A WEIGHTED STABLE TRACE FORMULA

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Abstract. We establish endoscopic and stable trace formulas whose discrete spectral terms are weighted by automorphic $L$-functions, by the use of basic functions that are incorporated into the global spectral and geometric coefficients. This extends earlier work which established the corresponding weighted invariant trace formula for noncompactly supported test functions. The meromorphic continuation of these weighted trace formulas would yield $r$-trace formulas, and can therefore be seen as precursors to them. As a corollary, we show that the meromorphic continuation of the weighted geometric coefficients implies the meromorphic continuation of automorphic $L$-functions. Along the way, we also formulate a weighted form of the Langlands-Shelstad transfer conjecture, generalizing the weighted fundamental lemma of Arthur. In the second part, we collect various results on the stable trace formula that we require, including an extension of the stable trace formula to noncompactly supported test functions.

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1. Introduction

1.1. Weighting the trace formula. Given a reductive group $G$ over a number field $F$, the stable trace formula is an identity of stable distributions on $G$, expressed as

$$ S(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Phi(M,V,\zeta)} b^M(\phi) S_M(\phi,f) d\phi $$

$$ = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\delta \in \Delta(M,V,\zeta)} b^M(\delta) S_M(\delta,f), $$

which has been used, for example, for the endoscopic classification of automorphic representations of various classical groups. To obtain it one first makes the trace formula invariant, and its stabilization in turn depends crucially upon the weighted fundamental lemma. This work is a continuation of [Won22], the goal of which is to weight the spectral side of the stable trace formula with automorphic $L$-functions by the use of certain basic functions. In [Won22] we established the extension of the invariant trace formula to allow for basic functions, and in this work we shall stabilize it. Altogether, this represents an important technical step in the Beyond Endoscopy program of incorporating automorphic $L$-functions into the Arthur-Selberg trace formula.

Fixing a central induced torus $Z$ of $G$ with an automorphic character $\zeta$, we let $V$ be a large finite set of valuations $v$ of $F$ outside of which $G,Z,$ and $\zeta$ are unramified. Let

$$ G_V = \prod_{v \in V} G(F_v), \quad G^V = \prod_{v \not\in V} G(F_v). $$

The stable linear form $S(f)$ established by Arthur is valid only for test functions of the form

$$ \hat{f} = f \times u^V, $$

where $f$ is a $\zeta^{-1}$-equivariant, compactly-supported function on $G_V$ and $u^V$ is the unit element of the $\zeta^{-1}$-equivariant Hecke algebra of $G^V$. In order to weight the trace formula with the associated $L$-functions, we shall instead use test functions of the form

$$ f^r_s = f \times b^V, $$

where $f$ is a $\zeta^{-1}$-equivariant, noncompactly-supported function on $G_V$ as in [Won22, §2], the space of which we denoted by $\mathcal{C}_\infty(G,V,\zeta)$, and $b^V$ is constructed from the basic function that depends on an irreducible complex finite-dimensional representation $r$ of the $L$-group $L^G$ of $G$, and a complex number $s$ with $\text{Re}(s)$ large enough. We shall assume both of these to be fixed. The main result of [Won22] is then the existence of an invariant linear form

$$ I^r_s(f) = I(f^r_s), \quad f \in \mathcal{C}_\infty(G,V,\zeta) $$

valid for $\text{Re}(s)$ large enough, which comes with the parallel expansions

$$ \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(M,V,\zeta)} a^M_{\gamma,s}(\pi) I_M(\pi,f) d\pi $$

$$ = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M,V,\zeta)} a^M_{\gamma,s}(\gamma) I_M(\gamma,f). $$
The global coefficients $a^M_r(\pi)$ and $a^M_r(\gamma)$ that occur here are weighted forms of the coefficients $a^M(\pi)$ and $a^M(\gamma)$ that occur in Arthur’s invariant trace formula [Art88a]. In particular, the spectral coefficient $a^M_r(\pi)$ is related to the unramified automorphic $L$-function $L^V(s, \pi, r)$. This can be seen as a step towards an $r$-invariant trace formula in the sense of Arthur [Art17]. If the weighted coefficients $a^M_r, s(\pi)$ and $a^M_r, s(\gamma)$ can be shown to have meromorphic continuation just to the left of the line $\text{Re}(s) = 1$, then in principle the residue theorem can be applied to establish an $r$-trace formula without removing the contribution of the nontempered spectrum, what is generally seen to be the key obstruction in obtaining an $r$-trace formula [FLN10]. Indeed, the $L$-functions of nontempered automorphic representations are expected still to have meromorphic continuation, and thus the method that we propose here will establish a trace formula that carry information about these $L$-functions, which one might hope will be treated in the comparison of trace formulas.

1.2. Stabilization. In this work, we proceed to stabilize the linear form $I_s^r(f)$. We shall establish a decomposition generalizing the stable trace formula [Art02],

$$I_s^r(f) = \sum_{G'} \iota(G, G') \hat{S}^r_{s}(f'),$$

for stable distributions $\hat{S}^r_{s} = \hat{S}^r_{G'}$ on the endoscopic groups $G'$ for $G$, where $f \to f'$ is the ordinary Langlands-Shelstad transfer and $\iota(G, G')$ is the coefficient defined in [Kot84] Theorem 8.3.1. The transfer $f'$ is determined only up its stable orbital integrals and the distribution $f \to \hat{S}^r_{s}(f')$ that is indexed by $G' = G^*$, the quasisplit inner form of $G$, can be regarded as the stable part of $I_s^r(f)$. Assume inductively that $\hat{S}^r_{s}$ exists and is stable. In the case that $G$ is quasisplit, in which case $G^* = G$, we define

$$\hat{S}^r_{G}(f) = I_s^r(f) - \sum_{G \neq G^*} \iota(G, G') \hat{S}^r_{s}(f').$$

The problem then is to show that the right-hand side is stable. If $G$ is not quasisplit, all the terms on the right-hand side are defined inductively. The decomposition of $I_s^r(f)$ above then represents an identity to be proved.

Let $\mathcal{C}^\circ(G, V, \zeta) = \mathcal{C}^\circ(G^*_V, \zeta_V)$ be the space of $\zeta^{-1}$-equivariant noncompactly supported test functions on $G^*_V$ in [Won22 §2.1] and §2.4. It is a subspace of the $\zeta^{-1}$-equivariant Harish-Chandra Schwartz functions $\mathcal{E}(G, V, \zeta)$, and is the class of functions to which Finis-Lapid-Müller extended the noninvariant trace formula. Our first main result is then the following.

Theorem 1. Assume Corollary 3.4.

(a) If $G$ is arbitrary, the invariant linear form

$$I_r^\varphi(f) = I_r^\varphi(f^r), \quad f \in \mathcal{C}^\circ(G, V, \zeta)$$

converges absolutely for $\text{Re}(s)$ large enough, and has the parallel expansions

$$\sum_{M \in \mathcal{Z}} |W_0^M||W_0^G|^{-1} \int_{\Gamma(M, V, \zeta)} a^M_{r, s}(\pi) I_r^\varphi(\pi, f) d\pi = \sum_{M \in \mathcal{Z}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M, V, \zeta)} a^M_{r, s}(\gamma) I_r^\varphi(\gamma, f).$$
(b) If $G$ is quasisplit, the stable linear form

$$S^G_{r,s}(f) = S(f^r), \quad f \in \mathcal{C}^\infty(G,V,\zeta)$$

converges absolutely for $\text{Re}(s)$ large enough, and has the parallel expansions

$$\sum_{M \in \mathcal{L}} |W_0^M| |W_0^{G^r}|^{-1} \int_{\mathfrak{P}(M,V,\zeta)} b^M_{r,s}(\phi) S_M(\phi, f) d\phi = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^{G^r}|^{-1} \sum_{\delta \in \Delta(M,V,\zeta)} b^M_{r,s}(\delta) S_M(\delta, f).$$

Similar formulas hold unconditionally if we replace $f^r$ with $f \times u^V_S \times b^S$.

The global coefficients $b^M_{r,s}(\delta)$ and $b^M_{r,s}(\phi)$ are now weighted forms of the stable coefficients $b^M(\delta)$ and $b^M(\phi)$ that occur in the usual stable trace formula, which are naturally constructed from the ‘unstable’ coefficients, and similarly for the endoscopic coefficients $a^M_{r,s}(\gamma)$ and $a^M_{r,s}(\pi)$. Corollary 3.4 here is the weighted transfer of stable orbital integrals of basic functions, which is possible to circumvent at the cost of using less elegant test functions, as explained below in §1.3. Moreover, the assumption that $\text{Re}(s)$ is sufficiently large is a matter of the choice of representation $r$ of $L^r G$. In the cases where meromorphic continuation is known, for example the $L$-functions arising from the Langlands-Shahidi method, this restriction can also be relaxed. Along with the expectation of meromorphic continuation of automorphic $L$-functions, we also expect these weighted coefficients to also have meromorphic continuation to the entire complex plane.

Our proof generally follows Arthur’s strategy in the stabilization of the trace formula. In [Art02, §6,7], this takes the form of a pair of main global theorems and main local theorems, regarding nature of the global coefficients and local distributions that appear on either side of the expansions. It is perhaps worthy of note that the local distributions that occur in the weighted trace formula remain the same, and what is needed is simply an extension to $\mathcal{C}^\infty(G,V,\zeta)$. The analogues of the main local theorems will be proved in Part 2, whereas Part 1 is dedicated to stabilizing the new weighted global coefficients.

1.3. A weighted transfer conjecture. The assumption of the main theorem above is convenient but not necessary, and we explain its appearance here. Along the way to stabilization, we formulate a weighted form of the Langlands-Shelstad transfer conjecture, namely the endoscopic transfer of weighted orbital integrals of the basic function. It is a natural joint extension of the Langlands-Sehlstad transfer and the weighted Fundamental Lemma, and we refer to Conjecture 3.3 for a precise statement.

We use this transfer to streamline the stabilization of the unramified geometric terms in the trace formula, where before what was needed was the transfer of weighted orbital integrals of the unit element of the spherical Hecke algebra. The latter led to the weighted Fundamental Lemma conjectured by Arthur [Art02, Conjecture 5.1], and which is now proved for split groups by the combined works of Waldspurger, Chaudouard, and Laumon [Wal09, CL12]. Given that the (twisted)

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1This may seem a little disjointed to the reader. The simple reason for this is because the results of Chapter 9 were proved in an earlier version of [Won22], but have been deferred here instead.
unweighted Langlands-Shelstad transfer follows from the (twisted) unweighted Fundamental Lemma, it seems natural to pose such an analogue. Indeed, Waldspurger has suggested such a possibility in the Lie algebra case [Wal95]. The weighted fundamental lemma itself is of course the strongest evidence for this conjecture. We can also take as evidence the transfer of weighted (relative) orbital integrals of basic functions recently obtained by Huajie Li in the context of the Guo-Jacquet relative trace formula [Li21, Theorem 1.5]. To summarize, we have the following diagram that relates what is known with our conjecture:

\[(\text{twisted) Fundamental Lemma} \implies (\text{twisted) Transfer}) \quad ? \quad \text{weighted Fundamental Lemma} \implies \text{weighted Transfer (Conjecture 3.3)}\]

While our result can be stated in these terms, we emphasise that the Conjecture 3.3 (in fact the special case Corollary 3.4) only streamlines our result; the assumption can be dropped by instead using functions of the form

\[(1.2) \quad f \times u_S \times b^S\]

in place of \( f \times b^V \) in (1.1), where \( S \supset V \) is a large finite set of places depending on the support of \( f \), and \( u_S^V \) is the \( \zeta^{-1} \)-equivariant characteristic function of \( K_S^V = \prod_{v \in S \setminus V} K_v \), where \( K_v \) is a fixed hyperspecial maximal compact subgroup of \( G_v \). This would be sufficient for capturing the poles of unramified automorphic \( L \)-functions on the spectral side, but comes at the expense of a slightly more complicated formula than the one obtained above.

1.4. Application to automorphic \( L \)-functions. The construction of the weighted endoscopic and stable global coefficients are built from the global coefficients in \( I_{r,s}(f) \), and their meromorphic continuation follows from that of the original coefficients. As a result of Theorem 8.4, Corollaries 8.3 and 8.6, we can relate the weighted geometric and spectral coefficients relate to each other in this sense.

**Theorem 2.** If \( a_{r,s}^M(\gamma) \) has meromorphic continuation for all \( M \in \mathcal{L}, \gamma \in \Gamma(M, V, \zeta) \) and \( s \in \mathbb{C} \), then so does \( b_{r,s}^M(\phi) \) for \( \pi \in \Pi(M, S, \zeta) \), and also the spectral and geometric expansions of \( S_{r,s}^M(f) \).

We recall that the weighted spectral coefficients \( b_{r,s}^M(\phi) \) contain information related to unramified automorphic \( L \)-functions, hence showing their meromorphic continuation to \( s = 1 \) amounts to doing the same for the \( L \)-functions. As is typical with the trace formula, rather than doing this directly we can consider the geometric coefficients \( a_{r,s}^M(\gamma) \) instead. Moreover, by their inductive definitions, the properties of the coefficients easily reduce to the ‘unstable’ ones. As an application of the weighted trace formula we show the following (Corollary 8.5).

**Theorem 3.** If \( a_{r,s}^M(\gamma) \) has meromorphic continuation for all \( M \in \mathcal{L}, \gamma \in \Gamma(M, V, \zeta) \) and \( s \in \mathbb{C} \), then for any cuspidal automorphic representation \( \tilde{\pi} \) of \( G(\mathbb{A}) \) unramified outside of \( S \), we have that \( L^S(s, \pi, r) \) has meromorphic continuation in \( s \in \mathbb{C} \).

Furthermore, by Theorem 8.2, the meromorphy of \( a_{r,s}^M(\gamma) \) can be reduced to unipotent coefficients, and thus to that of the unipotent contribution to the trace formula \( J_{unip}(f_s^r) \). This leads to a question of studying the unipotent part of the trace formula, which we can only comment on briefly here.

Arthur’s refinement of the coarse geometric expansion of the trace formula relies ultimately on expressing the unipotent contribution to the geometric side as a
linear combination of weighted orbital integrals. The global distribution $J^T_s(f)$ is evaluated at a distinguished point $T = T_0$ that lies a priori outside the range of absolute convergence, and therefore cannot be interpreted a priori as a measure, and in particular, in terms of orbital integrals (we extend this range to all $T$ in Lemma 8.1). Arthur previously circumvented this difficulty at the cost of restricting to test functions that are trivial at almost all places and global geometric coefficients that are not explicit.

While this has not impeded the comparison trace formulas up until now, interest remained in making these coefficients explicit for other applications of the trace formula. Nonetheless, there are established ideas of Hoffmann [Hof16] and Chaudouard [Cha17, Cha18] regarding the unipotent coefficients in the trace formula (see also [Lap10, §1.3]) and their expected connection to prehomogeneous vector spaces. The idea to express the unipotent terms in terms of zeta integrals and regularized orbital integrals goes back to the work of Selberg, and to date has been carried out in various special cases. The general case remains unsolved, but it is our hope that the latter application will motivate renewed interest in this problem.

1.5. The larger picture. Theorem 1 is a step towards the $r$-stable trace formula described in [Art17], whereby the cuspidal spectral terms are nonzero only if $L^S(s, \pi, r)$ has a pole at $s = 1$. Establishing it would amount to showing that the distribution $S^G_{r,s}(f)$ has meromorphic continuation to the point $s = 1$, and obtaining an explicit formula for its residue there. From this it will follow that the residue distribution

$$S^G_r(f) = -\frac{1}{2\pi i} \int_{\text{Re}(s)=1} S^G_{r,s}(f)x^s \frac{ds}{s}, \quad x > 0,$$

can be seen to be an $r$-trace formula, in the sense that the contribution of the cuspidal tempered terms for which the associated $L^S(s, \pi, r)$ is holomorphic in the half plane $\text{Re}(s) \geq 1$ will be zero. Here the contour integral should be taken with a small semi-circle to the left of $s = 1$ to detect the possible pole there. Crucially, this approach circumvents the difficulties of presented by the nontempered terms by allowing the contribution of the nontempered discrete spectrum to remain. Thus the distribution $S^G_r(f)$ will retain information about nontempered representations whose $L$-functions have poles in the half plane $\text{Re}(s) \geq 1$ for a given $r$. Indeed, by the principle of functoriality one also expects such $L$-functions to have meromorphic continuation, so the distribution $S^G_r(f)$ will be well-defined, and thus justifies calling $S^G_r(f)$ an $r$-trace formula. An approximation of this was constructed by Getz for $GL(n)$ using certain nonabelian Fourier transforms following ideas of Braverman and Kazhdan [Get18, §5].

We also mention briefly a potential path to a primitive trace formula that does not directly require the meromorphic continuation of $S^G_r$. Recall that we are seeking a decomposition approximately of the form

$$S^G_r(f) = \sum_{G'} \iota(r, G') \hat{P}^{G'}(f'),$$

according to [Art17 (2.3)]. The sum over $G'$ should run over certain transfer data (or so-called ‘beyond endoscopic data’), and $\iota(r, G)$ is the product of the usual invariant $\iota(G, G')$, appearing in the stable trace formula, with the dimension datum $m_{G'}(r)$ of $G'$ at $r$. The primitive linear form $\hat{P}^{G'}$ may then be defined inductively
as 
\[ P^G(f) = S^G_1(f) - \sum_{G \neq G'} \iota(1, G') \hat{P}^G(f'). \]

In Arthur’s formulation, these linear forms should only be supported on the cuspidal tempered spectrum. Using our weighted trace formula \( S^r_s(f) \), we can propose the following possibility: for appropriate coefficients \( j(r, G) \), we define inductively the linear form 
\[ P^G_s(f) = S^G_{1,s}(f) - \sum_{r \neq 1} j(r, G) S^G_{r,s}(f), \]

which is also meromorphic at \( s = 1 \), but for simpler reasons (related to the Riemann zeta function). The latter sum over \( r \) might be interpreted as the analogue of \( \kappa \)-orbital integrals in the stabilization of the trace formula, wherein the sum over \( \kappa \) is expressed in terms of stable orbital integrals on endoscopic groups. In the present scenario, the problem of primitisation can then be recast as expressing the sum over \( r \) (or rather, its residue at \( s = 1 \)) as a sum over \( G' \) of primitive forms \( \hat{P}^G \) involving the stable transfer \( f' \) from \( G \) to \( G' \). We would then have a primitive trace formula of the form 
\[ P^G_s(f) = \text{Res}_{s=1}(P^G_s(f)). \]

This would then establish the primitisation of the stable linear form \( S^G_1(f) \) and, as Arthur notes in [Art17], represents the basic case of the more general decomposition \( S^G_r(f) \) above.

1.6. Outline. We conclude with a summary of the contents. Part 1 deals with the main arguments of the stabilization. After reviewing basic notation and definitions in Chapter 2, we formulate in Chapter 3 the aforementioned weighted transfer conjecture, which we use in Chapter 4 to stabilize the unramified terms on the geometric side of the trace formula. In Chapter 5, we stabilize the unramified terms on the spectral side of the trace formula. These unramified terms occur as the new weighted terms in the global coefficients on either side, and represent the main technical innovations of Part 1, aside the weighted transfer conjecture, which we believe to be of independent interest.

The rest of the argument follows the method of Arthur’s stabilization, with special attention paid to the new spaces of functions involved. We formulate the analogues of Arthur’s main global theorems in Theorems 4.3 and 5.4, whose proofs take up the rest of Part 1. In Chapter 6, we prove a descent formula for the weighted global geometric coefficients along the lines of [Art01], which reduces the study of the geometric coefficients to unipotent elements. In Chapter 7, we complete the proofs of the global theorems and thereby establish the weighted endoscopic and stable trace formulas, using the results that are proved in Part 2. Finally, in Chapter 8 we discuss the relation between the global weighted spectral and geometric coefficients, including the application to automorphic \( L \)-functions.

