathrm{disc}}(\dot\pi) = 0$$

for any $\dot\pi \in \Pi_{t,\mathrm{disc}}(G,\zeta)$, and in the case that G is quasisplit,

$$b^G_{\mathrm{disc}}(\dot\phi) = 0$$

for any $\dot\phi$ in the complement of $\Phi_{t,\mathrm{disc}}(G,\zeta)$ in $\Phi^{\mathscr{E}}_{t,\mathrm{disc}}(G,\zeta)$, and the desired result follows.

Finally, we can conclude from these general remarks the extension of the endoscopic and stable trace formulae, with the required expansions

$$\begin{aligned} I^{\mathscr{E}}(f) &= \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in\Gamma^{\mathscr{E}}(M,V,\zeta)} a^{M,\mathscr{E}}(\gamma) I^{\mathscr{E}}_M(\gamma,f) \\ &= \sum_t \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi^{\mathscr{E}}_t(M,V,\zeta)} a^{M,\mathscr{E}}(\pi) I^{\mathscr{E}}_M(\pi,f)d\pi \end{aligned}$$

and

$$\begin{aligned} S(f) &= \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\delta\in\Delta(M,V,\zeta)} b^M(\delta) S_M(\delta,f) \\ &= \sum_t \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Phi_t(M,V,\zeta)} b^M(\phi) S_M(\phi,f)d\phi \end{aligned}$$

for $f \in \mathscr{C}^\circ(G,V,\zeta)$. □

We state the local consequences explicitly, which we shall require in the stabilization of the form $I^r_s(f)$. We record them separately according to the geometric and spectral sides, respectively.

Corollary 6.3. *Let F be a number field, and let V be a finite set of valuations $V_{ram}(G,\zeta)$.*

(a) *If G is arbitrary,*

$$I^{\mathscr{E}}_M(\gamma,f) = I_M(\gamma,f), \qquad \gamma\in\Gamma(M^Z_V,\zeta_V), f\in\mathscr{C}^\circ(G,V,\zeta)$$

(b) *Suppose that G is quasisplit, and that δ' belong to $\Delta((\tilde M'_V)^{\tilde Z'},\tilde\zeta'_V)$ for some $M'\in\mathscr{E}_{\mathrm{ell}}(M,V)$. Then the linear form*

$$f \to S^G_M(M',\delta',f), \qquad f\in\mathscr{C}^\circ(G,V,\zeta)$$

vanishes unless $M = M^*$, in which case it is stable.

Corollary 6.4. *Let F be a number field, and let V be a finite set of valuations $V_{ram}(G,\zeta)$.*

(a) *If G is arbitrary,*

$$I^{\mathscr{E}}_M(\pi,f) = I_M(\pi,f), \qquad \gamma\in\Pi(M^Z_V,\zeta_V), f\in\mathscr{C}^\circ(G,V,\zeta)$$

(b) *Suppose that G is quasisplit, and that ϕ' belong to $\Phi((\tilde{M}'_V)^{\tilde{Z}'}, \tilde{\zeta}'_V)$ for some $M' \in \mathscr{E}_{\text{ell}}(M,V)$. Then the linear form*

$$f \to S^G_M(M', \phi', f), \qquad f \in \mathscr{C}^\circ(G,V,\zeta)$$

vanishes unless $M = M^*$, in which case it is stable.

Proof. The statements concern the compound linear forms that arise on either side of the endoscopic and stable trace formulas. In the proof of Theorem 6.2, we established the analogues of the local theorems in [Art02] which can be stated for a local field F and test functions $f \in \mathscr{C}^\circ(G,\zeta)$. The result will follow from the necessary splitting and descent formulas in the same manner as for functions $f \in \mathscr{H}(G,V,\zeta)$ described in [Art02, §6]. On the geometric side, we conclude that if G is arbitrary, then

$$I^{\mathscr{E}}_M(\gamma, f) = I_M(\gamma, f), \qquad \gamma \in \Gamma_{G\text{-reg,ell}}(M,\zeta), f \in \mathscr{C}^\circ(G,\zeta)$$

whereas if G is quasisplit and δ' belongs to $\Delta_{G\text{-reg}}(\tilde{M}', \tilde{\zeta}')$ for some $M' \in \mathscr{E}_{\text{ell}}(M)$, then the linear form

$$f \to S^G_M(M', \delta', f), \qquad f \in \mathscr{C}^\circ(G,\zeta)$$

vanishes unless $M' = M^*$, in which case it is stable. The desired result follows from the splitting and descent formulas for these distributions, which are proved in this setting in Propositions 3.3 and 3.4 of [Wona], and are also stated in a different form in [MW16b].

Similarly, on the spectral side, we conclude from the proof of Theorem 6.2 that if G is arbitrary, then

$$I^{\mathscr{E}}_M(\pi, f) = I_M(\pi, f), \qquad \pi \in \Pi_{G\text{-reg,ell}}(M,\zeta), f \in \mathscr{C}^\circ(G,\zeta)$$

whereas if G is quasisplit and ϕ' belongs to $\Phi_{G\text{-reg}}(\tilde{M}', \tilde{\zeta}')$ for some $M' \in \mathscr{E}_{\text{ell}}(M)$, then the linear form

$$f \to S^G_M(M', \phi', f), \qquad f \in \mathscr{C}^\circ(G,\zeta)$$

vanishes unless $M' = M^*$, in which case it is stable. The desired result follows from the splitting and descent formulas for these distributions, which second of which is stated in this setting in the proof of Propositions 4.1 of [Wonb], and are also stated in a different form in [MW16b].

□

The modifications of the endoscopic and stable linear forms in [Wonb] then applies to this setting also. We recall that in [Wonb] we constructed modified distributions $\tilde{I}^{\mathscr{E}}$ and $\tilde{S}$ which lead to purely geometric expansions for the unitary part of the spectral expansion in terms of modified linear forms $\tilde{I}^{\mathscr{E}}_M(\gamma)$ and $\tilde{S}_M(\delta)$ defined in [Wonb, §3]. The modifications apply to our extension of the endoscopic and stable linear forms on $\mathscr{C}^\circ(G,V,\zeta)$ also.

Corollary 6.5. *The modified distributions $\tilde{I}^{\mathscr{E}}$ and $\tilde{S}$ extend to continuous linear forms on $\mathscr{C}^\circ(G,V,\zeta)$.*

(a) *If G is arbitrary,*

$$I^{\mathscr{E}}_{\text{unit}}(f) = \sum_{M \in \mathscr{L}} |W^M_0||W^G_0|^{-1} \sum_{\gamma \in \Gamma(M,V,\zeta)} a^{M,\mathscr{E}}(\gamma) \tilde{I}^{\mathscr{E}}_M(\gamma, f).$$

(b) *If G is quasisplit,*

$$S_{\mathrm{unit}}(f) = \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\delta \in \Delta(M,V,\zeta)} b^M(\delta)\tilde{S}_M(\delta, f).$$

Proof. This follows from the fact that the maps τ_M and $\iota_M^{\mathscr{E}}$ used to construct the modified distributions are defined for functions in $\mathscr{C}(G, V, \zeta)$ (though we only have need of the smaller space $\mathscr{H}_{\mathrm{ac}}(G, V, \zeta)$ in the application to f_s^r), whence we may apply Theorem 6.2 and argue as in the first part of the proof of Proposition 3.5. □

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