In Part 2, we prove various properties regarding the stable trace formula which are relatively self-enclosed, where in particular the first three chapters are auxiliary statements required in the course of Part 1. In Chapter 9, we extend the usual stable trace formula to the class of nonstandard test functions \( \mathcal{C}^\infty(G, V, \zeta) \), and as a result obtain the analogous main local theorems of [Art02] needed for the stabilisation. In Chapter 10, we prove splitting and descent formulas for singular stable orbital integrals along the lines of Arthur’s work [Art99b]. These results can
in principle be deduced from the work of Moeglin and Waldspurger on the twisted stable trace formula [MW16a], but we find it easier to prove them directly. In Chapter [1] we give a treatment of the stable germ expansions for orbital integrals, of the kind used in [Art02] but remain unpublished. This builds in part on recent work of Arthur [Art16]. Good control over the stable germ expansions is needed to study the asymptotics of the Fourier transform of orbital integrals, which we hope will be useful in the later study of the trace formula (as suggested, for example, in [Art16] and [Art17] §3.11)). These results on stable orbital integrals that are known to experts but do not appear in the literature, which we require in Part 1 (and not to mention [Art02] Art01, Art03, and [Won22]), appearing there as unpublished citations).

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Part 1. Weighting the stable trace formula

2. Definitions

In order to formulate our statements precisely, we first fix some necessary notation. Most of the notation and constructions that we employ throughout are chosen to be consistent with those of [Art02] Art01, Art03 (and [Won22]), while the main variations occur where the basic functions are involved.

2.1. Preliminaries. Let G be a quasisplit connected reductive group over a field F of characteristic zero. Let \( \mathcal{L} \) be the set of Levi subgroups of G containing a fixed F-rational minimal Levi subgroup \( M_0 \) of G. Also let K be a fixed maximal compact subgroup of \( G(F) \), which we shall assume to be special in the case that \( F \) is nonarchimedean. We shall also fix a minimal parabolic subgroup \( P_0 \) with Levi component \( M_0 \), and with this convention denote by \( \mathcal{F}(M) \) the set of parabolic subgroups of G containing \( M \in \mathcal{L} \), and \( \mathcal{P}(M) \) the subset of parabolic subgroups with Levi component equal to \( M \).

For any \( M \in \mathcal{L} \), we shall write \( a_M = \text{Hom}(X(M)_F, \mathbb{R}) \), where \( X(M)_F \) is the group of F-rational homomorphisms from \( M \) to \( \mathbb{R} \). There is a surjective homomorphism \( H_M : M(F) \to a_M \) defined by

\[
e^{\chi(H_M(m))} = e^{(H_M(m), \chi)} = |\chi(m)|, \quad m \in M(F), \quad \chi \in X(M)_F.
\]

We also define the dual vector space \( a_M^* = X(M)_F \otimes \mathbb{R} \), and its complexification \( a_{M,C}^* = X(M)_F \otimes \mathbb{C} \). If \( P \) is a parabolic subgroup of \( G \) with Levi component equal to \( M_P \), then we also have a continuous mapping \( H_P : G(F) \to a_{M_P} \) given by

\[
H_P(nmk) = H_{M_P}(m) \quad n \in N_P(F), m \in M_P(F), k \in K,
\]

for any \( x = nmk \) in \( G(F) \), where \( N_P \) is the unipotent radical of \( P \). Moreover, if \( F \) is global we use the same notation to denote the map \( H_G : G(\mathcal{A}) \to a_G \) defined in an analogous fashion, and denote by \( G(\mathcal{A})^1 \) the kernel of \( H_G \) in \( G(\mathcal{A}) \). Let \( A_G \) be the maximal F-split torus in the center of \( G \), and \( A_G(\mathbb{R})^0 \) the connected component of \( A_G(\mathbb{R}) \). Then \( G(\mathcal{A}) \) is the direct product of \( G(\mathcal{A})^1 \) and \( A_G(\mathbb{R})^0 \). For any two \( M, L \in \mathcal{L} \) such that \( M \subset L \), there is a natural map from \( a_M \) to \( a_L \), whose kernel we denote by \( a^L_M \). Also, we set \( a_0 = a_{M_0} \) and \( H_0 = H_{P_0} \).
2.2. \textbf{K-groups.} We will work in the setting of a multiple group $G$, which is an algebraic variety whose components $G_{\alpha}$ are reductive groups over a field $F$,

$$G = \prod_{\alpha \in \pi_0(G)} G_{\alpha},$$

with an equivalence class of frames

$$(\psi, u) = \{(\psi_{\alpha\beta}, u_{\alpha\beta}) : \alpha, \beta \in \pi_0(G)\},$$

where $\psi_{\alpha\beta} : G_{\beta} \to G_{\alpha}$ is an $F$-isomorphism and $u_{\alpha\beta} : \Gamma_F \to G_{\alpha,sc}$ is a locally constant function from the absolute Galois group $\Gamma_F$ of $F$ to the simply connected cover $G_{\alpha,sc}$ of the derived group of $G_{\alpha}$. The class of $(\psi, u)$ satisfies compatibility conditions described in [Art02] §4. If $F$ is a local field, we call a multiple group $G$ a $K$-group if the $u_{\alpha\beta}$ are 1-cocycles and for each $\alpha$, the image of the map from $\{u_{\alpha\beta} : \beta \in \pi_0(G)\}$ to $H^1(F, G_{\alpha})$ is in bijection with the image of $H^1(F, G_{\alpha,sc})$ in $H^1(F, G_{\alpha})$. If $F$ is a global field, we call a multiple group $G$ a $K$-group if it satisfies the preceding properties, and it has a local product structure, i.e., there is a family of local $K$-groups $(G_v, F_v)$ indexed by the valuations of $F$, and a family of homomorphisms $G \to G_v$ over each completion $F_v$, whose restricted direct product $G \to \prod G_v$ is an isomorphism of schemes over the adele ring $\mathbb{A}$ of $F$. Such a structure determines a surjective map

$$\alpha \to \alpha_v = \prod_{v \in V} \alpha_v, \quad \alpha \in \pi_0(G), \alpha_v \in \pi_0(G_v)$$

of components, for any finite set of valuations $V$ of $F$, and is bijective if $V$ contains the infinite places $V_\infty$ of $F$. Suppose that $G$ is a $K$-group, and that $G^*$ is a quasisplit inner twist of $G$, which determines a quasisplit inner twist $G^*_v$ of each of the local $K$-groups $G_v$. We call $G$ an inner $K$-form of $G^*$, and we call $G$ quasisplit if at least one of its components is quasisplit over the global field $F$.

We shall fix a central induced torus $Z$ of $G$, and a character $\zeta$ of $Z(\mathbb{A})/Z(F)$. It comes with central embeddings $Z \to Z_\alpha$ over $F$ for each $\alpha \in \pi_0(G)$ that are compatible with the isomorphisms $\psi_{\alpha\beta}$. Locally, we have pairs $Z_v$ and $\zeta_v$ where $\zeta_v$ is a character on $Z_v$. We then write $G^*_v$ for the set of $x \in G_V = \prod_{v \in V} G_v$ such that $H_G(x)$ lies in the image of the canonical map from $a_Z$ to $a_G$. We shall generally assume that the finite set $V$ contains the set places $V_{\text{tam}}(G, \zeta)$ over which $G, Z$, and $\zeta$ are ramified. We also have natural notions of Levi subgroups and parabolic subgroups of a $K$-group $G$, whence we denote by $\mathcal{L}(M)$ to be the collection of Levi subgroups of $G$ containing $M$, $\mathcal{L}^0(M)$ the subset of proper Levi subgroups in $\mathcal{L}(M)$, and $\mathcal{P}(M)$ the collection of parabolic subgroups of $G$ containing $M$. We shall fix a minimal Levi $M_0$ of $G$, and write $\mathcal{L} = \mathcal{L}(M_0)$ and $\mathcal{L}^0 = \mathcal{L}^0(M_0)$. For any $M \in \mathcal{L}$, we have the real vector space $a_M = \text{Hom}(X(M)_F, \mathbb{R})$, and the set

$$a_{M,V} = \{H_M(m) : m \in M(F_V)\}$$

is a subgroup of $a_M$, and $F_V = \prod_{v \in V} F_v$. It is equal to $a_M$ if $V$ contains an archimedean place, and is a lattice in $a_M$ otherwise.

2.3. \textbf{Hecke algebras.} Now, suppose we have fixed a maximal compact subgroup $K_v$ of $G_v$ at each place $v$. Following [Art02] §1, we define

$$\mathcal{H}(G_V^*, \zeta_V) = \mathcal{H}(G, V, \zeta)$$
to be the $\zeta_{V}^{-1}$-equivariant Hecke algebra of $G_{V}^{G}$, consisting of smooth, compactly-supported $K_{V}$-finite functions on $G_{V}$. We similarly define $\mathcal{H}(G_{v}, \zeta_{v})$ to be the local $\zeta_{v}^{-1}$-equivariant Hecke algebra. If $\mathcal{H}(G_{v})$ is the usual Hecke algebra of $G_{v}$, then one has a natural projection onto $\mathcal{H}(G_{v}, \zeta_{v})$ given by sending any $f_{v} \in \mathcal{H}(G_{v})$ to the function

$$
(2.1) \quad f_{v}^{\zeta}(x) = \int_{Z_{v}} f_{v}(zx) \zeta_{v}(z) dz, \quad x \in G_{v}
$$

in $\mathcal{H}(G_{v}, \zeta_{v})$.

We also require the larger space of almost compactly-supported functions $\mathcal{H}_{ac}(G_{v})$ introduced in [Art89] §11, defined in the following manner. Let $\Gamma$ be a finite subset of irreducible representations of $K_{v}$, and let $\mathcal{H}(G_{v})_{\Gamma}$ be the subspace of functions in $\mathcal{H}(G_{v})$ that transform on each side under $K_{v}$ according to representations in $\Gamma$. We then define $\mathcal{H}_{ac}(G_{v})_{\Gamma}$ to be the space of functions on $G_{v}$ such that for every $g \in C_{c}^{\infty}(a_{G_{v}})$ the function

$$
\begin{align*}
 f^{h}(x) &= f(x)g(H_{G}(x)) \\
 \text{belongs to } \mathcal{H}(G_{v})_{\Gamma}. \text{ We then form the direct limit over finite subsets } \Gamma, \\
 \mathcal{H}_{ac}(G_{v}) &= \lim_{\Gamma} \mathcal{H}_{ac}(G_{v})_{\Gamma}.
\end{align*}
$$

We may also view $\mathcal{H}_{ac}(G_{v})$ as the space of uniformly smooth functions $f$ in $\mathcal{H}(G_{v})$ such that for any $X \in a_{G_{v}}$, the restriction of $f$ to the preimage of $X$ in $G_{v}$ is compactly supported. By uniformly smooth, we mean that the function $f$ is bi-invariant under an open compact subgroup of $G_{v}$. We can then define

$$
\mathcal{H}_{ac}(G_{v}, \zeta_{v}) = \{ f^{\zeta} : f \in \mathcal{H}(G_{v}, \zeta_{v}) \}
$$

to be the image of $\mathcal{H}_{ac}(G_{v})$ under the mapping $f \mapsto f^{\zeta}$ given by (2.1), containing $\mathcal{H}(G_{v}, \zeta_{v})$. Similarly, we define

$$
\mathcal{H}_{ac}(G_{V}^{G}, \zeta_{V}) = \mathcal{H}_{ac}(G, V, \zeta) = \{ f^{\zeta} : f \in \mathcal{H}(G, V, \zeta) \}.
$$

We topologise these spaces in the manner described in [Art89] §11. As remarked in [Art05] §23, the invariant linear forms $I_{M}(\gamma, f)$ and $I_{M}(\pi, f)$ that we shall study are determined by their restriction to $\mathcal{H}(G_{V})$ of $\mathcal{H}_{ac}(G_{V})$, and moreover that $I_{M}(\gamma, f)$ can be extended continuous to a linear form on the Schwartz space $\mathcal{E}(G_{V})$. We also define the analogous spaces $\mathcal{E}(G_{V}^{G}, \zeta_{V})$ and its variants above.

2.4. Basic functions. We can now introduce the basic functions that enter into the unramified terms in the trace formula. Let $r$ be an irreducible finite-dimensional complex representation of the $L$-group $L_{G}$ of $G$. Suppose that $V$ contains $V_{\infty}$ and $V_{\text{ram}}(G, \zeta)$, and that $r$ is unramified outside of $V$. Let

$$
\mathcal{C} = \{ c_{v} : v \notin V \}
$$

be a family of semisimple conjugacy classes in the local $L$-group $L_{G_{v}} = \hat{G} \rtimes W_{F_{v}}$ whose image in the Weil group $W_{F_{v}}$ is a Frobenius element. We also assume that the image of $c_{v}$ under the projection $L_{G_{v}} \to L_{Z_{v}}$ is the conjugacy class in $L_{Z_{v}}$ associated to $\zeta_{v}$, and that for every $\hat{G}$-invariant polynomial $A$ on $L_{G}$, we have $|A(c_{v})| \leq q_{v}^{-\alpha}$ for each $v \notin V$ and for some constant $\alpha > 0$ depending only on $A$. To such an element we may associate the unramified local $L$-function

$$
L_{v}(s, c, r) = \det(1 - r(c_{v})q_{v}^{-s})^{-1}, \quad s \in \mathbb{C}
$$

where
which is analytic in some right half plane. It can be expressed as the formal series
\[ \sum_{k=0}^{\infty} \text{tr}(\text{Sym}^k r)(c_v)q_v^{-ks}, \]
where \( q_v \) is the cardinality of the residue field of \( F_v \). Let us write \( \mathcal{H}(G_v, K_v) \) for the spherical Hecke algebra of \( G_v \) with respect to \( K_v \), and define \( \mathcal{H}_{ac}(G_v, K_v) \) as the space of bi-\( K_v \)-invariant functions in \( \mathcal{H}_{ac}(G_v) \). Suppose we have fixed a Borel pair \((B_v, T_v)\) of \( G_v \) defined over \( F_v \) such that the torus \( T_v \) splits over a finite unramified extension of \( F_v \). We shall also assume that \( K_v \) is a hyperspecial maximal compact subgroup that lies in the apartment of \( T \) and write \( K_{T_v} = K_v \cap T_v \). Let \( W_v = N_{G_v}/T_v \) be the Weyl group of \( T_v \) in \( G_v \). We recall that the Satake isomorphism gives a bijection
\[ \mathcal{H}(G_v, K_v) \to \mathcal{H}(T_v, K_{T_v})^{W_v}, \]
where the right hand side can be identified with the coordinate algebra of \( W_v \)-invariant regular functions on the complex torus \( T_v = X(T_v) \otimes \mathbb{C} \). The isomorphism extends to the slightly larger space as a consequence of [LM] Proposition 2.7,
\[ \mathcal{H}_{ac}(G_v, K_v) \to \mathcal{H}_{ac}(T_v, K_{T_v})^{W_v}, \tag{2.2} \]
where the left hand side now consists of \( W_v \)-invariant formal series on the same torus.

The basic function \( b_v \) belongs to \( \mathcal{H}_{ac}(G_v, K_v) \) by [Won22, Lemma 2.1], which maps to an element in \( \mathcal{H}_{ac}(G_v, \zeta_v) \) by (2.1). It also belongs to the Fréchet algebra
\[ \mathcal{C}^\infty(G, V, \zeta) = \mathcal{C}^\infty(G^\mathbb{C}, \zeta) \]
defined in [Won22] §2.1, modeled after the space introduced by Finis, Lapid, and Müller to extend the trace formula. It is topologised using a family of seminorms on right \( K_V \)-invariant functions and then taking an inductive limit. The Hecke algebra \( \mathcal{H}(G, V, \zeta) \) is dense in \( \mathcal{C}^\infty(G, V, \zeta) \), which is in turn contained in \( \mathcal{C}(G, V, \zeta) \) as a subalgebra, though \( \mathcal{C}^\infty(G, V, \zeta) \) does not contain \( \mathcal{H}_{ac}(G, V, \zeta) \). Most importantly, the results of Arthur on local distributions on \( \mathcal{C}(G_v, \zeta_v) \) will apply to \( \mathcal{C}^\infty(G_v, \zeta_v) \) also. (In fact, the exact space that we are most interested is what is now called the \( r \)-Schwartz space, after Ngô [Ngo16] §4, which is precisely the space of functions of the form (1.1), which includes (1.2) as a special case.)

If \( G \) is a connected reductive group, then \( b_v = b_{v,s}^e \) is the unramified spherical function in \( \mathcal{H}_{ac}(G_v, \zeta_v) \) whose character at the representation \( \pi_v = \pi(c_v) \) associated to \( c_v \) satisfies
\[ b_{G_v}(c_v) = \text{tr}(\pi_v(b_v)) = L_v(s, c, r) \]
for any \( s \in \mathbb{C} \) with real part large enough. If \( G \) is a \( K \)-group, we note that each of the \( G_\alpha \) are related by inner twists and therefore share a common dual group \( \hat{G} \). We can thus write \( b_v = \oplus_{\alpha} b_{v,\alpha} \) and to each \( c_v \) we have the identity
\[ b_{G_v}(c_v) = |\pi_0(G_v)|L_v(s, c_v, r). \tag{2.3} \]
Throughout, we shall assume \( r \) and \( s \) to be fixed with \( \text{Re}(s) \) large unless otherwise specified.

**Remark 2.1.** As an aside, we note that if \( G \) is semisimple, one can also embed \( \hat{G} \) into a reductive group \( \hat{G}_r \) whose center \( Z(\hat{G}_r) \) is isomorphic to \( \mathbb{C}^\times \) [Cas17], and
extend \( r \) to a representation of \( \hat{G}_r \) such that \( r(z) = zI \), namely,

\[
\hat{G}_r = \frac{\mathbb{C}^\times \times \hat{G}}{\{ (r(z), z^{-1}) : z \in Z(\hat{G}) \}}.
\]

This provides a homomorphism \( \det_v : G_{r,v} \to F_v^\times \) for \( v \not\in V \), in which case \( b_{r,v}^s \) is equal to the product of \( b_{r,0}^s \) and \( | \det_v |^s \). We caution that while the local \( L \)-factor, and hence \( b_{r,v}^s \) may be defined at \( s = 0 \), it is by no means clear that the product over all \( v \not\in V \) converges, as it amounts to the meromorphic continuation of the unramified global \( L \)-function.

Finally, we assume that the invariant measure chosen on \( G(\mathbb{A}) \) or its subgroups satisfy any obvious compatibility conditions. For example, for any standard parabolic subgroup, we require that

\[
\int_{G(\mathbb{A})} f(x) \, dx = \int_K \int_{MP(\mathbb{A})} \int_{AP(\mathbb{R})^0} \int_{N_P(\mathbb{A})} f(mank) \, dn \, da \, dm \, dk,
\]

whenever the integral converges. We may also assume for simplicity that the Haar measures on \( K \) and \( N_P(\mathbb{A}) \) are normalized so that \( K \) and \( N_P(\mathbb{A}) \) have volume 1. The Haar measure \( dx \) on \( G(\mathbb{A}) \) is then determined by \( dm \) and \( da \), and the exponential map induces a measure \( dH \) on \( \mathfrak{a}_P \). The measure on \( AP(\mathbb{R})^0 \) then is obtained from that on \( \mathbb{R} \), which we take to be the usual Lebesgue measure.

2.5. Endoscopic data. The endoscopic datum for \( G \) consists of a connected quasisplit group \( G' \) over \( F \), embedded in a larger datum \((G', \mathcal{G}', s', \xi')\), where we recall that \( G' \) is a quasisplit group over \( F \), \( \mathcal{G}' \) is a split extension of \( W_F \) by a dual group \( \hat{G}' \) of \( G' \), \( s' \) is a semisimple element in \( \hat{G}' \), and \( \xi' \) is an \( L \)-embedding of \( \mathcal{G}' \) into \( L \)-torus [LS87 (1.3)]. It is required that \( \xi'(G') \) be equal to the connected centralizer of \( s' \) in \( \hat{G}' \), and that \( \xi'(u')s' = s'\xi'(u')a(u') \) for any \( u' \in \mathcal{G}' \) and locally trivial cocycle \( a \in H^1(W_F, Z(\hat{G})) \). We say that an endoscopic datum \( G' \) is elliptic if the connected component of the identity in the centralizer of \( \xi'(\mathcal{G}) \) in \( \hat{G} \) is trivial, which is the same as saying that the image of \( \xi' \) in \( L \)-torus is not contained in \( L \)-torus for any proper Levi subgroup \( M \) of \( G \) over \( F \).

Let \( G_{v,\alpha_v} \) be a connected component of \( G_v \). We write \( \Gamma_{\text{reg}}(G_{v,\alpha_v}) \) for the set of strongly regular conjugacy classes in \( G_{v,\alpha_v} \), and also

\[
\Gamma_{\text{reg}}(G_v) = \prod_{\alpha_v \in \pi_0(G_v)} \Gamma_{\text{reg}}(G_{\alpha_v,v}).
\]

We shall write \( \mathcal{E}(G) \) for the set of isomorphism classes of endoscopic data for \( G \) over \( F \) that are locally relevant to \( G \), in the sense that for every \( v \), \( G'_{v} \) has a strongly \( G \)-regular element that is an image of some class in \( \Gamma_{\text{reg}}(G_v) \). If \( V \) is a finite set of valuations of \( F \) that contains \( V_{\text{ram}}(G) \), we write \( \mathcal{E}(G, V) \) for the subset of elements \( G' \in \mathcal{E}(G) \) that are unramified outside of \( V \). We also write \( \mathcal{E}_{\text{cel}}(G) \) and \( \mathcal{E}_{\text{cel}}(G, V) \) for the subset of elements in \( \mathcal{E}(G) \) and \( \mathcal{E}(G, V) \), respectively, that are elliptic over \( F \). Given any \( G' \in \mathcal{E}_{\text{cel}}(G) \), we shall fix auxiliary data \((\hat{G}', \xi')\) where \( \hat{G}' \to G' \) is a central extension of \( G' \) by an induced torus \( \hat{G}' \) and \( \xi' : \mathcal{G}' \to L \hat{G}' \) is an \( L \)-embedding satisfying the conditions of [Art96 Lemma 2.1].
2.6. Transfer factors. To discuss the endoscopic transfer of functions, we have to define the transfer factors that we shall use, following [Art02 §4]. Let $\gamma_v$ be a strongly regular element of $G_v$ and $\delta'_v$ a strongly $G$-regular element in $\tilde{G}'_v$. Suppose that $V$ contains $V_{\text{ram}}(G)$. We shall use the local transfer factors for $K$-groups

$$\Delta(\delta'_v, \gamma_v) = \Delta_K(\delta'_v, \gamma_v),$$

normalized according to [Art02 §4]. In particular, they depend a priori on a choice of local base points $\delta'_v \in \tilde{G}'(F)$ and $\gamma_v \in G_\alpha(F)$ for some $\alpha \in \pi_0(G)$, but can in fact be normalized to depend only on the choice of hyperspecial maximal compact subgroup $K_v$ of $G_v$. We recall that two elements $\gamma \in G_{\alpha,V}$ and $\delta \in G_{\beta,V}$ are called stably conjugate if $\psi_{\alpha\beta}(\delta)$ is stably conjugate to $\gamma$ in $G_{\alpha,V}$. Let $\Delta_{G,\text{reg}}(\tilde{G}'_v)$ be the the set of $G$-regular stable conjugacy classes in $\tilde{G}'_v$. The transfer factor for $\gamma_V \in \Gamma_{\text{reg}}(G_V)$ and $\delta'_V \in \Delta_{G,\text{reg}}(\tilde{G}'_V)$ is then defined as the product

$$\Delta(\delta'_V, \gamma_V) = \prod_{v \in V} \Delta(\delta'_v, \gamma_v)$$

of local transfer factors, which depends only the hyperspecial maximal compact subgroup $R^V = \prod_{v \in V} K_v$. We shall also consider local endoscopic data $G'_v$ of the local $K$-groups $G_v$, and write $\mathcal{E}(G_V)$ for the set of products $G'_V = \prod_{v \in V} G'_v$ of $G'_v \in \mathcal{E}(G_v)$. For such a $G'_V$, the transfer factor

$$\Delta(\delta'_V, \gamma_V), \quad \delta'_V \in \prod_{v \in V} \Delta_{G,\text{reg}}(G'_v), \quad \gamma_V \in \prod_{v \in V} \Gamma_{\text{reg}}(G_v)$$

is a local object that does not depend on the local base point or the corresponding auxiliary data $(\tilde{G}'_V, \tilde{\xi}'_V)$.

Let us also define $\tilde{\Delta}^{\mathcal{E}}_{\text{reg}}(G_V)$ to be the quotient of the family of elements

$$\{ \{G'_V, \tilde{x}'_V, \delta'_V\} : G'_V \in \mathcal{E}(G_V), \tilde{x}'_V : \mathcal{E}' \to L\tilde{G}'_V, \delta'_V \in \Delta_{G,\text{reg}}(\tilde{G}'_V) \}$$

that are $G_V$-images, up to the equivalence defined in [Art96 §2] for the set that was denoted as $\tilde{\Gamma}^{\mathcal{E}}(G_v)$ therein. We similarly define $\tilde{\Delta}^{\mathcal{E}}_{\text{reg}}(G_V)$ for the quotient of the family of elements in $(G'_V, \delta'_V)$ that are $G_V$-images, where $G'_V \in \mathcal{E}(G_V)$ and $\delta'_V \in \Delta_{G,\text{reg}}(G'_V)$. The transfer factor depends only on the image $\delta'_V$ of $\delta'_v$ in $\tilde{\Delta}^{\mathcal{E}}_{\text{reg}}(G_V)$. We can therefore regard the transfer factor

$$\Delta(\delta'_V, \gamma_V) = \Delta(\delta'_V, \gamma_V)$$

as a function on $\tilde{\Delta}^{\mathcal{E}}_{\text{reg}}(G_V) \times \Gamma_{\text{reg}}(G_V)$.

2.7. Orbital integrals. Suppose for the moment that $G$ is a connected reductive group over $F$. Given an element $\gamma_V \in G_{\gamma,V}$, let $G_{\gamma,V} = \prod_v G_{\gamma_v}$ be its centralizer and $D(\gamma_V) = \prod_v D(\gamma_v)$ the Weyl discriminant for some choice of invariant measure on the quotient $G_{\gamma,V} \cap G_{\mathfrak{X}} \backslash G_{\mathfrak{E}}$. Then for any $f \in \mathcal{E}(G'_{\mathfrak{E}})$, we define the orbital integral of $f$ at $\gamma_V$ as

$$f_G(\gamma_V) = |D(\gamma_V)|^{\frac{1}{2}} \int_{G_{\gamma,V} \cap G_{\mathfrak{X}} \backslash G_{\mathfrak{E}}} f(x^{-1} \gamma_V x) dx$$

and the $\zeta_V$-equivariant analogue

$$\int_{\mathfrak{Z}_V} \zeta_V(z) f_G(z \gamma_V) dz.$$
The latter belongs to the space of $\zeta_V$-equivariant distributions $\mathcal{D}(G^+_V, \zeta_V)$ that are invariant under $G^+_V$-conjugation and supported on the preimage in $G^+_V$ of a finite union of conjugacy classes in $G^+_V / \mathbf{Z}_V$. We write $\mathcal{D}_{\text{orb}}(G^+_V, \zeta_V)$ for the subspace of $\mathcal{D}(G^+_V, \zeta_V)$ spanned by (2.4). The spaces are equal if $V$ contains only nonarchimedean places, and contain more general distributions otherwise. We shall fix a basis $\Gamma(G^+_V, \zeta_V)$, satisfying natural compatibility conditions described in [Art02, p.186]. For example, the intersection $\Gamma_{\text{orb}}(G^+_V, \zeta_V) = \Gamma(G^+_V, \zeta_V) \cap \mathcal{D}_{\text{orb}}(G^+_V, \zeta_V)$ is required to be again a basis of $\mathcal{D}_{\text{orb}}(G^+_V, \zeta_V)$. Given an element $\gamma_V$ in $G^+_V$, we can identify it with an element in $\mathcal{D}_{\text{orb}}(G^+_V, \zeta_V)$ by sending $f \in \mathcal{C}(G^+_V)$ to the $\zeta_V$-equivariant orbital integral of $f$ at $\gamma_V$.

We can further define the weighted orbital integral at a strongly regular conjugacy class $\gamma_V$ of $M^+_V$ as

$$J_M(\gamma_V, f) = |D(\gamma_V)|^{1/2} \int_{G^+_V \setminus G^+_V} f(x^{-1} \gamma_V x) v_M(x) dx,$$

where $v_M(x)$ is the volume of a certain convex hull depending on $x$ and $M$. The weighted orbital integrals at singular elements are more complicated to define, and are described in [Art16, §1] for real groups and [Art88c, (6.2)] for $p$-adic groups. The $\zeta_V$-equivariant analogues are then defined in a similar fashion as (2.4), and the extension of these definitions to a $K$-group $G$ is largely formal (cf. [Art99b, §3]).

Let $S$ be a large finite set of valuations of $V$ containing $V \supset V_{\infty}$. The unramified weighted orbital integrals will be defined at the places $v$ in $S - V$. Let us write

$$G^+_S = \prod_{v \in S - V} G_v, \quad Z^+_S = \prod_{v \in S - V} Z_v, \quad K^+_S = \prod_{v \in S - V} K_v,$$

and define $\mathcal{K}(G^+_S)$ to be the set of conjugacy classes in $G^+_S = G^+_S / \mathbf{Z}_S$ that are bounded, meaning that for each $v \in S - V$, the image of each element lies in a compact subgroup of $G_v$. Any element $k \in \mathcal{K}(G^+_S)$ determines a distribution $\gamma^+_S(k)$ in $\mathcal{D}_{\text{orb}}(G^+_S, \zeta_S)$. Given $\gamma \in \Gamma(G^+_S, \zeta_S)$ and $k \in \mathcal{K}(G^+_S)$, we write $\gamma \times k = \gamma \times \gamma^+_S(k)$ for the associated element in $\Gamma(G^+_S, \zeta_S)$. Let $u_v$ be the function on $G_v$ supported on $K_v Z_v$ such that $u_v(k z) = \zeta^{-1}(z)$ for any $k \in K_v$ and $z \in Z_v$. We define the unramified weighted orbital integral

$$r^+_M(k) = J_M(\gamma^+_S(k), u^+_S)$$

on $\mathcal{K}(M^+_S)$.

2.8. **Stable distributions.** We are almost ready to state the weighted transfer conjecture. Suppose that $c$ belongs to the set of semisimple conjugacy class $\Gamma_{\text{ss}}(G^+_V)$ of $G^+_V$. We write $\mathcal{D}_c(G^+_V, \zeta_V)$ for the subspace of distributions in $\mathcal{D}(G^+_V, \zeta_V)$ whose preimages in $G^+_V$ are supported on conjugacy classes with semisimple part equal to $c$. We then write $\Gamma_{\text{ss}}(G^+_V, \zeta_V)$ for the classes $c$ such that $\mathcal{D}_c(G^+_V, \zeta_V)$ is nonzero. As with [Art02, §5], we shall assume for simplicity that $\Gamma_{\text{ss}}(G^+_S, \zeta_S)$ is equal to $\Gamma_{\text{ss}}(G^+_V)$, so that there is an injection from $\mathcal{K}(G^+_S)$ into $\Gamma(G^+_S, \zeta_S)$.

We call a distribution on $G^+_S$ stable if it lies in the closed linear span of the strongly regular, stable orbital integrals

$$f^G(\delta_V) = \sum_{\gamma_V} f_G(\gamma_V),$$

where $\delta_V$ is any strongly regular, stable conjugacy class in $G^+_V$ and the sum runs over the finite set of conjugacy classes in $\delta_V$. Let $S \mathcal{D}(G^+_V, \zeta_V)$ be the subspace
of stable distributions in $\mathcal{D}(G^*_V, \zeta_V)$. We can identify any strongly regular element $\delta \in \Delta_{G_{-\text{reg}}}(G^*_V)$ with a subset $\Delta_{G_{-\text{reg}}}(G^*_V)$ of $S\mathcal{D}(G^*_V, \zeta_V)$ generated by $f^G(\delta_V)$ as in [Art02 §1]. Similarly we fix a subset $\Delta_{G_{-\text{reg}}}(G^*_V, \zeta')$ of $G$-regular elements in $S\mathcal{D}(G^*_V, \zeta_V)$. The transfer factor can then be converted to a function on $\Delta_{G_{-\text{reg}}}(G^*_V, \zeta') \times \Gamma_{G_{-\text{reg}}}(G^*_V, \zeta')$ by [Art02 §4]. (See also §9.2).

### 3. A Weighted Transfer Conjecture

The stable trace formula established by Arthur depends on a weighted Fundamental Lemma, which is a generalization of the Fundamental Lemma originally conjectured by Langlands and Shelstad [LS87]. The unweighted version was proved by Ngô [Ngô10] for Lie algebras in positive characteristic, from which the required result follows. It implies the Langlands-Shelstad transfer of orbital integrals [Wal97a], which is required in the various endoscopic constructions necessary for the stabilization of the trace formula. On the other hand, the weighted Fundamental Lemma for Lie algebras has been proved by Chaudouard and Laumon in the split case and in positive characteristic [CL12], and by Waldspurger this yields the weighted Fundamental Lemma in characteristic zero [Wal09]. Similar methods are expected apply to the general case of quasisplit groups, which the full stabilization of the trace formula is conditional upon. In our case, the stabilization of the unramified terms in the trace formula will require a transfer of weighted orbital integrals of functions other than the unit element.

#### 3.1. Fundamental Lemma.

For this section, let $F$ be a nonarchimedean local field, and assume that $G, Z, \zeta$ are unramified over $F$. Following [Art02 §5], we write $\mathcal{K}_{G_{-\text{reg}}}(G)$ for the set of strongly regular conjugacy classes of $G(F) = G(F)/Z(F)$ that are bounded, and $k \to \gamma(k)$ for the canonical injection from $\mathcal{K}_{G_{-\text{reg}}}(G)$ to $\Gamma_{G_{-\text{reg}}}(G, \zeta) = \Gamma_{G_{-\text{reg}}}(G^*_V, \zeta_V)$. We also write $\mathcal{L}_{G_{-\text{reg}}}(G)$ for the set of strongly regular stable conjugacy classes in $G(F)$ that are bounded, and $\ell \to \delta(\ell)$ for the corresponding injection from $\mathcal{L}_{G_{-\text{reg}}}(G)$ to $\Delta_{G_{-\text{reg}}}(G, \zeta) = \Delta_{G_{-\text{reg}}}(G^*_V, \zeta_V)$. If $G'$ is any local endoscopic datum for $G$, the normalized transfer factor $\Delta_K(\delta', \gamma)$ attached to the hyperspecial maximal compact subgroup $K$ of $G(F)$ is a canonical function on $\Delta_{G_{-\text{reg}}}(G^*_V, \zeta') \times \Gamma_{G_{-\text{reg}}}(G^*_V, \zeta_V)$. It depends on the choice of auxiliary data $(G', \xi')$ attached to $G'$. If $G'$ is unramified, there is a canonical class of admissible embeddings of $L^G$ into $L_G$, so that we may set $G' = G'$. We set

$$\Delta_K(\ell', k) = \Delta_K(\delta'(\ell'), \gamma(k)), \quad \ell' \in \mathcal{L}_{G_{-\text{reg}}}(G'), k \in \mathcal{K}_{G_{-\text{reg}}}(G')$$

for the unramified endoscopic datum $G'$. It is independent of the character $\zeta$ and the choice of $\xi'$.

Now suppose that $M$ is a Levi subgroup of $G$ in good position relative to $K$. We write $r_M^G(k) = J_M(k, u)$ as before for $k \in \mathcal{K}_{G_{-\text{reg}}(M)}$, and note that it is also independent of the central datum $(Z, \zeta)$. If $M'$ is an unramified endoscopic datum of $M$, we then have the transfer factor

$$\Delta_{K \cap M}(\ell', k), \quad \ell' \in \mathcal{L}_{G_{-\text{reg}}}(M'), k \in \mathcal{K}_{G_{-\text{reg}}(M')}.$$

We define $\delta_{M'}(G)$ to be the set of endoscopic data $G'$ for $G$, taken up to translation of $s'$ by $Z(G)^F$ with $\Gamma = \Gamma_F$ the absolute Galois group of $F$, in which $s'$ lies in $s_M^G(Z(M)^F, G')$ is the connected centralizer of $s'$ in $G$, $\Phi'$ equals $\Phi G'$, and $\xi'$ is the identity embedding of $G'$ in $L_G$. For each $G' \in \mathcal{E}_{M'}(G)$, we fix an embedding $M' \subset G'$ for which $M' \subset G'$ is a dual Levi subgroup. We also fix auxiliary data
\[(\tilde{G}', \tilde{\xi}') \text{ for } G', \text{ where } \tilde{G}' \to G' \text{ and } \tilde{\xi}' : \mathcal{G}' \to L\tilde{G}', \text{ that restrict to auxiliary data } (M', \tilde{\xi}'_M) \text{ for } M', \text{ where } \tilde{M}' \to M' \text{ and } \tilde{\xi}'_M : \mathcal{G}' \to L\tilde{M}', \text{ and whose central character data } (\tilde{Z}', \tilde{\zeta}') \text{ coincide. Note that } G^* \text{ belongs to } \mathcal{E}_{M'}(G) \text{ if and only if } M' = M^*.\]

Finally, we define the coefficient
\[t_{M'}(G, G') = |Z(\tilde{M})^\Gamma / Z(\tilde{M})^\Gamma| |Z(\tilde{G})^\Gamma / Z(\tilde{G})^\Gamma|^{-1}\]

for any \(G' \in \mathcal{E}_{M'}(G)\).

We can then state the weighted Fundamental Lemma [Art02 Conjecture 5.1], which we recall is now a theorem in the case of split groups \(G\) due to [Wal09 Théorème 8.7] and [CL12 Théorème 1.4.1]. We may assume that the central datum \((Z, \zeta)\) is trivial, since the functions involved are independent of it.

**Theorem 3.1.** For each \(G\) quasisplit and \(M\), there is a function
\[s_M^G(\ell), \quad \ell \in \mathcal{L}_{G, \text{reg}}(M)\]

with the property that for any \(K\), any unramified elliptic endoscopic datum \(M'\) for \(M\), and any element \(\ell' \in \mathcal{L}_{G, \text{reg}}(M')\), the transfer
\[r_{M'}^G(\ell') = \sum_{k \in E_{G, \text{reg}}(M)} \Delta_{K \cap M}(\ell', k) r_M^G(k)\]
equals
\[\sum_{G' \in \mathcal{E}_{M'}(G)} t_{M'}(G, G') s_{M'}^{G'}(\ell').\]  

(3.1)

The function \(s_M^G(\ell)\) is uniquely determined by the required identity. If \(M' = M\), the quasisplit group \(G\) belongs to \(\mathcal{E}_{M'}(G)\), and the identity becomes
\[s_M^G(\ell) = \sum_{k \in E_{G, \text{reg}}(M)} \Delta_{K \cap M}(\ell', k) r_M^G(k) - \sum_{G' \neq G} t_{M'}(G, G') s_{M'}^{G'}(\ell'),\]

which serves as an inductive definition of \(s_M^G(\ell)\). When \(M = G\), the expression \(s_G^G(\ell') = s_{G'}^{G'}(\ell')\), and if \(G' \neq G\) the required identity reduces to the standard Fundamental Lemma, whereas if \(G' = G\) there is nothing to prove. On the other hand, if \(M\) is the minimal Levi subgroup, then \(M' = M\) is the only endoscopic datum and the statement is trivial.

**Remark 3.2.** We note that implicit in the statement of the weighted fundamental lemma is that this sum of *noninvariant* weighted orbital integrals is not only invariant but stably invariant. This expectation will continue to hold in our formulation of the weighted transfer below.

### 3.2. Transfer

Let us return to \(F\) being a global field. Let us write
\[
\mathcal{I}(G_V, \zeta_V), \quad \mathcal{I}(G_V, \zeta_V)
\]
for the space of functions generated by ordinary orbital integrals \(f_G\) and stable orbital integrals \(f^G\) respectively of functions \(f \in \mathcal{H}(G_V, \zeta_V)\). We also define the larger spaces
\[
\mathcal{I}_{ac}(G_V, \zeta_V), \quad \mathcal{I}_{ac}(G_V, \zeta_V)
\]

corresponding to \(f \in \mathcal{H}_{ac}(G_V, \zeta_V)\). Similarly, if we define \(\mathcal{C}(G_V, \zeta_V)\) to be the space of \(\zeta_V^{-1}\)-equivariant Schwartz functions on \(G_V\), we write
\[
I\mathcal{C}(G_V, \zeta_V), \quad S\mathcal{C}(G_V, \zeta_V)
\]
for the spaces generated by \( f_G \) and \( f^G \) respectively for all \( f \in \mathcal{C}(G_V, \zeta_V) \). The 
topologies on these spaces are chosen such that the maps \( f \to f_G \) and \( f \to f^G \) 
respectively are open and continuous. We also note that the topology in \( \mathcal{C}(G_V, \zeta_V) \) 
is taken to be the usual one induced by the family of seminorms used to define the 
Schwartz space. Finally, we also denote the analogous subspaces \( I_{\mathcal{C}^0}(G_V, \zeta_V) \) 
and \( S\mathcal{C}^0(G_V, \zeta_V) \).

The Langlands-Shelstad transfer then asserts that for any \( G'_V \in \mathcal{E}(G_V) \), the 
map that sends \( f \in \mathcal{H}(G_V, \zeta_V) \) to the function

\[
f'(\delta') = f^G(\delta') = \sum_{\gamma \in \Gamma_{\text{reg}}(G_V, \zeta_V)} \Delta(\delta', \gamma) f_G(\gamma)
\]

where \( \delta' \in \Delta_{G, \text{reg}}(G'_V, \zeta'_V) \), is a continuous map from \( \mathcal{H}(G_V, \zeta_V) \) to \( \mathcal{S}(G'_V, \zeta'_V) \) 
[Art02 §5]. As a consequence of Lemma 11.1 this extends to a map from \( \mathcal{C}(G_V, \zeta_V) \) 
to \( S\mathcal{C}(G'_V, \zeta'_V) \). (Note that Shelstad proved the archimedean transfer for Schwartz 
functions [She82] long ago.) Moreover, by continuity with respect to the topology 
above, we see that the transfer also extends to a continuous map from \( \mathcal{H}_{\text{ac}}(G_V, \zeta_V) \) 
to \( \mathcal{S}_{\text{ac}}(G'_V, \zeta'_V) \). That is, as described in in [Art02, p.224], the Langlands-Shelstad 
transfer \( f \mapsto f' \) sends \( \mathcal{H}(G_V, \zeta_V) \) continuously \( \mathcal{S}(G'_V, \zeta'_V) \). Given the topologies 
on the associated spaces, the extension of the transfer to a map from \( \mathcal{C}(G_V, \zeta_V) \) to 
\( \mathcal{S}_{\text{ac}}(G'_V, \zeta'_V) \) then follows from the density properties above.

We are interested in the transfer of weighted orbital integrals. We recall the 
invariant linear forms on \( \mathcal{H}_{\text{ac}}(G'_V, \zeta_V) \) defined inductively by

\[
I_M(\gamma, f) = J_M(\gamma, f) - \sum_{L \in \mathcal{L}(M)} I_M^L(\gamma, \phi_L(f))
\]

where \( \phi_L \) is the map on \( \mathcal{H}_{\text{ac}}(G'_V, \zeta_V) \) defined in [Art02 (2.3)] and [Art88a §2]. We 
recall that they are determined by their restriction to the subspace \( \mathcal{H}(G'_V, \zeta_V) \) of 
\( \mathcal{H}_{\text{ac}}(G'_V, \zeta_V) \), and extend to continuously to \( \mathcal{C}(G'_V, \zeta_V) \). Suppose that we have fixed 
so-called endoscopic and stable bases \( \Delta^\mathcal{E}(G'_V, \zeta_V) \) and \( \Delta(G'_V, \zeta_V) \) of the spaces 
\( \mathcal{D}(G'_V, \zeta_V) \) and \( \mathcal{S}\mathcal{D}(G'_V, \zeta_V) \) respectively following the requirements of [Art02 §9] 
(see also §9.2); for example, they are chosen to be subsets of the corresponding bases 
\( \Delta^\mathcal{E}(G_V, \zeta_V) \) and \( \Delta(G_V, \zeta_V) \) of the spaces \( \mathcal{D}(G_V, \zeta_V) \) and \( \mathcal{S}\mathcal{D}(G_V, \zeta_V) \) respectively. Assume inductively that for each \( G' \in \mathcal{E}_{M'}(G) \) we have defined stable linear forms 
\( S_M^G(\delta', f') \) on \( \mathcal{S}_{\text{ac}}(G'_V, \zeta'_V) \) with \( \delta' \in \Delta(M'_V, \zeta'_V) \). Let

\[
\mathcal{E}^0_{M'}(G) = \begin{cases} 
\mathcal{E}_{M'}(G) - \{G^*\}, & \text{if } G \text{ is quasisplit} \\
\mathcal{E}_{M'}(G), & \text{otherwise,}
\end{cases}
\]

and let \( \varepsilon(G) = 1 \) if \( G \) is quasisplit and equal to 0 otherwise. We then define linear 
forms \( I_M^G(\delta', f) \) and \( S_M^G(M', \delta', f) \) inductively by the formula

\[
I_M^G(\delta', f) = \sum_{G' \in \mathcal{E}^0_{M'}(G)} I_{M'}(G', f') S_M^G(\delta', f') + \varepsilon(G) S_M^G(M', \delta', f)
\]

together with the supplementary requirement that

\[
I_M^G(\delta', f) = I_M(\delta, f)
\]
in the case that $G$ is quasisplit and $\delta'$ maps to the element $\delta \in \Delta^G(M_{\ell}^Z, \zeta_V)$, where

$$I_M(\delta, f) = \sum_{\gamma \in \Gamma(M_{\ell}^G, \delta, \zeta_V)} \Delta_M(\delta, \gamma) I_M(\gamma, f)$$

and $\Delta_M(\delta, \gamma)$ is the generalized transfer factor for $M$ that is compatible with the generalized transfer factor for $G$. Namely, if $\mu \rightarrow \mu^G$ and $\gamma \rightarrow \gamma_M$ are the induction and restriction maps respectively between the spaces $\mathcal{D}(G, \zeta_V)$ and $\mathcal{D}(M, \zeta_V)$ such that $f_G(\mu^G) \rightarrow f_M(\mu)$ and

$$\sum_{\gamma \in \Gamma(G, \zeta_V)} a_M(\gamma_M)b_G(\gamma) = \sum_{\mu \in \Gamma(M, \zeta_V)} a_M(\mu)b_G(\mu)$$

respectively for any linear functions $a_M$ on $\mathcal{D}(M, \zeta_V)$ and $b_G$ on $\mathcal{D}(G, \zeta_V)$, then we have that

$$\Delta_G(\nu^G, \gamma) = \Delta_M(\nu, \gamma_M), \quad \nu \in \Delta^G(M_{\ell}, \zeta_V), \gamma \in \Gamma(G, \zeta_V),$$

and

$$\Delta_G(\delta, \mu^G) = \Delta_M(\delta_M, \mu), \quad \delta \in \Delta^G(G_{\ell}, \zeta_V), \mu \in \Gamma(M, \zeta_V)$$

respectively. In the case that $G$ is quasisplit and $M' = M^*$, we have that $\delta' = \delta^*$ belongs to $\Delta((M_{\ell}^G)^*, \zeta_V)$ and the image $\delta$ of $\delta'$ in $\Delta^G(M_{\ell}^G, \zeta_V)$ lies in the subset $\Delta(M_{\ell}^G, \zeta_V)$. It follows from Corollary 9.3(b), a mild extension of the main local theorem [Art02], that the form

$$S^G_M(\delta, f) = S^G_M(M', \delta', f)$$

is stable, and vanishes unless $M' = M^*$. We thus have a linear form $S^G_M(\delta^*, f^*) = S^G_M(\delta, f)$ on $\mathcal{S}(\zeta_V)$ that is the analogue for $(G^*, M^*)$ of the terms $S^G_M(\delta', f')$.

Let us now state our weighted form of the Langlands-Shelstad transfer conjecture, formulated over a local field $F$. The preceding objects that we have just defined have natural analogues in this case.

**Conjecture 3.3.** For each $G$ and $M$, the stable linear form

$$\bar{S}^G_M(\delta, f), \quad \delta \in \Delta_{G, \text{reg}}(M, \zeta),$$

on $\mathcal{S}(\zeta_V)$ has the property that for any unramified local elliptic endoscopic datum $M'$ for $M$, and any element $\delta' \in \Delta_{G, \text{reg}}(M', \zeta_V)$, the transfer

$$J_M(\delta', f) = \sum_{\gamma \in \Gamma_{G, \text{reg}}(M, \zeta)} \Delta_M(\delta', \gamma) J_M(\gamma, f)$$

equals

$$\sum_{\gamma' \in \Delta_{G}(G)} \iota_{M'}(\gamma') \bar{S}^G_M(\delta', f').$$

As with the case of the ordinary Langlands-Shelstad transfer, one may also conjecturally specify the transfer $f'$ in the case that $f$ belongs to the spherical Hecke algebra by means of the $L$-embedding $\xi'$. We remind here that $f'$ is unique only up to its weighted orbital integral, just as in the case of ordinary transfer where $f' = f^G$ is identified with its stable orbital integral. We have formulated a more general statement than strictly necessary, however, as for our purposes we only need the following special case, that is, only for nonarchimedean local fields $F_v$ for which $v \not\in V \supset V_{\text{ram}}(G, \zeta)$ and where $f = b$. As in the case of Theorem 3.1, we may again assume that $(Z, \zeta)$ is trivial.
Corollary 3.4. Assume Conjecture 3.3. For each $G, M$ and $b$, there is a function

$$s_{M}^{G}(\ell, b), \quad \ell \in \mathcal{L}_{G, \text{reg}}(M)$$

with the property that for any $K$, any unramified elliptic endoscopic datum $M'$ for $M$, and any element $\ell' \in \mathcal{L}_{G, \text{reg}}(M')$, the transfer

$$r_{M}^{G}(\ell', b) = \sum_{k \in K_{G, \text{reg}}(M)} \Delta_{K \cap M}(\ell', k)r_{M}^{G}(k, b)$$

equals

$$\sum_{G' \in \mathcal{E}_{M}(G)} t_{M'}(G, G')s_{M'}^{G'}(\ell', b').$$

In particular, the functions $s_{M}^{G'}(\ell', b')$ here would be given by the linear form $S_{M'}^{G'}(\delta', b')$ where $b'$ is the transfer of $b$. Evidence for the conjecture is of course the weighted Fundamental Lemma itself, which is the case $f = u$, and the unweighted transfer which is the case of $M = G$. Finally, we note that one can also formulate an analogue of Conjecture 3.3 for Lie algebras, as Waldspurger has pointed out that it should be possible to formulate an analogous Lie algebra statement for the transfer of weighted orbital integrals [Wal95, VIII.7], which we can take as further support for the conjecture. (We note that already in the ordinary weighted Fundamental Lemma, it is a sum of noninvariant weighted orbital integrals that is shown to be stably invariant.)

Remark 3.5. We remind the reader that this conjecture is only needed to streamline the stabilisation of the unramified terms at a finite number of places, and if we do not assume this conjecture, the main results hold unconditionally, only for a slightly smaller class of test functions of the form $\{1, 2\}$.  

4. The unramified geometric terms

4.1. The unramified terms. We can now turn to the unramified geometric terms that arise in the stable trace formula. We shall fix a suitably large finite set of places $S \supset V$. We shall assume that $V$ contains $V_{\infty}$, so the places in $S - V$ are nonarchimedean. We may thus assume that any distribution on $\Gamma(G_{S}^{V}, \zeta_{S}^{V})$ is defined by a signed measure on the preimage in $G_{S}^{V}$ of a conjugacy class in $\hat{G}_{S}^{V}$. We also assume that $\Gamma_{\text{ram}}(G_{S}^{V}, \zeta_{S}^{V})$ is equal to $\Gamma_{\text{ram}}(\hat{G}_{S}^{V})$, which holds if $V$ contains $V_{\text{ram}}(G, \zeta)$, so that there is an injection $k \to \gamma_{S}^{V}(k)$ from $\mathcal{K}(\hat{G}_{S}^{V})$ from $\mathcal{K}(G_{S}^{V})$ into $\Gamma(\hat{G}_{S}^{V}, \zeta_{S}^{V})$. Following [Won22, (4.11)], we define the unramified weighted orbital integrals

$$r_{M}^{G}(k, b) = J_{M}(\gamma_{S}^{V}(k), b_{S}^{V}), \quad k \in \mathcal{K}(\hat{M}_{S}^{V}).$$

Furthermore, we define the subset $\mathcal{L}(\hat{G}_{S}^{V})$ of $\Delta(\hat{G}_{S}^{V}, \zeta_{S}^{V})$ consisting of formal linear combinations of classes in $\mathcal{K}(\hat{G}_{S}^{V})$ corresponding to distributions in $\Delta(\hat{G}_{S}^{V}, \zeta_{S}^{V})$ under the linear extension of the map $\gamma_{S}^{V}$. It can be identified with a subset of the corresponding family $\mathcal{L}(G^{*})$ for $G^{*}$ by means of a canonical embedding $\ell \to \ell^{*}$. Similarly, we define the subset $\mathcal{L}(\hat{G}_{S}^{V})$ in $\Delta^{\mathcal{E}}(\hat{G}_{S}^{V}, \zeta_{S}^{V})$ as the quotient of $G$-relevant pairs in

$$\{(G', \ell') : G' \in \mathcal{E}(\hat{G}_{S}^{V}), \ell' \in \mathcal{L}(\hat{G}_{S}^{V})\}$$

with an injection $\ell \to \delta_{S}^{V}(\ell)$ into $\Delta^{\mathcal{E}}(\hat{G}_{S}^{V}, \zeta_{S}^{V})$, sending the subset $\mathcal{L}(\hat{G}_{S}^{V})$ into $\Delta(G_{S}^{V}, \zeta_{S}^{V})$. Suppose now that $V$ contains $V_{\text{ram}}(G, \zeta)$. The unramified function
\( r^G_M(k, b) \) depends on a choice of hyperspecial maximal compact subgroup \( K^G_S \) of \( G^S_M \), which we shall assume to be in good position relative to \( M^G_S \). The intersection \( K^G_S \cap M^G_S \) is also a hyperspecial maximal compact subgroup of \( M^G_S \). Following \cite{Art02} §8, we define the normalized transfer factor

\[
\Delta_{K^G_S, M^G_S}(\ell, k) = \Delta_{K^G_S \cap M^G_S}(\delta^G_S(\ell), \gamma^G_S(k)) = \prod_{v \in V - S} \Delta_{K_v, M_v}(\delta_v, \gamma_v)
\]

for \( k \in K(M^G_S) \) and \( \ell \in \mathcal{L}^e(M^G_S) \), and we use this to form the function

\[
(4.1) \quad r^G_M(\ell, b) = \sum_{k \in K(M^G_S)} \Delta_{K^G_S, M^G_S}(\ell, k)r^G_M(k, b).
\]

on \( \mathcal{L}^e(M^G_S) \), which is independent of the choice of \( K^G_S \). We note that the sets \( \mathcal{L}(G^G_S) \) and \( \mathcal{L}^e(G^G_S) \) are independent of \( \zeta^G_S \), and hence so is \( \Delta_{K^G_S, M^G_S}(\ell, k) \). In the following we shall consider triples \( (G, M, \zeta) \) where \( G \) is a reductive \( K \)-group over \( F \), \( M \) a Levi subgroup of \( G \), and \( \zeta \) a character of \( Z(\mathbb{A})/Z(F) \) for a central induced torus \( Z \) of \( G \).

**Proposition 4.1.** Assume Conjecture \[3, 5\] For each triple \( (G, M, \zeta) \) with \( G \) quasisplit, there is a function

\[
s^G_M(\ell, b) = s^G_M(\ell^*, b^*)
\]

which vanishes unless \( V \) contains \( V_{\text{ram}}(G) \), and such that for any elliptic endoscopic datum \( M' \) of \( M \) and any \( \ell' \in \mathcal{L}(M') \), with image \( \ell \in \mathcal{L}^e(M^G_S) \), we have

\[
r^G_M(\ell, b) = \sum_{G' \in \mathcal{E}_{M'}(G)} \iota_M'(G, G')s^G_M(\ell, b')
\]

where \( \mathcal{E}_{M'}(G) \) is the set of elliptic endoscopic data for \( G' \) and \( M' \).

**Proof.** If \( V \) does not contain \( V_{\text{ram}}(G) \), we set \( s^G_M(\ell, b) = 0 \). Otherwise, we define \( s^G_M(\ell, b) \) inductively by setting

\[
s^G_M(\ell, b) = r^G_M(\ell, b) - \sum_{G' \in \mathcal{E}_{M'}(G)} \iota_M'(G, G')s^G_{M'}(\ell^*, b^*).
\]

The sum is finite since the coefficient \( \iota_M'(G, G') \) vanishes unless \( G' \) is elliptic. We then have to show that \( r^G_M(\ell, b) \) equals the endoscopic expression

\[
r^G_M(\ell', b') = \sum_{G' \in \mathcal{E}_{M'}(G)} \iota_M'(G, G')s^G_{M'}(\ell', b').
\]

We shall assume that \( S \) is large enough to contain \( V_{\text{ram}}(M') \). If \( V \) does not contain \( V_{\text{ram}}(M') \), then \( M' \) ramifies at some places \( v \in S - V \). In that case, the functions \( s^G_{M'}(\ell', b) \) vanish for all \( G' \in \mathcal{E}_{M'}(G) \) by definition, so we have to show that \( r^G_M(\ell, b) \) also vanishes.

By the usual splitting formulas for weighted orbital integrals, we can reduce to the case where \( S - V = \{v\} \). Moreover, the germ expansions and descent formula allow us to reduce to the case where \( \ell \) is elliptic and the groups \( G, M, M' \) are replaced by the local objects \( G_v, M_v, M'_v \). If \( M' \) is ramified at \( v \), by \cite{Art02} Proposition 8.1 we see that \( r^G_M(\ell, b) \) vanishes as an application of \cite{Kot80} Proposition 7.5. We note that the latter result is stated for functions in the spherical Hecke algebra of \( G \), but the argument holds identically for Schwartz functions. If \( M' \) is unramified at \( v \), we have to show that \( r^G_M(\ell, b) = r^{G, e}_M(\ell, b) \). In this case, the required identity...
will follow from an application of weighted transfer. Namely, for any $G, M$, and $b$
there is a function
$$s^G_M(\ell, b), \quad \ell \in \mathcal{L}_{G, \text{reg}}(M_v)$$
such that for any $K, \text{any unramified elliptic endoscopic datum } M'_v$ of $M_v$, and any
$$\ell' \in \mathcal{L}_{G, \text{reg}}(M'_v)$$
the transfer
$$\sum_{k \in K_{G, \text{reg}}(M_v)} \Delta_{K, \cap M_v}(\ell', k) r^G_M(k, b)$$
is equal to
$$\sum_{G' \in \mathcal{E}(G)} \iota_{M'}(G, G') s^{G'}_{M'}(\ell', b).$$
The function $s^G_M(\ell, b)$ is uniquely defined by this identity. The existence here then
follows from Corollary 3.4. □

Let $(b^V_S)^\prime$ be the transfer of $b^V_S$ in $\mathcal{J}_{\text{ac}}(G^V, \zeta^V_S)$, and write
$$\hat{f}^b_S = f \times b^V_S$$
for any $f \in \mathcal{C}(G^V, \zeta^V_S)$. We then have the following analogue of [Art02, Corollary 8.2].

**Corollary 4.2.** For any pair $(G, \zeta)$ such that $V$ contains $V_\infty$, any endoscopic
datum $G' \in \mathcal{E}_{\text{cl}}(G)$, and any function $f \in \mathcal{C}(G^V, \zeta^V_S)$, we have

$$(\hat{f}^b_S)^\prime = \begin{cases} f' \times (b^V_S)^\prime, & \text{if } V \supset V_{\text{ram}}(G, \zeta), \\ 0, & \text{otherwise}. \end{cases}$$

In particular, the function $(\hat{f}^b_S)^\prime$ vanishes unless $G'$ belongs to $\mathcal{E}_{\text{cl}}(G, V)$.

**Proof.** This follows from the case $M = G$ in the preceding proposition. We note
that this case relies only on the ordinary Langlands-Shelstad transfer. □

### 4.2. The geometric coefficients

We shall first construct the global geometric coefficients, which will be the terms on the geometric side that depend on the basic function. Following [Art02, §2], we write $\Gamma_{\text{ell}}(G, S, \zeta)$ for the set of $\gamma$ in $\Gamma_{\text{orb}}(G^Z_S, \zeta_S)$
such that there is a $\gamma \in G(F)$ such that

(i) the semisimple part of $\gamma F$ is $F$-elliptic in $G$,
(ii) the conjugacy class of $\gamma$ in $G_V$ maps to $\gamma$,
(iii) $\gamma$ is bounded at each $v \not\in S$.

Then let $\mathcal{K}_{\text{ell}}(M, S)$ denote the elements of $\mathcal{K}(G^Z_S)$ such that $\gamma \times k$ belongs to $\Gamma_{\text{ell}}(G, S, \zeta)$ for some $\gamma$. Let $A_M$ be the maximal split torus of a Levi subgroup $M$ of $G$. We then identify the Weyl group of $(G, A_M)$ with the quotient of the normaliser of $M$ by $M$, thus $W^G(M) = \text{Norm}_G(M)/M$, and set $W^G_0 = W^G(M_0)$. We then define for any $\gamma \in \Gamma(G^Z_v, \zeta^V_V)$, the geometric coefficient

$$a^G_{r, s}(\gamma) = \sum_{M \in \mathcal{L}} |W^M_0| |W^G_0|^{-1} \sum_{k \in \mathcal{K}_{\text{ell}}(M, S)} a^M_{r, s, \text{ell}}(\gamma M \times k) r^G_M(k, b)$$

where the elliptic coefficient $a^M_{r, s, \text{ell}}(\gamma M \times k)$ is the one constructed in [Won22 (4.10)].

The coefficient $a^G_{r, s}(\gamma)$ is supported on the set $\Gamma_{\text{ell}}(G, S, \zeta)$, where $S$ is any finite set of valuations of containing $V$ such that $\gamma \times K^V$ is $S$-admissible in the sense of [Art02 §1]. It is supported on the discrete subset $\Gamma(G, V, \zeta)$ of $\Gamma(G^Z_v, \zeta^V_V)$ given by
the union of induced distributions $\mu^G$ where $\mu$ runs over elements in $\Gamma \ell(M, V, \zeta)$ and $M$ runs over Levis in $\mathcal{L}'$. We shall write $a^G_{r,s,\ell}(\gamma, S)$ for the term $M = G$ in the expansion of $a^G_{r,s}(\gamma)$. That is,

\begin{equation}
(4.3) \quad a^G_{r,s,\ell}(\gamma, S) = \sum_{k \in \mathcal{K}_\ell(G, S)} a^G_{r,s,\ell}(\gamma \times k)r_G(k, b).
\end{equation}

We note that whereas $a^G_{r,s,\ell}(\gamma, S)$ depends on $S$, by [Won22, Corollary 4.7] the coefficient $a^G_{r,s}(\gamma)$ does not.

We next construct parallel families of endoscopic and stable geometric coefficients on the domains $\Gamma(G^Z, \zeta_V)$ and $\Delta^\theta(G^Z, \zeta_V)$. For any $\gamma \in \Gamma(G^Z, \zeta_V)$, we set

\begin{equation}
(4.4) \quad a^G_{r,s}(\gamma) = \sum_{G'} \sum_{\delta'} \iota(G, G')b^G_{r,s}(\delta')\Delta_G(\delta', \gamma) + \varepsilon(G) \sum_{\delta} b^G_{r,s}(\delta)\Delta_G(\delta, \gamma)
\end{equation}

with $G'$, $\delta'$, and $\delta$ summed over $\mathcal{E}_{\ell\ell}(G, V)$, $\Delta((\tilde{G}'^V, \tilde{\zeta}'_V))$ and $\Delta^\theta(G^Z, \zeta_V)$ respectively, and the coefficients $b^G_{r,s}(\delta)$ are defined inductively by the requirement that

\begin{equation}
\forall \gamma \in \Gamma(G^Z, \zeta_V) : a^G_{r,s}(\gamma) = a^G_{r,s}(\gamma)
\end{equation}

in the case that $G$ is quasisplit. Moreover, we set

\begin{equation}
b^G_{r,s}(\delta') = b^G_{r,s}(\delta), \quad \delta \in \Delta(G^Z, \zeta_V)
\end{equation}

where $b^G_{r,s}(\delta)$ is obtained as a function on $\Delta^\theta(G^Z, \zeta_V)$ by the local inversion formula [Art02 (5.5)]. The coefficients $a^G_{r,s}(\gamma)$ and $b^G_{r,s}(\delta)$ are in fact supported on the discrete subsets $\Gamma^\theta(G, V, \zeta)$ and $\Delta^\theta(G, V, \zeta)$ respectively, which are constructed in a manner parallel to $\Gamma(G, V, \zeta)$ [Art02 §7]. For example, we inductively define the set $\Delta^{\ell\ell}(G, V, \zeta)$ to be the collection of $\delta \in \Delta^\theta(G^Z, \zeta_V)$ such that either $\Delta(\gamma, \delta) \neq 0$ for some $\gamma \in \Gamma^{\ell\ell}(G, V, \zeta)$, or $\delta$ is the image in $\Delta^\theta(G^Z, \zeta_V)$ of an element $\delta'$ in the subset $\Delta^{\ell\ell}(\tilde{G}'^V, \tilde{\zeta}'_V)$ of $\Delta((\tilde{G}'^V, \tilde{\zeta}'_V))$ for some $G' \in \mathcal{E}_{\ell\ell}(G, V)$. We set

\begin{equation}
\Delta^{\ell\ell}(G, V, \zeta) = \Delta^{\ell\ell}(G, V, \zeta) \cap \Delta(G^Z, \zeta_V),
\end{equation}

and define $\Delta(G, V, \zeta)$ again to be the union of induced classes $\mu^G$ where $\mu \in \Delta^{\ell\ell}(M, V, \zeta)$ for some $M \in \mathcal{L}'$. Then the sums over $\delta'$ and $\delta$ in (4.4) can be taken over the smaller sets $\Delta(\tilde{G}', V, \zeta')$ and $\Delta^\theta(G, V, \zeta)$ respectively.

We can now state the main global theorem concerning the geometric coefficients. It is the analogue of the main Global Theorem 1' of [Art02 §7], and will be proved by a series of reductions. We state it here in order to use the necessary induction hypotheses for the reduction.

**Theorem 4.3.** (a) If $G$ is arbitrary, we have

\begin{equation}
a^G_{r,s}(\gamma) = a^G_{r,s}(\gamma), \quad \gamma \in \Gamma^\theta(G, V, \zeta).
\end{equation}

(b) If $G$ is quasisplit, we have that

\begin{equation}
b^G_{r,s}(\delta), \quad \delta \in \Delta^\theta(G, V, \zeta),
\end{equation}

is supported on the subset $\Delta(G, V, \zeta)$ of $\Delta^\theta(G, V, \zeta)$.
We shall also define the endoscopic and stable analogues of the elliptic coefficients $a_{r,s,\text{ell}}^G(\gamma S)$. For any admissible elements $\gamma S \in \Gamma_{\text{ell}}^G(G,S,\zeta)$ and $\delta S \in \Delta_{\text{ell}}^G(G,S,\zeta)$ with $S \supset V_{\text{ram}}(G,\zeta)$, we set
\begin{equation}
(4.5)
a_{r,s,\text{ell}}^G(\gamma S) = \sum_{G'} \sum_{\tilde{\delta}} \iota(G,G') b_{r,s,\text{ell}}^{G'}(\tilde{\delta})\Delta_G(\tilde{\delta},\gamma S) + \varepsilon(G) \sum_{\tilde{\delta}} b_{r,s,\text{ell}}^{G}(\tilde{\delta})\Delta_G(\tilde{\delta},\gamma S),
\end{equation}
with $G'$, $\tilde{\delta}$, and $\Delta_G$ summed over $\Delta_{\text{ell}}^G(G,S,\tilde{\gamma}')$ and $\Delta_{\text{ell}}^G(G,S,\zeta)$ respectively, and the coefficients $b_{r,s,\text{ell}}^{G'}(\delta)$ are defined inductively by the requirement that
\begin{equation}
(4.6)
a_{r,s,\text{ell}}^G(\gamma S) = a_{r,s,\text{ell}}^G(\gamma S)
\end{equation}
and
\begin{equation}
(4.7)
b_{r,s,\text{ell}}^{G'}(\delta) = b_{r,s,\text{ell}}^{G}(\delta)
\end{equation}
in the case that $G$ is quasisplit.

Finally, let us also define analogues of (4.3). We write $\delta \times \ell = \delta \times \ell'(\ell)$ for the element in $\Delta_{\ell'}^G(G,\zeta)$ associated to a pair $\delta \in \Delta_{\ell}^G(G,\zeta)$ and $\ell \in \mathcal{L}_{\ell}^G(G')$. We also write $\mathcal{L}_{\ell}^{\text{ell}}(G,\mathcal{S})$ for the set of $\ell \in \mathcal{L}_{\ell}^{\text{ell}}(G',\mathcal{S})$ such that $\delta \times \ell$ belongs to $\Delta_{\ell'}^{\text{ell}}(G,\zeta)$ for some $\delta \in \Delta_{\ell}^G(G',\zeta)$, and $\mathcal{L}_{\ell}^{\text{ell}}(G,\mathcal{S})$ for the intersection of $\mathcal{L}_{\ell}^{\text{ell}}(G,\mathcal{S})$ with $\mathcal{L}_{\ell}^{\text{ell}}(G,\mathcal{S})$. We also write $K_{\text{ell}}^{\mathcal{V}}(G,\mathcal{S})$ for the set of $k$ in $K(G,\mathcal{S})$ such that $\gamma \times k$ belongs to $\Gamma_{\text{ell}}^G(G,\zeta)$ for some $\gamma$. We then define the endoscopic and stable analogues of (4.3).

\begin{equation}
(4.8)
a_{r,s,\text{ell}}^G(\gamma,S) = \sum_{k \in K_{\text{ell}}^{\mathcal{V}}(G,\mathcal{S})} a_{r,s,\text{ell}}^G(\gamma \times k)r_G(k,b)
\end{equation}
for $G$ arbitrary and $\gamma \in \Gamma_{\text{ell}}^G(G,\mathcal{S})$, and
\begin{equation}
(4.9)
b_{r,s,\text{ell}}^G(\delta,S) = \sum_{\ell \in \mathcal{L}_{\ell}^{\text{ell}}(G,\mathcal{S})} b_{r,s,\text{ell}}^G(\delta \times \ell)r_G(k,b)
\end{equation}
for $G$ quasisplit and $\delta \in \Delta_{\text{ell}}^G(G,\mathcal{V},\zeta)$. These definitions will allow us to define endoscopic and stable variants of the geometric expansion of the linear form $I_M^G(f)$.

### 4.3. The elliptic and orbital parts.
Recall that the geometric expansion of $I_M^G(f)$ is given in [Won22] Theorem 4.6 by
\begin{equation}
(4.10)
I_M^G(f) = \sum_{M \in \mathcal{L}} |W^M_0| |W^G_0|^{-1} \sum_{\gamma \in \Gamma(M,\mathcal{V},\zeta)} a_{r,s}^M(\gamma)f_M(\gamma, f)
\end{equation}
that is valid for any $f \in \mathcal{E}^\circ(G,\mathcal{V},\zeta)$. We shall examine this more closely. Let us first define
\begin{equation}
(4.11)
I_{r,s,\text{ell}}(\hat{f}_S) = \sum_{\gamma \in \Gamma_{\text{ell}}(G,S,\mathcal{S})} a_{r,s,\text{ell}}^G(\gamma S)\hat{f}_S G(\gamma S),
\end{equation}
for $\hat{f}_S$ belonging to the subspace $\mathcal{E}^\circ_{\text{adm}}(G,S,\mathcal{S})$ of functions in $\mathcal{E}^\circ(G,S,\mathcal{S})$ whose support is an admissible subset of $G_S$, and
\begin{equation}
(4.12)
I_{r,s,\text{orb}}(f) = \sum_{\gamma \in \Gamma(G,V,\zeta)} a_{r,s}^G(\gamma)f_G(\gamma)
\end{equation}
for \( f \in \mathcal{C}(G,V,\zeta) \), corresponding to the term \( M = G \) in \([4.8]\) and is a linear combination of invariant orbital integrals. If we restrict to the elliptic coefficients, we obtain the linear form

\[
I_{r,s,\text{ell}}(f,S) = \sum_{\gamma \in \Gamma_{\text{ell}}(G,S,\zeta)} a_{r,s,\text{ell}}^G(\gamma,S) f_G(\gamma)
\]

(4.11)

which can be regarded as the elliptic part of \( I_r^G(f) \). It follows from the definitions that \( I_{r,s,\text{ell}}(f,S) = I_{r,s,\text{ell}}(f^b_S) \) for any \( S \) large enough such that \( f^b_S \) belongs to \( \mathcal{E}_r^{\text{ad}}(G,V,\zeta) \).

We define endoscopic and stable analogues of these by setting inductively

\[
I_{r,s,\text{ell}}(f_S) = \sum_{G' \in \mathcal{E}^0_\text{ell}(G,S)} \iota(G,G') \hat{S}^G_{r,s,\text{ell}}(f_S) + \varepsilon(G) S^G_{r,s,\text{ell}}(f_S)
\]

(4.12)

and

\[
I_{r,s,\text{orb}}(f) = \sum_{G' \in \mathcal{E}^0_\text{ell}(G,V)} \iota(G,G') \hat{S}^G_{r,s,\text{orb}}(f') + \varepsilon(G) S^G_{r,s,\text{orb}}(f)
\]

(4.13)

where we recall that the coefficient \( \iota(G,G') \) is the one given in \([\text{Kot84}] \) Theorem 8.3.1. Here \( \mathcal{E}^0_\text{ell}(G,S) \) is the complement of \( \{G\} \) in \( \mathcal{E}_\text{ell}(G,S,\zeta) \) and \( f \in \mathcal{E}_r(G,S,\zeta) \) for \( f \in \mathcal{E}_r^0(G,V,\zeta) \) respectively. The term \( \hat{S}^G_{r,s,\text{ell}} \) is a linear form on the image \( S^{\mathcal{E}^0_{\text{ell}}(G',S,\zeta)}_{\mathcal{E}^0_{\text{ell}}(G',S,\zeta)} \) of \( \mathcal{E}^0_{\text{ell}}(G',S,\zeta) \) in \( S^{\mathcal{E}^0_{\text{ell}}(G',S,\zeta)}_{\mathcal{E}^0_{\text{ell}}(G',S,\zeta)} \), and \( \hat{S}^G_{r,s,\text{orb}} \) is a linear form on \( S^{\mathcal{E}^0_{\text{ell}}(G',V,\zeta)}_{\mathcal{E}^0_{\text{ell}}(G',V,\zeta)} \). We furthermore require that

\[
I_{r,s,\text{ell}}(f_S) = I_{r,s,\text{ell}}(f_S)
\]

and

\[
I_{r,s,\text{orb}}(f) = I_{r,s,\text{orb}}(f)
\]

in the case that \( G \) is quasisplit, and the general induction hypothesis that \( S^G_{r,s,\text{ell}} \) and \( S^G_{r,s,\text{orb}} \) are stable for \( G' \) in \( \mathcal{E}^0_\text{ell}(G,S) \) and \( \mathcal{E}^0_\text{ell}(G,V) \) respectively.

If \( G \) is arbitrary, it follows by the same argument of \([\text{Art02}] \) Lemma 7.2] that

\[
I_{r,s,\text{ell}}(f_S) = \sum_{\gamma_S \in \Gamma^{\mathcal{E}^0_\text{ell}}(G,S,\zeta)} a_{r,s,\text{ell}}^{G,\mathcal{E}}(\gamma_S) f_{S,G}(\gamma_S)
\]

(4.14)

and

\[
I_{r,s,\text{orb}}(f) = \sum_{\gamma \in \Gamma^{\mathcal{E}^0_\text{ell}}(G,V,\zeta)} a_{r,s}^{G,\mathcal{E}}(\gamma) f_{G}(\gamma),
\]

(4.15)

and if \( G \) is quasisplit we have that

\[
S^G_{r,s,\text{ell}}(f_S) = \sum_{\delta_S \in \Delta^{\mathcal{E}^0_\text{ell}}(G,S,\zeta)} b_{r,s,\text{ell}}^{G,\mathcal{E}}(\delta_S) f_{S,G}(\delta_S)
\]

(4.16)

and

\[
S^G_{r,s,\text{orb}}(f) = \sum_{\delta \in \Delta^{\mathcal{E}^0_\text{ell}}(G,V,\zeta)} b_{r,s}^{G,\mathcal{E}}(\delta) f_{G}(\delta).
\]

(4.17)

Here

\[
f \to f_G^\mathcal{E}(\delta) = f'(\delta')
\]

is a linear form on \( \mathcal{E}_r(G_V,\zeta_V) \) for any \( \delta' \in \Delta_{G,\text{reg}}(G'_V,\zeta'_V) \) with image \( \delta \in \Delta_{\text{reg}}^\mathcal{E}(G_V,\zeta_V) \). Let \( I^{\mathcal{E}}(G_V,\zeta_V) \) and \( I^{\mathcal{E}_r}(G_V,\zeta_V) \) be the spaces spanned by \( f_G \).
and \( f_G^\ell \) respectively for any \( f \in C^\circ(G_V, \zeta_V) \). The map \( f_G \to f_G^\ell \) provides an isomorphism between these two spaces.

We also set

\[
(4.18) \quad I_{r,s,\text{ell}}^\ell(f, S) = \sum_{G' \in \mathcal{E}_\text{ell}(G)} \nu(G, G') \hat{S}_{r,s,\text{ell}}^{G'}(f', S) + \epsilon(G) S_{r,s,\text{ell}}^G(f, S)
\]

for linear forms \( \hat{S}_{r,s,\text{ell}}^{G'}(f', S) \) on \( C^\circ(\hat{G}', S, \hat{\zeta}') \) which are defined inductively by requiring that

\[
I_{r,s,\text{ell}}^\ell(f, S) = I_{r,s,\text{ell}}(f, S)
\]

in the case that \( G \) is quasisplit. Now suppose that \( S \) is large enough so that \( \hat{f}_S \) belongs to \( C_{\text{adm}}(G, S, \zeta) \). It then follows inductively from Corollary \[4.2 \] and \[4.12 \] that

\[
I_{r,s,\text{ell}}^\ell(f, S) = I_{r,s,\text{ell}}(\hat{f}_S)
\]

and

\[
S_{r,s,\text{ell}}^G(f, S) = S_{r,s,\text{ell}}(\hat{f}_S),
\]

moreover from the expansions \[4.14 \] and \[4.10 \] we conclude that

\[
(4.19) \quad I_{r,s,\text{ell}}^\ell(f, S) = \sum_{\gamma \in \mathcal{C}_\text{ell}(G, S, \zeta)} a_{r,s,\text{ell}}^{\gamma,G}(\gamma, S) f_G(\gamma)
\]

and

\[
(4.20) \quad S_{r,s,\text{ell}}^G(f, S) = \sum_{\delta \in \Delta_{\text{ell}}(G, S, \zeta)} b_{r,s,\text{ell}}^{\gamma,G}(\delta, S) f_G^\ell(\delta).
\]

These formulae represent a stabilization of the term with \( M = G \) in \[4.2 \].

Having made these preliminary manipulations, we can now establish the geometric expansion of \( I_{r,s}^\ell(f) \) and \( S_{r,s}^G(f) \). It follows the broad argument of \[Art02, \text{Theorem 10.1} \]. We set

\[
(4.21) \quad I_{r,s}^\ell(f) = \sum_{G' \in \mathcal{E}(G, V)} \nu(G, G') \hat{S}_{r,s}^{G'}(f') + \epsilon(G) S_{r,s}^G(f), \quad f \in C^\circ(G, V, \zeta)
\]

for linear forms \( \hat{S}_{r,s}^{G'} = \hat{S}_{r,s}^{G'} \) on \( C^\circ(\hat{G}', V, \hat{\zeta}') \), defined inductively by the supplementary requirement that \( I^\ell(f) = I(f) \) in the case that \( G \) is quasisplit. We shall also assume inductively that if \( G \) is replaced by a quasisplit inner \( K \)-form \( \hat{G}' \), the corresponding analogue of \( S^G \) is defined and stable.

**Proposition 4.4.** Let \( f \in C^\circ(G, V, \zeta) \).

(a) If \( G \) is arbitrary,

\[
I_{r,s}^\ell(f) - I_{r,s,\text{orb}}^\ell(f) = \sum_{M \in \mathcal{X}_0} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma^\ell(M, V, \zeta)} a_{r,s}^{M,\ell}(\gamma) I^\ell(\gamma, f).
\]

(b) If \( G \) is quasisplit,

\[
S_{r,s}^G(f) - S_{r,s,\text{orb}}(f) = \sum_{M \in \mathcal{X}_0} |W_0^M||W_0^G|^{-1} \sum_{M' \in \mathcal{E}(M, V)} \nu(M, M') \sum_{\delta' \in \Delta(M', V, \zeta')} b_{r,s}^{M'}(\delta') S_{r,s}^G(M', \delta', f).
\]

**Proof.** The proof follows the same argument as \[Art02, \text{Theorem 10.1} \]. □
The next proposition concerns the coefficients $a^{G,\epsilon}_{r,s}(\gamma)$ and $b^{G}_{r,s}(\delta)$, generalising [Art02, Proposition 10.3], so we provide a detailed proof here.

**Theorem 4.5.** (a) If $G$ is arbitrary and $\gamma \in \Gamma^{G}(G,V,\zeta)$, then

$$
(4.22) \quad a^{G,\epsilon}_{r,s}(\gamma) - a^{G,\epsilon}_{r,s,\ell}(\gamma, S) = \sum_{M \in \mathcal{L}^0} |W^{M}_0||W^{G}_0|^{-1} \sum_{k \in K^{V,\epsilon}_{r,s,\ell}(\bar{M}, S)} a^{M,\epsilon}_{\ell}(\gamma_{M} \times k) r^{G}_{M}(k, b).
$$

(b) If $G$ is quasisplit and $\delta \in \Delta^{G}(G,V,\zeta)$, then

$$
(4.23) \quad b^{G}_{r,s}(\delta) - b^{G}_{r,s,\ell}(\delta, S) = \sum_{M \in \mathcal{L}^0} |W^{M}_0||W^{G}_0|^{-1} \sum_{k \in L^{V,\epsilon}_{\ell}(M, S)} b^{M}_{r,s,\ell}(\delta_{M} \times \ell) s^{G}_{M}(\ell, b)
$$

if $\delta$ lies in the subset $\Delta(G,V,\zeta)$ of $\Delta^{G}(G,V,\zeta)$ and is zero otherwise.

**Proof.** Let us first consider the differences of (4.15) and (4.19),

$$
I^{G}_{r,s,\text{orb}}(f) - I^{G}_{r,s,\ell}(f, S) = \sum_{\gamma \in \Gamma^{G}(G,V,\zeta)} (a^{G,\epsilon}_{r,s}(\gamma) - a^{G,\epsilon}_{r,s,\ell}(\gamma, S)) f_{G}(\gamma),
$$

and (4.17) and (4.20),

$$
S^{G}_{r,s,\text{orb}}(f) - S^{G}_{r,s,\ell}(f, S) = \sum_{\delta \in \Delta^{G}(G,V,\zeta)} (b^{G}_{r,s}(\delta) - b^{G}_{r,s,\ell}(\delta, S)) f^{G}_{\ell}(\delta).
$$

Substituting the right-hand side of (4.22), we obtain the linear form

$$
(4.24) \quad \sum_{M \in \mathcal{L}^0} |W^{M}_0||W^{G}_0|^{-1} \sum_{\gamma \in \Gamma^{G}(G,V,\zeta)} \sum_{k \in K^{V,\epsilon}_{r,s,\ell}(M, S)} a^{M,\epsilon}_{\ell}(\gamma \times k) f_{M}(\gamma) r^{G}_{M}(k, b),
$$

which we will denote by $I^{G,0}_{r,s,\text{orb}}(f, S)$. We may vary the function $f \in C^{0}(G,V,\zeta)$ independent of $S$ as long as the support of $f$ remains $S$-admissible, hence the required identity (4.22) is equivalent to showing that

$$
(4.25) \quad I^{G,0}_{r,s,\text{orb}}(f) - I^{G,0}_{r,s,\ell}(f, S) = I^{G}_{r,s,\text{orb}}(f) - I^{G}_{r,s,\ell}(f, S).
$$

On the other hand, if $G$ is quasisplit, substituting the right-hand side of (4.23), we obtain the linear form

$$
(4.26) \quad \sum_{M \in \mathcal{L}^0} |W^{M}_0||W^{G}_0|^{-1} \sum_{\delta \in \Delta^{G}(M,V,\zeta)} \sum_{\ell \in L^{V,\epsilon}_{\ell}(M, S)} b^{M}_{r,s,\ell}(\delta \times \ell) f^{M}(\delta) s^{G}_{M}(\ell, b),
$$

which we will denote by $S^{G,0}_{r,s,\text{orb}}(f, S)$, if $\delta \in \Delta(G,V,\zeta)$, and is zero otherwise. Then the required identity (4.23) is equivalent to showing that

$$
(4.27) \quad S^{G,0}_{r,s,\text{orb}}(f) - S^{G,0}_{r,s,\ell}(f, S) = S^{G}_{r,s,\text{orb}}(f) - S^{G}_{r,s,\ell}(f, S).
$$

On the other hand, from the definitions (4.13) and (4.18), the difference

$$
(4.28) \quad (I^{G}_{r,s,\text{orb}}(f) - I^{G}_{r,s,\ell}(f, S)) - \varepsilon(G)(S^{G}_{r,s,\text{orb}}(f) - S^{G}_{r,s,\ell}(f, S))
$$

is equal to

$$
\sum_{G' \in \mathcal{G}(G,V)} i(G,G') (\tilde{S}^{G'}_{r,s,\text{orb}}(f') - \tilde{S}^{G'}_{r,s,\ell}(f', S)).
$$
We can assume inductively that \( \tilde{S}_{r,s,orb}^{G'}(f') - \tilde{S}_{r,s,ell}^{G'}(f', S) \) equals \( \tilde{S}_{r,s,orb}^{G',0}(f, S) \) for any \( G' \in \mathcal{E}_{ell}(G, V) \). Replacing it with \( \tilde{S}_{r,s,ell}^{G'}(f', S) \) above, we may use [Art102] Lemma 10.2 to rearrange the sum as

\[
\sum_{R \in \mathcal{L}^*} |W_{R}^{G'}||W_{G'}^{G'}|^{-1} \sum_{R' \in \mathcal{E}_{ell}(R, V)} \iota(R, R') \sum_{\sigma'_{S}} b_{r,s,ell}^{G'}(\sigma'_{S}) B_{R'}(\sigma'_{S}, f)
\]

where \( \sigma'_{S} \) is summed over the product of \( \sigma' \in \Delta_{ell}(\tilde{R}', V, \tilde{\zeta}') \) and \( \ell' \in \mathcal{L}_{ell}^{\gamma}(\tilde{R}', S) \), \( \mathcal{L}^* = \mathcal{L}^{G'} \) runs over Levi subgroups of \( G' \), and we shall express as the difference

\[
B_{R'}(\sigma'_{S}, f) = \sum_{G' \in \mathcal{E}_{ell}^{0}(G)} \iota(G, G') f_{R'}^{G'}(\sigma') s_{G'}^{G}((\ell', b')),
\]

which we shall express as the difference

\[
\sum_{G' \in \mathcal{E}_{ell}(G)} \iota(G, G') s_{G'}^{G}((\ell', b')) f_{R'}^{G'}(\sigma') - \varepsilon(G)s_{R}^{G}(R', \sigma'_{S}, b, f),
\]

where \( s_{G}^{G}(R', \sigma'_{S}, b, f) \) equals \( s_{R}^{G}(\sigma'_{S}, f) \) if \( R' = R \) and \( \sigma'_{S} = \sigma \), and is zero otherwise. The contribution of the second term to (4.28) is \(-\varepsilon(G)\times(4.26)\), which is \( S_{r,s,orb}^{G,0}(f, S) \). The contribution of the first term is zero if \( R \in (\mathcal{L}^*)^{0} \) does not come from \( G \) by definition of \( f_{R'}^{G'}(\sigma') \), whereas if \( (R, R', \sigma') \) lies in the \( W_{0}^{G'} \)-orbit of a triplet \( (M, M', \delta') \) that comes from \( G \), then it follows from Proposition 4.1 that the first term equals

\[
r_{M}^{G}(\ell, b) f_{M'}^{G'}(\delta') = r_{M}^{G}(\ell, b) f_{M}^{G}(\delta),
\]

where \( \delta \times \ell' \) is the image of \( \delta' \times \ell' \) in the product of \( \Delta_{ell}(M, V, \zeta) \) and \( \mathcal{L}^{V, \delta}(M, S) \). We can thus write the contribution of the first term to (4.28) as

\[
(4.29) \sum_{M \in \mathcal{L}^0} |W_{0}^{M}||W_{0}^{G'}|^{-1} \sum_{M' \in \mathcal{E}_{ell}(M, V)} \iota(M, M') \sum_{\delta'} \sum_{\ell'} b_{r,s,ell}^{M'}(\delta' \times \ell') r_{M}^{G}(\ell, b) f_{M}^{G}(\delta),
\]

where \( \delta' \) and \( \ell' \) are summed over \( \Delta_{ell}(M', V, \tilde{\zeta}') \) and \( \mathcal{L}^{V}(M', S) \). The sum over \( \mathcal{E}_{ell}(M, V) \) can actually be taken over \( \mathcal{E}_{ell}(M, S) \), since by Proposition 4.1 the contribution of \( M' \) in the complement of \( \mathcal{E}_{ell}(M, V) \) of \( \mathcal{E}_{ell}(M, S) \) is zero. Moreover, using the definitions \( r_{M}^{G}(\ell, b) \) in (4.1) and of \( f_{M}^{G}(\delta) \), we may express their product as

\[
\sum_{\gamma \in \Gamma^{\mathcal{L}^0}(M, V, \zeta)} \sum_{k \in \mathcal{K}_{\mathcal{L}^0}(M, S)} \Delta_{M}(\delta' \times \ell', \gamma \times k) r_{M}^{G}(k, b) f_{M}(\gamma).
\]

If \( G \) is quasisplit, the general induction hypothesis implies that Theorem 4.3(b) holds for any \( M \in \mathcal{L}^0 \) in (4.29). We may then write (4.3) as

\[
a_{r,s,ell}^{M,\delta}(\gamma_{S}) = \sum_{M' \in \mathcal{E}_{ell}(M, S)} \iota(M, M') \sum_{\delta_{S}} b_{r,s}^{M'}(\delta') \Delta_{M}(\delta'_{S}, \gamma_{S}),
\]

and hence the inner sum on \( M' \) in (4.29) is equal to

\[
\sum_{\gamma_{S}} a_{r,s,ell}^{M,\delta}(\gamma_{S}) r_{M}^{G}(k, b) f_{M}(\gamma).
\]
Then we conclude that (4.29) is equal to \( I_{r,s,\text{orb}}^\epsilon(f, S) \) and (4.28) is then equal to the difference
\[
I_{r,s,\text{orb}}^\epsilon(f, S) - \epsilon(G) S_{r,s,\text{orb}}^\epsilon(f, S).
\]

Now if \( \epsilon(G) = 0 \), the required identity (4.25) follows from the preceding assertion. On the other hand, if \( \epsilon(G) = 1 \), then
\[
I_{r,s,\text{orb}}^\epsilon(f) - I_{r,s,\text{ell}}^\epsilon(f, S) = I_{r,s,\text{orb}}^\epsilon(f) - I_{r,s,\text{ell}}^\epsilon(f, S)
\]
by definition, and
\[
\sum_{M \in \mathcal{F}_0} |W_0^M||W_0^G|^{-1} \sum_{\gamma_S} a_{r,s,\text{ell}}^\epsilon(\gamma_S) r_M^G(k, b) f_M(\gamma).
\]
This again fulfills (4.25), and the remaining terms thus imply that the required identity (4.27) is satisfied as well.

The preceding proposition provides the first reduction of study of the global coefficients to the basic elliptic coefficients. We record it as the following statement for later use.

**Corollary 4.6.** Suppose that
(a) if \( G \) is arbitrary, we have
\[
a_{r,s,\text{ell}}^{G,\epsilon}(\hat{\gamma}_S) = a_{r,s,\text{ell}}^G(\hat{\gamma}_S),
\]
for any admissible element \( \hat{\gamma}_S \in \Gamma_{\text{ell}}^\epsilon(G, S, \zeta) \), and
(b) if \( G \) is quasisplit, we have that
\[
b_{\text{ell}}^{G}(\hat{\delta}_S), \quad \hat{\delta}_S \in \Delta_{\text{ell}}^\epsilon(G, S, \zeta),
\]
vanishes for any admissible element \( \hat{\delta}_S \) in the complement of \( \Delta_{\text{ell}}^\epsilon(G, S, \zeta) \) of \( \Delta_{\text{ell}}^\epsilon(G, S, \zeta) \).

Then Theorem 4.3 holds.

5. The unramified spectral terms

5.1. Endoscopic \( L \)-functions. For the moment, let \( G \) be a connected reductive group over a number field \( F \), and fix an induced central torus \( Z \) in \( G \) with an automorphic character \( \zeta \). Also let \( V \) be a finite set of places containing \( V_{\text{ram}}(G, \zeta) \). We consider families \( c = \{ c_v : v \notin V \} \) of semisimple conjugacy classes \( c_v \) in the local \( L \)-group \( L G_v \) of \( G_v \), whose image in the local Weil group \( W F_v \) is a Frobenius element. Then by abuse of notation, let \( \mathcal{E}(G^V, \zeta^V) \) be the set of families \( c \) such that the image of each \( c_v \) under the projection \( L G_v \to L Z_v \) gives the unramified Langlands parameter of \( \zeta_v \), and for any \( G \)-invariant polynomial \( A \) on \( L G \) we have
\[
|A(c_v)| \leq q_v^{-r_A}, \quad v \notin V
\]
for some \( r_A > 0 \), where again \( q_v \) is order of the residue field of \( F_v \). Given \( c \in \mathcal{E}(G^V, \zeta^V) \) and a finite-dimensional representation \( \rho \) of \( L G \), we form the Euler product
\[
L(s, c, \rho) = \prod_{v \notin V} \det(1 - \rho(c_v) q_v^{-s})^{-1}, \quad s \in \mathbb{C},
\]
\footnote{The expression (10.13) indicated at the beginning of [Art02, p.272] should be equal to \( I_{\text{orb}}^\epsilon(f) \) and not \( I_{\text{ell}}^\epsilon(f) \).}
which converges to an analytic function of $s$ in some right half plane. There is a natural action of $\mathfrak{a}_{G,Z}^{c} \otimes \mathbb{C}$ on $\mathcal{C}(G^{V}, \zeta^{V})$ given by

$$c \mapsto c_{\lambda} = \{ c_{v,\lambda} = c_{v}q_{v}^{-\lambda} : v \notin V \}, \quad \lambda \in \mathfrak{a}_{G,Z}^{c} \otimes \mathbb{C},$$

whereby $L(s, c, \rho)$ is analytic for Re$(s) \gg$ Re$(\lambda)$. Our main interest will be in the following case: given a Levi subgroup $M$ of $G$ with dual group $\hat{M}$, there is a bijection $P \to \hat{P}$ from $\mathcal{P}(M)$ to the set $\mathcal{P}(\hat{M})$ of $\Gamma$-stable parabolic subgroups of $\hat{G}$ with Levi component $\hat{M}$. For any $P, Q \in \mathcal{P}(M)$, let $\rho_{Q|P}$ be the adjoint representation of $L^\Gamma M$ on the Lie algebra of the intersection of unipotent radicals of $\hat{P}$ and $\hat{Q}$. The $L$-functions $L(s, c, \rho_{Q|P})$ will be used to construct the unramified spectral terms.

Let $M'$ stand for an elliptic endoscopic datum $(M', \mathcal{M}', s_{M}', \xi_{M}')$ that is unramified outside of $V$. Recall that $\mathcal{M}'$ is a split extension of $W_{F}$ by $M'$ that need not be $L$-isomorphic to $L^\Gamma M'$. We therefore fix a central extension $\tilde{M}'$ of $M'$ by an induced torus $\tilde{G}'$ over $F$, together with an $L$-embedding $\tilde{\xi} : \mathcal{M}' \to \tilde{L}^\Gamma M'$. If $W_F \to \tilde{\mathcal{M}}$ is any section, the composition

$$W_F \to \tilde{\mathcal{M}} \to \tilde{L}^\Gamma M' \to \tilde{L} \tilde{G}'$$

is a global Langlands parameter that is dual to a character $\tilde{\eta}'$ of $\tilde{G}(F) \tilde{C}(\mathbb{A})$. We may assume that $M'$ and $\eta'$ are also unramified outside of $V$.

Given $c' \in \mathcal{C}(\tilde{L}^\Gamma M', (\tilde{\zeta})^{V})$, the projection of $c'_{v}$ onto $L^\Gamma \tilde{C}'_{v}$ for any $v \notin V$ is the conjugacy class that corresponds to the Langlands parameter of the unramified representation $\tilde{\eta}'_{v}$. Thus $c' = \tilde{\xi}_{M}(c'_{v})$ for some semisimple conjugacy class $\tilde{\xi}_{M}$ in $\mathcal{M}'$. Let us write $c_{v} = \tilde{\xi}_{M,\tilde{\xi}}(c'_{v})$ for the image of $c'_{v}$ in $L^\Gamma \tilde{M}'$, so that the family $c = \{ c_{v} : v \notin V \}$ belongs to $\mathcal{C}(M^{V}, \zeta^{V})$. This gives a map

$$(5.1) \quad \mathcal{C}(\tilde{M}'^{V}, (\tilde{\zeta})^{V}) \to \mathcal{C}(M^{V}, \zeta^{V}).$$

Now let $\mathcal{C}_{\text{ant}}(G, \zeta)$ be the set of $c$ for which there exists an irreducible representation $\pi_{V}$ of $G_{V}$ such that $\pi_{V} \otimes \pi^{V}(c)$ is an automorphic representation of $G(\mathbb{A})$, where $\pi^{V}(c)$ is the product over all $v \notin V$ of unramified representations $\pi_{v}(c) = \pi(c_{v})$ of $G_{v}$ determined by each $c_{v}$. By Langlands' functoriality principle, we expect that the map (5.1) descends to a map of subsets

$$\mathcal{C}_{\text{ant}}(\tilde{M}', \zeta) \to \mathcal{C}_{\text{ant}}(M, \zeta),$$

and while this is not known, it will be enough to use the result of Arthur [Art99a, Proposition 1] that if $c$ is the image of a family $c' \in \mathcal{C}(M^{V}, \zeta^{V})$, the $L$-functions $L(s, c, \rho_{Q|P})$ have meromorphic continuation.

Suppose that $\psi_{a} : M \to M_{a}$ is an inner twist over $F$ that is unramified outside of $V$. Let $(Z_{a}, \zeta_{a})$ be the image of $(Z, \zeta)$ and let $\psi_{a}^{*} : L M_{a} \to L M$ be an $L$-isomorphism dual to $\psi_{a}$. We assume inductively that for any elliptic endoscopic datum $M'$ for $M$ that is proper in the sense that it is not equal to a quasisplit inner form of $M$, the set $\mathcal{C}_{\text{ant}}(M', \zeta')$ is defined. We then define $\mathcal{C}_{\text{ant}}(M, \zeta)$ to be the union over all such $M'$ of the images in $\mathcal{C}(M^{V}, \zeta^{V})$ of $\mathcal{C}_{\text{ant}}(M', \zeta')$, together with the union over all $M_{a}$ as above of the images in $\mathcal{C}(M^{V}, \zeta^{V})$ of the sets $\mathcal{C}_{\text{ant}}(M_{a}, \zeta_{a})$.

Let $(Z(\tilde{M}))^{0}$ be the identity component of the $\Gamma$-invariant elements in the center of $\tilde{M}$. Let $a$ be a nontrivial character of $(Z(\tilde{M}))^{0}$, and let $\rho_{a}$ be the representation
of $L^M$ on the root space $\mathfrak{g}_a$ of $a$ on the Lie algebra of $\hat{G}$. For any $c \in \mathcal{C}(M^V, \zeta^V)$, the endoscopic $L$-function

$$L_G(s, c, a) = L(s, c, \rho_a)$$

is an analytic function of $s$ in some right half plane, and is equal to 1 unless $a$ is a root of $(\hat{G}, (Z(\hat{M}^\Gamma))^0)$. If $\Sigma(\hat{P})$ denotes the set of roots attached to a parabolic subgroup $P \in \mathcal{P}(M)$, we then have

$$L(s, c, \rho_{Q|P}) = \prod_{a \in \Sigma(\hat{P}) \cap \Sigma(\hat{Q})} L_G(s, c, a).$$

It has meromorphic continuation to the complex plane for any $c \in \mathcal{C}_{\text{aut}}(M, \zeta)$.

Suppose $a$ is a nontrivial character of $(Z(M)^\Gamma)^0$. Then the kernel $Z_a$ of $a$ acts by translation on $Z(M)^\Gamma/Z(\hat{G})^\Gamma$, which is in bijection with $\mathcal{E}(M)^\Gamma$ by \cite[Corollary 3]{Art99a}. Let $\mathcal{E}(M)^\Gamma/Z_a$ be the set of orbits, and recall that there is an isomorphism

$$(Z(M)^\Gamma)^0/(Z(\hat{M})^\Gamma)^0 \cap Z(\hat{G})^\Gamma \sim (Z(\hat{M})^\Gamma)^0/(Z(\hat{M})^\Gamma)^0 \cap Z(\hat{G})^\Gamma.$$ 

If $a$ is trivial on $(Z(M)^\Gamma)^0 \cap Z(\hat{G})^\Gamma$, let $a'$ be the unique character on $(Z(M)^\Gamma)^0$ that is trivial on $(Z(\hat{M})^\Gamma)^0 \cap Z(\hat{G})^\Gamma$. If not, we take $a'$ to be any character on $(Z(M)^\Gamma)^0$ whose restriction to $(Z(M)^\Gamma)^0$ is equal to $a$. Hence the character $a$ determines a family of $L$-functions $L_G(s, c', a')$ for each $c' \in \mathcal{C}((M)^\Gamma, \zeta_V^V)$ and $G' \in \mathcal{E}(M)^\Gamma$. If $c$ is the image in $\mathcal{C}(M^V, \zeta^V)$ of some $c'$ in $\mathcal{C}(M^V, \zeta_V^V)$, we have the decomposition

$$L_G(s, c, a) = \prod_{G' \in \mathcal{E}(M)^\Gamma/Z_a} L_{G'}(s, c', a')$$

by \cite[Lemma 4]{Art99a}.

The complexification $a_{M,Z}^* \otimes \mathbb{C}$ can be identified with a subspace of the Lie algebra of $Z(M)^\Gamma$. If $da$ is the linear form on $a_{M}^* \otimes \mathbb{C}$ associated with the character $a$ on $(Z(M)^\Gamma)^0$, we then have

$$L_G(s, c_{\lambda}, a) = L_G(s + (da)(\lambda), c, a).$$

It follows from this that for any fixed $s \in \mathbb{C}$ and $c \in \mathcal{C}_{\text{aut}}(M, \zeta)$, the $L$-function $L_G(s, c, a)$ is a meromorphic function of $\lambda$.

5.2. The unramified terms. We are now ready to define the unramified normalizing factors. The preceding discussion concerned only connected groups. To extend the constructions to a $K$-group $G$, we simply take the union of the corresponding sets attached to each of the components $G_\alpha$ of $G$. Given $c \in \mathcal{C}(M, \zeta)$, the quotients

$$r(c_\lambda, a) = L_G(0, c_\lambda, a)L_G(1, c_\lambda, a)^{-1}$$

and

$$r_{Q|P}(c_\lambda) = \prod_{a \in \Sigma(\hat{P}) \cap \Sigma(\hat{Q})} r(c_\lambda, a)^{-1}$$

$$= L_G(0, c_\lambda, \rho_{Q|P})L_G(1, c_\lambda, \rho_{Q|P})^{-1}$$
are meromorphic functions of \( \lambda \in \mathfrak{a}_{M,Z}^* \otimes \mathbb{C} \), for any \( P,Q \in \mathcal{P}(M) \). Then
\[
    r_Q(\lambda, c_{\lambda}) = r_Q|_Q(c_{\lambda})^{-1} r_Q|_Q(c_{\lambda + \frac{1}{2} \lambda}), \quad Q \in \mathcal{P}(M)
\]
is a \((G,M)\)-family of functions of \( \lambda \in \mathfrak{a}_{M,Z}^* \). The limit
\[
    r^G_M(c_{\lambda}) = \lim_{\lambda \to 0} \sum_{Q \in \mathcal{P}(M)} r_Q(\lambda, c_{\lambda}) \theta_Q(\Lambda)^{-1},
\]
with
\[
    \theta_Q(\Lambda) = \text{vol}(a_{M}^G/\mathbb{Z}(\Delta_Q^G))^{-1} \prod_{\alpha \in \Delta_Q} \Lambda(\alpha^\vee),
\]
is defined as a meromorphic function of \( \lambda \in \mathfrak{a}_{M,Z}^* \otimes \mathbb{C} \) as in [Art81 §2].

Now consider the space \( \mathcal{F}(G^F_V, \zeta_V) \) of finite, complex linear combinations of irreducible characters on \( G^F_V \) with \( \mathbb{Z}_V \)-central character equal to \( \zeta_V \). We identify an element \( \pi \in \mathcal{F}(G^F_V, \zeta_V) \) with a linear form \( f \to f_G(\pi) \) on \( \mathcal{C}(G,V,\zeta) \). There is an additional map \( f \to f_M \) factoring through this map, by which we define an induction operation from \( \mathcal{F}(M_V, \zeta_V) \) to \( \mathcal{F}(G_V, \zeta_V) \), given by \( \rho \to \rho^G \) satisfying
\[
    f_G(\rho^G) = f_M(\rho).
\]
In particular, this holds in the case that \( V \) contains a single element. In [Won22 (5.1)], using the basic function \( b = b^*_{\pi} \) we introduced the unramified character
\[
    r^G_M(c, b) = r^G_M(c) b_M(c)
\]
that takes the place of the unramified spectral terms appearing in the stable trace formula [Art81 §3]. Here \( b_M(c) \) is the character of the induced representation of \( \pi_v(c) \) evaluated at the function \( b \). We note that these unramified spectral terms differ from Arthur’s \( r^G_M(c) \) in that \( r^G_M(c) \) are scalars, whereas \( r^G_M(c, b) \) can be understood as scalar multiples of induced characters, which we shall have to stabilize.

**Lemma 5.1.** For any elliptic endoscopic datum \( M' \) of \( M \) unramified outside of \( V \supset V_\infty \), and any \( c' \in \mathcal{C}^V,\mathcal{E}(M', \zeta') \) with image \( c \in \mathcal{C}^V,\mathcal{E}(M, \zeta) \), we have
\[
    b_M(c) = b'_M(c').
\]

**Proof.** Since we are assuming that \( M \) is unramified outside of \( V \), there is a canonical class of \( L \)-embeddings from \( L^M_{M'} \) to \( L^M_V \), which we shall also denote by \( \xi_M \), for each \( v \not\in V \), and we can take \( Z_v \) to be trivial. The transfer \( b' \) of \( b \) on the level of functions is prescribed by the canonical map on spherical Hecke algebras
\[
    \mathcal{H}_{ac}(M_V, K_v \cap M_v) \to \mathcal{H}_{ac}(M'_{M'}, K'_v \cap M'_{M'}),
\]
or rather its \( \zeta^{-1} \)-equivariant analogue, induced by \( \xi_M' \) and the generalized Satake isomorphism. By construction, it commutes with the map sending \( c' \) to \( c \) given by [5.1].

If \( r \) is a finite-dimensional representation of \( L^M_V \), we write \( r' \) for the corresponding representation of \( L^M_{M'} \). In particular, we have
\[
    r(c) = r(\xi_M'(c')) = r'(c'),
\]
and the required identity then follows from the equality unramified local \( L \)-functions \( L_v(s, c, r) = L_v(s, c', r') \) and the definition of \( b \).

We now prove the following statement for triples \((G, M, \zeta)\) as above, for a finite set of places \( V \) of \( F \).
Proposition 5.2. For each triple \((G, M, \zeta)\) with \(G\) quasisplit and each \(c \in \mathcal{C}^*\) \((M, \zeta)\), there is a function
\[
s^G_M(c, b) = n^G_M(c, b), \quad \lambda \in \mathfrak{a}^*_M, \mathcal{C}
\]
that is analytic in some right half plane, such that for any elliptic endoscopic datum \(M'\) of \(M\) and any \(c' \in \mathcal{C}^*\) \((M', \zeta')\) with image \(c \in \mathcal{C}^*\) \((M, \zeta)\), the identity
\[
r^G_M(c, b) = \sum_{G' \in E_{G}(G)} \lambda_M(G, G') s^G_M(c', b')
\]
holds.

Proof. If \(V\) does not contain \(V_{\text{ram}}(G)\), we set \(s^G_M(c, b) = 0\). Otherwise, we define \(s^G_M(c, b)\) inductively by setting
\[
s^G_M(c, b) = r^G_M(c, b) - \sum_{G' \in E_{G}(G)} \lambda_M(G, G') s^G_M(c, b).
\]
The sum is finite since the coefficient \(\lambda_M(G, G')\) vanishes unless \(G'\) is elliptic. We then have to show that \(r^G_M(c, b)\) equals the endoscopic expression
\[
r^G_M(c', b') = \sum_{G' \in E_{G}(G)} \lambda_M(G, G') s^G_M(c', b').
\]
We shall prove this in a slightly more general setting, along the lines of \[Art99a, \S 4\].

Suppose that \(A\) is a finite set of continuous characters on \((Z(M)^{\Gamma})^0\). Consider the product of \(b_M(c, \lambda)\) with
\[
r_Q(\Lambda, c, b, A) = \prod_{a \in A \setminus \Sigma(Q)} r(c, a) r(c+\frac{\lambda}{\Lambda}, a), \quad Q \in \mathcal{P}(M),
\]
which is a \((G, M)\)-family of functions of \(\Lambda \in i\mathfrak{a}^*_M, Z\), with values in the space of functions of \(\lambda\). Then the limit
\[
r^G_M(c, b, A) = \lim_{\Lambda \to 0} \sum_{Q \in \mathcal{P}(M)} r_Q(\Lambda, c, b, A) b_M(c, \lambda) \theta_Q(\Lambda)^{-1}
\]
is an analytic function of \(\lambda\) in some right half plane. If \(A' = \{a' : a \in A\}\), we define inductively
\[
s^G_M(c, b, A) = r^G_M(c, b, A) - \sum_{G' \in E_{G}(G)} \lambda_M(G, G') s^G_M(c', b', A)
\]
for \((G, M, \zeta)\) quasisplit, and
\[
r^G_M(c', b', A') = \sum_{G' \in E_{G}(G)} \lambda_M'(G, G') s^G_M(c', b', A')
\]
in general. In fact, if \((G^*, M^*, \zeta^*)\) is a quasisplit inner twist of \((G, M, \zeta)\), there is a bijection \(c \to c^*\) from \(\mathcal{C}^*\) \((M, \zeta)\) to \(\mathcal{C}^*\) \((M^*, \zeta^*)\) such that
\[
r^G_M(c, b, A) = r^G_{M^*}(c^*, b^*, A),
\]
where \(b^* = b^{G^*}\). Since \(r^G_{M^*}(c^*, b', A')\) is equal to \(r^G_M(c', b', A')\), we can therefore assume that our triple \((G, M, \zeta)\) is quasisplit. We shall show that \(r^G_M(c', b', A')\) equals \(r^G_M(c, b, A)\) by induction on \(A\).
Assume first that $A = \{a\}$. The function $r^G_M(c_\lambda, b, a)$ vanishes unless $a$ is a root of $(\hat{G}, (Z(M)^\Gamma)^0)$ and spans the kernel $a_M^G$ of the natural map $a_M \to a_G$. The same assertion holds inductively for the functions $s^G_M(c'_\lambda, b', a')$, and hence also for $r^G_M(c_\lambda, b', a')$. We can therefore assume that $M$ is a maximal Levi subgroup, in which case $r^G_M(c_\lambda, a)$ is a logarithmic derivative $r(c_\lambda, a)$. Then by [5.2] and Lemma 5.1 we have
\[
\sum_{c'_\lambda \in E_l(G)} r^G_M(c'_\lambda, b', a') b_M(c_\lambda).
\]
On the other hand, define the expression
\[
* \sum_{c'_\lambda \in E_l(G)} \iota_M(G, G') \ast s^G_M(c'_\lambda, b', a')
\]
where
\[
* s^G_M(c'_\lambda, b', a') = |Z_a / Z_{a'} \cap Z(G)^\Gamma|^{-1} r^G_M(c'_\lambda, b', a').
\]
It suffices then to show that $* r^G_M(c'_\lambda, b', a')$ equals $r^G_M(c_\lambda, b, a)$, for if $M' = M$ this establishes inductively that $s^G_M(c_\lambda, a) = s^G_M(c_\lambda, a)$, whereby for $M'$ arbitrary we conclude that
\[
r^G_M(c'_\lambda, b', a') = * r^G_M(c'_\lambda, b', a') = r^G_M(c_\lambda, b, a)
\]
as required. From [Art99a, p. 1146], it follows that the identity
\[
\iota_M(G, G') \ast s^G_M(c'_\lambda, b, a') = |Z_a / Z_{a'} \cap Z(G)^\Gamma|^{-1} r^G_M(c'_\lambda, b', a')
\]
holds, and substituting into (5.3) we therefore have
\[
* r^G_M(c'_\lambda, b', a') = \sum_{c'_\lambda \in E_l(G)/Z_a} r^G_M(c'_\lambda, b', a') = r^G_M(c_\lambda, b, a),
\]
since $r^G_M(c'_\lambda, b', a')$ depends only on the orbit of $Z_a$ in $E_l(G)$, and since the stabilizer of $G'$ in $Z_a$ is $Z_a \cap Z(G)^\Gamma$.

Now suppose that $A$ is the disjoint union of nonempty proper subsets $A_1$ and $A_2$. Assume inductively that
\[
l^{G}_{M}(c_\lambda, b', a') = l^G_{M}(c_\lambda, b, A), \quad L_i \in \mathcal{L}(M)
\]
for $i = 1, 2$. We would like to use splitting formulas to reduce the case of $A$ to those of $A_1$ and $A_2$. While $r_Q(A, c, A)$ is certainly a product a $(G, M)$-families, we also note that $r^G_M(c_\lambda, b, A)$ can be viewed as a weighted character, whose weight factor is simply the limit of $(G, M)$ families defined by $r^G_M(c_\lambda, A)$, rather than the usual intertwining operators that form the usual weighted characters in the trace formula. We therefore have the splitting formula
\[
r^G_M(c_\lambda, b, A) = \sum_{L_1, L_2 \in \mathcal{L}(M)} d^G_M(L_1, L_2) r^L_{M}(c_\lambda, b, A_1) r^L_{M}(c_\lambda, b, A_2)
\]
where the coefficient $d^G_M(L_1, L_2)$ is defined as in [Art88a, §2], analogous to [Art88a, Proposition 9.4]. We similarly have a splitting formula for $r^G_M(c'_\lambda, b', A')$, obtained in the same way as in the proof of [Art99a, Theorem 5]. It follows then from our induction assumption that $r^G_M(c'_\lambda, b', A')$ equals $r^G_M(c_\lambda, b, A)$, and the proposition follows by taking $A$ to be the set of roots of $(\hat{G}, (Z(M)^\Gamma)^0)$. \qed
For the application to the trace formula, we may work with a slightly smaller subset of \( \mathcal{C}(G^V, \zeta^V) \). Define \( \mathcal{C}^{\text{disc}}_\kappa (G, \zeta) = \mathcal{C}^{\text{disc}}_{\kappa, \kappa'} (G^*, \zeta^*) \) inductively as the union over all inner \( K \)-forms \( G_1 \) of \( G^* \), of the sets \( \mathcal{C}^{\text{disc}}_\kappa (G_1, \zeta_1) \) defined below, together with the union over all elliptic endoscopic data \( G' \in \mathcal{E}_{\kappa, \kappa'} (G^*, V) \) of the images in \( \mathcal{C}(G^V, \zeta^V) \) of \( \mathcal{C}^{\text{disc}}_\kappa (G', \zeta') \). It is a proper subset of \( \mathcal{C}_{\text{aut}, \kappa} (G, \zeta) \). The following corollary then follows from the same proof of [Art02, Corollary 8.4].

**Corollary 5.3.** Let \( c \in \mathcal{C}^{\text{disc}}_\kappa (M, \zeta) \). Then \( r^G_{\mathcal{C}} \) is an analytic function of \( \lambda \in i\mathfrak{a}_{M, Z} \) in some right-half plane satisfying

\[
\int_{i\mathfrak{a}_{M, Z}} r^G_{\mathcal{C}} (c, \lambda, b) (1 + \|\lambda\|)^{-N} d\lambda < \infty
\]

for some \( N \), and similarly for \( s^G_{\mathcal{C}} (c, \lambda, b) \) if \( G \) is quasisplit.

By [Art02] Corollary 8.4, the functions \( r^G_{\mathcal{C}} (c, \lambda, b) \) and \( s^G_{\mathcal{C}} (c, \lambda) \) themselves are analytic in \( \lambda \), and as we are assuming that the implied parameter \( s \) of \( b \) has real part large enough, we can in fact evaluate \( r^G_{\mathcal{C}} (c, \lambda, b) \) and \( s^G_{\mathcal{C}} (c, \lambda, b) \) at the point \( \lambda = 0 \).

**5.3. Spectral transfer factors.** We now turn to the spectral side of the trace formula. We shall first recall the basic definitions of the objects that appear on the spectral side as in [Art02] §3. Recall the space \( \mathcal{F}(G^V, \zeta_V) \) of finite complex linear combinations of irreducible characters on \( G^V \). It has a canonical basis \( \Pi(G^V, \zeta_V) \) given by irreducible characters with \( Z_V \)-central character equal to \( \zeta_V \). As before, we identify elements \( \pi \in \mathcal{F}(G^V, \zeta_V) \) with the linear form

\[
f \mapsto f (\pi) = \operatorname{tr}(f) \]

on \( \mathcal{H}(G^V, \zeta_V) \). We write \( \Pi_{\text{unit}} (G^V, \zeta_V) \) for the subset of unitary characters in \( \Pi(G^V, \zeta_V) \). The orbits of the action \( \pi \mapsto \pi_\lambda \) of \( i\mathfrak{a}^*_{G, Z} \) on \( \pi \in \Pi_{\text{unit}} (G^V, \zeta_V) \) can be identified with the set \( \Pi_{\text{unit}} (G^V, \zeta_V) \). The induction operation is given by the induced characters

\[
f_M (\pi) = f (\pi^G) = \operatorname{tr}(f_\mathcal{G}(\pi)), \quad \pi \in \Pi_{\text{unit}} (M_V, \zeta_V).
\]

If moreover we restrict to the subset of tempered characters \( \Pi_{\text{temp}} \) of \( G^V, \zeta_V \), we may then take \( f \in \mathcal{C}(G^V, \zeta_V) \).

We regard \( f_G \) as a linear form on both \( \mathcal{D}(G^V, \zeta_V) \) and \( \mathcal{F}(G^V, \zeta_V) \), which are in turn determined by its restriction to the subsets \( \Gamma^\text{reg} (G^V, \zeta_V) \) and \( \Pi_{\text{temp}} (G^V, \zeta_V) \) respectively. The functions determine each other, so we may consider the subspace of stable distributions \( S \mathcal{F}(G^V, \zeta_V) \) in \( \mathcal{F}(G^V, \zeta_V) \). Parallel to the geometric side, we consider a basis \( \Phi(G^V, \zeta_V) \) of \( S \mathcal{F}(G^V, \zeta_V) \), and the subset \( \Phi(G^V, \zeta_V) = \Phi^\text{\delta} (G^V, \zeta_V) \cap S \mathcal{F}(G^V, \zeta_V) \) that forms a basis of \( S \mathcal{F}(G^V, \zeta_V) \). It is defined in terms of the abstract bases \( \Phi_{\text{disc}} (M_v, \zeta_v) \) of a certain cuspidal subspace \( S \mathcal{C}_{\text{cusp}} (M_v, \zeta_v) \) parallel to [Art90], and their analogues for endoscopic groups \( M' \) of \( M \). (See also §9.3) Then if \( \phi_\nu \) is an element of \( \Phi(G^V, \zeta_V) \) with image \( \phi_\nu' \in \Phi_{\text{disc}} (G_{\nu}, \zeta_{\nu}) \), we have spectral transfer factors

\[
\Delta (\phi_\nu, \pi_{\nu}) = \Delta (\phi_\nu', \pi_{\nu}), \quad \pi_{\nu} \in \Pi(G_{\nu}, \zeta_{\nu})
\]

such that the linear form \( f \mapsto f^G_{\nu} (\phi_\nu) = f (\phi_\nu') \) has an expansion

\[
f^G_{\nu} (\phi_\nu) = \sum_{\pi \in \Pi (G_{\nu}, \zeta_{\nu})} \Delta (\phi_\nu, \pi_{\nu}) f (\pi_{\nu}).
\]
Then we form the extended transfer factor
\[ \Delta(\phi, \pi) = \prod_{\nu \in V} \Delta(\phi_\nu, \pi_\nu), \quad \phi \in \Phi^\ell(G_V, \zeta_V), \quad \pi \in \Pi(G_V, \zeta_V). \]

Following [Art02, §5], we can assume that we have fixed quotients \( \Phi^\ell(G_V^\ell, \zeta_V^\ell) \) of \( \Phi^\ell(G_V, \zeta_V) \) and \( \Phi(G_V^\ell, \zeta_V^\ell) \) of \( \Phi(G_V, \zeta_V) \) that form bases of \( \mathcal{F}(G_V^\ell, \zeta_V^\ell) \) and \( S\mathcal{F}(G_V^\ell, \zeta_V^\ell) \) respectively. If \( \phi \) and \( \pi \) have unitary central characters, we can then identify them with the orbits \( \phi_\lambda \) and \( \pi_\lambda \) of \( i\alpha_{G,Z}^* \) in \( \Phi^\ell(G_V^\ell, \zeta_V^\ell) \) and \( \Pi(G_V^\ell, \zeta_V^\ell) \) respectively. We then define transfer factors on \( \Phi^\ell(G_V^\ell, \zeta_V^\ell) \times \Pi(G_V^\ell, \zeta_V^\ell) \) as the average
\[ \Delta(\phi, \pi) = \sum_{\lambda \in i\alpha_{G,Z}^*} \Delta(\phi_\lambda, \pi_\lambda), \]
where the sum only has one nonzero term.

5.4. The spectral coefficients. We shall now construct the global spectral coefficients, which will be the terms on the spectral side that depend on the basic function, following [Art02] §7. We can identify any element \( c \in \mathcal{E}(G^V, \zeta^V) \) with a \( K^V \)-unramified representation \( \pi^V(c) \in \Pi(G^V, \zeta^V) \). Given \( \pi \in \Pi(G_V^\ell, \zeta_V^\ell) \), we write \( \pi \times c = \pi \times \pi^V(c) \) for the associated representation in \( \Pi(G(A)^\ell, \zeta) \). We shall identify \( \pi \) with its representative in \( \Pi(G^\ell, \zeta) \), and \( \pi \times c \) with the associated representation in \( \Pi(G(A)^\ell, \zeta) \). We recall the basic spectral coefficient \( a(\hat{\pi}) \) in [Art02, §3] that is supported on the discrete subset \( \Pi_{\text{disc}}(G) \) of unitary representations \( \Pi_{\text{unit}}(G(A)^\ell) \). Let \( \Pi_{\text{disc}}(G, \zeta) \) be the subset of those representations whose central character on \( Z(A)^\ell \) is equal to \( \zeta \). We then define \( \Pi_{\text{disc}}(G, \zeta) \) to be the set of representations \( \pi \in \Pi_{\text{unit}}(G^\ell, \zeta) \) such that \( \pi \times c \) belongs to \( \Pi_{\text{disc}}(G, \zeta) \) for some \( c \in \mathcal{E}(G^V, \zeta^V) \). We also define \( \mathcal{E}_{\text{disc}}(G, \zeta) \) to be the set of \( c \in \mathcal{E}(G^V, \zeta^V) \) such that \( \pi \times c \) belongs to \( \Pi_{\text{disc}}(G, \zeta) \) for some \( \pi \in \Pi_{\text{disc}}(G, \zeta) \), which we used to define the sets \( \mathcal{E}_{\text{disc}}(G_1, \zeta_1) \) earlier.

We then define for any \( \pi \in \Pi(G^\ell, \zeta_V^\ell) \), the spectral coefficient
\[ a_{r,s}(\pi) = \sum_{L \in \mathcal{L}} \|W^G_0 \| W^G_{0}^{-1} \sum_{\bar{c} \in \mathcal{E}_{\text{disc}}(M, \zeta)} a^M_{\text{disc}}(\pi_M \times c)r^G_M(c, b), \]
where \( \pi_M \times c \) is a finite sum of representations \( \hat{\pi} \in \Pi_{\text{unit}}(M(A), \zeta) \), and \( a^M_{\text{disc}}(\pi_M \times c) \) is the sum of corresponding values \( a^G_{r,s}(\hat{\pi}) \). Let \( \Pi_{\text{disc}}(M, V, \zeta) \) be the preimage of \( \Pi_{\text{disc}}(M, V, \zeta) \) in \( \Pi_{\text{unit}}(M_V, \zeta_V) \), and let \( \Pi_{\text{disc}}^G(M, V, \zeta) \) be the set of \( i\alpha_{M,Z}^* \)-orbits in \( \Pi_{\text{disc}}(M, V, \zeta) \). There is a free action \( \rho \rightarrow \rho_\lambda \) of \( i\alpha_{M,Z}^* \) on \( \Pi_{\text{disc}}^G(M, V, \zeta) \) whose orbits can be identified with \( \Pi_{\text{disc}}(M, V, \zeta) \). Any element \( \rho \in \Pi_{\text{disc}}^G(M, V, \zeta) \) is an irreducible representation of \( M_V \cap G_V^\ell \), from which one can form the parabolically induced representation \( \rho^G \) of \( G_V^\ell \). Finally, let \( \Pi(G, V, \zeta) \) be the union over \( M \in \mathcal{L} \) and \( \rho \in \Pi_{\text{disc}}^G(M, V, \zeta) \) of irreducible constituents of \( \rho^G \). It follows then from the definitions that \( a_{r,s}^G(\pi) \) is supported on the subset \( \Pi(G, V, \zeta) \) of \( \Pi(G^\ell, \zeta_V^\ell) \). We again write \( a^G_{r,s,\text{disc}}(\pi) \) for the term \( M = G \) in the expansion of \( a_{r,s}^G(\pi) \). That is,
\[ a^G_{r,s,\text{disc}}(\pi) = \sum_{\bar{c} \in \mathcal{E}_{\text{disc}}(G, \zeta)} a^G_{\text{disc}}(\pi \times c)r_G(c, b), \]
and we note that \( r_G(c, b) \) is equal to \( L(s, c, r) \).

We next construct parallel families of endoscopic and stable spectral coefficients on certain subsets \( \Phi_{\text{disc}}^\ell(G, V, \zeta), \Phi_{\text{disc}}(G, V, \zeta), \) and \( \Pi_{\text{disc}}^\ell(G, V, \zeta) \) of \( \Phi^\ell(G_V^\ell, \zeta_V^\ell) \).
\( \Phi(G_{\ell \ell}^\phi, \zeta_{\ell \ell}) \) and \( \Pi(G_{\ell \ell}^\phi, \zeta_{\ell \ell}) \) respectively in the same way as \( \Pi_{\text{disc}}(G, V, \zeta) \). Similarly, from these discrete subsets we construct the larger subsets \( \Phi^\phi(G, V, \zeta) \), \( \Phi(G, V, \zeta) \), and \( \Pi^\phi(G, V, \zeta) \) again by the same induction process. For any \( \pi \in \Pi^\phi(G, V, \zeta) \), we set

\[
a_{G,s}^\phi (\pi) = \sum_{G^\prime} \sum_{\phi^\prime} \ell(G, G^\prime) b_{G,s}^\phi (\phi^\prime) \Delta_G (\phi^\prime, \pi) + \varepsilon(G) \sum_{\phi} b_{G,s}^\phi (\phi) \Delta_G (\phi, \pi)
\]

with \( G^\prime, \phi^\prime \), and \( \phi \) summed over \( \mathcal{E}_{\text{ecl}}(G, V) \), \( \Phi(\tilde{G}^\prime, V, \tilde{\zeta}^\prime) \) and \( \Phi^\phi(G, V, \zeta) \) respectively, and the coefficients \( b_{G,s}^\phi (\phi^\prime) \) are defined inductively by the requirement that

\[
a_{G,s}^\phi (\pi) = a_{G,s}^\phi (\pi)
\]

in the case that \( G \) is quasisplit. Moreover, we set

\[ b_{G,s}^\phi (\phi^*), \quad \phi \in \Phi^\phi(G, V, \zeta) \]

where \( b_{G,s}^\phi (\phi^* \phi) \) is obtained as a function on \( \Phi^\phi(G, V, \zeta) \) by the local inversion formula \([\text{Art02}, (5.8)]\).

We then state the main global theorem concerning the spectral coefficients. It is the analogue of the main Global Theorem 2' of \([\text{Art02}, \S 7]\), and will again be proved by a series of reductions. We state it here in order to use the necessary induction hypotheses.

**Theorem 5.4.** (a) If \( G \) is arbitrary, we have

\[
a_{G,s}^\phi (\pi) = a_{G,s}^\phi (\pi), \quad \pi \in \Pi^\phi(G, V, \zeta).
\]

(b) If \( G \) is quasisplit, we have that

\[
b_{G,s}^\phi (\phi), \quad \phi \in \Phi^\phi(G, V, \zeta),
\]

is supported on the subset \( \Phi(G, V, \zeta) \) of \( \Phi^\phi(G, V, \zeta) \).

We also define the stable and endoscopic analogues of the discrete coefficient \( a_{G,s,\text{disc}}(\hat{\pi}) \). That is, for any elements \( \hat{\pi} \in \Pi_{\text{disc}}(G, \zeta) \) and \( \phi \in \Phi_{\text{disc}}(G, \zeta) \), we set

\[
a_{G,s,\text{disc}}(\hat{\pi}) = \sum_{G^\prime} \sum_{\hat{\phi}^\prime} \ell(G, G^\prime) b_{G,s,\text{disc}}^\phi (\hat{\phi}^\prime) \Delta_G (\hat{\phi}^\prime, \hat{\pi}) + \varepsilon(G) \sum_{\hat{\phi}} b_{G,s,\text{disc}}^\phi (\hat{\phi}) \Delta_G (\hat{\phi}, \hat{\pi})
\]

with \( G^\prime, \hat{\phi}^\prime \), and \( \hat{\phi} \) summed over \( \mathcal{E}_{\text{disc}}(G) \), \( \Phi_{\text{disc}}(G^\prime, \hat{\zeta}^\prime) \) and \( \Phi_{\text{disc}}^\phi(G, \zeta) \) respectively, and the coefficients \( b_{G,s,\text{disc}}^\phi (\hat{\phi}^* \phi) \) are defined inductively by the requirement that

\[
a_{G,s,\text{disc}}(\hat{\pi}) = a_{G,s,\text{disc}}(\hat{\pi})
\]

and

\[
b_{G,s,\text{disc}}^\phi (\hat{\phi}^* \phi) = b_{G,s,\text{disc}}^\phi (\hat{\phi}^* \phi)
\]

in the case that \( G \) is quasisplit.

Finally, we define the endoscopic and stable analogues of \([5.5]\). For any \( c \in \mathcal{E}(G_{\ell \ell}^V, \zeta_{\ell \ell}^V) \), let \( \phi^V(c) \) be the corresponding product of unramified Langlands parameters \( \phi_v(c_v) \) over all \( v \notin V \), write \( \phi \times c = \phi \times \phi^V(c) \) for the associated element in \( \Phi^\phi(G(\mathbb{A})^Z, \zeta) \) for any \( \phi \in \Phi^\phi(G_{\ell \ell}^\phi, \zeta_{\ell \ell}) \). We then set

\[
e_{G,s,\text{disc}}^\phi (\pi) = \sum_{c \in \mathcal{E}_{\text{disc}}(G, \zeta)} a_{G,s,\text{disc}}^\phi (\pi \times c) r_G(c, b)
\]

for any \( \pi \in \Pi_{\text{disc}}(G, \zeta) \).
for $G$ arbitrary and $\pi \in \Pi^\ell_{\text{disc}}(G, V, \zeta)$, and
\[ b^G_{r,s,\text{el}}(\delta, S) = \sum_{c \in \Phi^\ell_{\text{disc}}(G, \zeta)} b^G_{r,s,\text{el}}(\phi \times c)r_G(c, b) \]
for $G$ quasisplit and $\phi \in \Phi^\ell_{\text{disc}}(G, V, \zeta)$. These definitions will allow us to define endoscopic and stable variants of the spectral expansion of the linear form $I_\pi^r(f)$.

5.5. **The discrete and unitary parts.** Recall that the spectral expansion of $I_\pi^r(f)$ is given in [Won22, Theorem 5.2] by
\[ I_\pi^r(f) = \sum_{M \in \mathcal{E}} |W_M^0||W_M^G|^{-1} \int_{\Pi(M, V, \zeta)} a^M_{r,s}(\pi)I_M(\pi, f)d\pi \]
for any $f \in \mathcal{E}_\pi^0(G, V, \zeta)$, where $d\pi$ is a natural Borel measure on the set $\Pi(M, V, \zeta)$. Let us define
\[ I_{r,s,\text{disc}}(\hat{f}) = \sum_{\pi \in \Pi_{\text{disc}}(G, \zeta)} a^G_{r,s,\text{disc}}(\pi)\hat{f}_G(\pi), \]
for $\hat{f} \in \mathcal{E}_\pi^0(G(\mathbb{A})^2, \zeta)$, and
\[ I_{r,s,\text{unit}}(f) = \int_{\Pi(G, V, \zeta)} a^G_{r,s}(\pi)f_G(\pi)d\pi, \]
for $f \in \mathcal{E}_\pi^0(G, V, \zeta)$, corresponding to the term $M = G$ in (5.7). We call it the purely unitary part of $I_\pi^r(f)$, which is a continuous linear combination of irreducible unitary characters. If we restrict to the discrete coefficients, we obtain the linear form
\[ I_{r,s,\text{disc}}(f) = \sum_{\pi \in \Pi_{\text{disc}}(G, V, \zeta)} a^G_{r,s,\text{disc}}(\pi)f_G(\pi) \]
which can be regarded as the discrete part of $I_\pi^r(f)$. It follows from the definitions that $I_{r,s,\text{disc}}(f) = I_{r,s,\text{disc}}(\hat{f})$ for $\hat{f} = f \times b^V$.

We define endoscopic and stable analogues of these by setting inductively
\[ I^\ell_{r,s,\text{disc}}(\hat{f}) = \sum_{G' \in \mathcal{E}^\ell_{\text{disc}}(G, V)} I^\ell_{r,s,\text{disc}}(\hat{f}')(G')S^G_{r,s,\text{disc}}(\hat{f}') + \varepsilon(G)S^G_{r,s,\text{disc}}(\hat{f}) \]
and
\[ I^\ell_{r,s,\text{unit}}(f) = \sum_{G' \in \mathcal{E}^\ell_{\text{unit}}(G, V)} I^\ell_{r,s,\text{unit}}(f')(G')S^G_{r,s,\text{unit}}(f') + \varepsilon(G)S^G_{r,s,\text{unit}}(f) \]
where $\hat{f}_S \in \mathcal{E}_\pi^0(G, S, \zeta)$ and $f \in \mathcal{E}_\pi^0(G, V, \zeta)$ respectively, and the terms $\hat{S}^G_{r,s,\text{disc}}$ and $\hat{S}^G_{r,s,\text{unit}}$ are linear forms on $\mathcal{E}^0_{\text{disc}}(G', V, \zeta)$. We furthermore require that
\[ I^\ell_{r,s,\text{disc}}(f) = I_{r,s,\text{disc}}(f) \]
and
\[ I^\ell_{r,s,\text{unit}}(f) = I_{r,s,\text{unit}}(f) \]
respectively in the case that $G$ is quasisplit, and the general induction hypothesis that $S^\ell_{\text{disc}}$ and $S^\ell_{\text{unit}}$ are stable for $G'$ in $\mathcal{E}^0_{\text{disc}}(G, S)$ and $\mathcal{E}^0_{\text{disc}}(G, V)$ respectively. If $G$ is arbitrary, it follows by the same argument of [Art02, Lemma 7.3] that
\[ I^\ell_{r,s,\text{disc}}(\hat{f}) = \sum_{\pi \in \Pi_{\text{disc}}(G, \zeta)} a^G_{r,s,\text{disc}}(\pi)\hat{f}_G(\pi) \]
and
\begin{equation}
I_{r,s,\text{unit}}^G(f) = \int_{\Pi^d(G,V,\zeta)} a_{r,s}^{G,E}(\pi)f_G(\pi)d\pi,
\end{equation}
and if $G$ is quasisplit we have that
\begin{equation}
S_{r,s,\text{disc}}^G(\hat{\phi}) = \sum_{\phi \in \Phi^d_{\text{disc}}(G,\zeta)} b_{r,s,\text{disc}}^G(\phi)f_G^d(\phi)
\end{equation}
and
\begin{equation}
S_{r,s,\text{unit}}^G(f) = \int_{\Phi^d(G,V,\zeta)} b_{r,s}^G(\phi)f_G^d(\phi)d\phi,
\end{equation}
where again we have natural Borel measures defined on the sets $\Phi^d_{\text{disc}}(G,\zeta)$ and $\Phi^d_{\text{disc}}(G,V,\zeta)$.

We also set
\begin{equation}
I_{r,s,\text{disc}}^G(f) = \sum_{G' \in E_0^d(G)} \iota(G,G') \hat{S}_{r,s,\text{disc}}^{G'}(f') + \varepsilon(G)S_{r,s,\text{disc}}^G(f)
\end{equation}
for linear forms $\hat{S}_{r,s,\text{disc}}^{G'}(f')$ on $\mathcal{S}^{\varepsilon}(G',V',\zeta')$ which are defined inductively by requiring that
\begin{equation}
I_{r,s,\text{disc}}^G(f) = I_{r,s,\text{disc}}^G(\hat{f})
\end{equation}
in the case that $G$ is quasisplit. If we take $\hat{f} = f \times b^V$, it then follows inductively from Corollary 5.3 and (5.11) that
\begin{equation}
I_{r,s,\text{disc}}^G(f) = I_{r,s,\text{disc}}^G(\hat{f})
\end{equation}
and
\begin{equation}
S_{r,s,\text{disc}}^G(f) = S_{r,s,\text{disc}}^G(\hat{f}),
\end{equation}
moreover from the expansions (5.13) and (5.15) we conclude that
\begin{equation}
I_{r,s,\text{disc}}^G(f) = \sum_{\pi \in \Pi^d_{\text{disc}}(G,V,\zeta)} a_{r,s,\text{disc}}^{G,E}(\pi)f_G(\pi)
\end{equation}
and
\begin{equation}
S_{r,s,\text{disc}}^G(f) = \sum_{\Phi^d_{\text{disc}}(G,V,\zeta)} b_{r,s,\text{disc}}^G(\phi)f_G^d(\phi)
\end{equation}
These formulae represent a stabilization of the term with $M = G$ in (5.4).

Having made these preliminary manipulations, we can now establish the spectral expansion of $I_{r,s}^G(f)$ and $S_{r,s}^G(f)$. We recall from (4.21) the definition
\begin{equation}
I_{r,s}^G(f) = \sum_{G' \in E_0^d(G,V)} \iota(G,G') \hat{S}_{r,s}^{G'}(f') + \varepsilon(G)S_{r,s}^G(f), \quad f \in \mathcal{S}^{\varepsilon}(G,V,\zeta)
\end{equation}
defined inductively by the supplementary requirement that $I_{r,s}^G(f) = I(f)$ in the case that $G$ is quasisplit. We are assuming inductively that if $G$ is replaced by a quasisplit inner $K$-form $\tilde{G}'$, the corresponding analogue of $S_{r,s}^G$ is defined and stable.
Proposition 5.5. Let $f \in \mathcal{E}^\circ(G, V, \zeta)$.

(a) If $G$ is arbitrary,

$$I_{r,s,\text{unit}}^\varepsilon(f) - I_{r,s,\text{disc}}^\varepsilon(f) = \sum_{M \in \mathcal{L}^0} |W_0^M||W_0^G|^{-1} \int_{\Pi^\varepsilon(M, V, \zeta)} a_{r,s,\varepsilon}^M(\pi) I_M^\varepsilon(\pi, f) d\pi.$$  

(b) If $G$ is quasisplit,

$$S_{r,s}^G(f) - S_{r,s,\text{unit}}^G(f) = \sum_{M \in \mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{M' \in \mathcal{L}^0(M, V)} \iota(M, M') \int_{\Phi(M', V, \zeta')} b_{r,s}^M(\phi') S_M^G(M', \phi', f) d\phi.$$  

Proof. The proof follows the same argument as [Art02, Theorem 10.6].

Theorem 5.6. (a) If $G$ is arbitrary and $\pi \in \Pi^\varepsilon(G, V, \zeta)$, then

$$a_{r,s,\varepsilon}^G(\pi) - a_{r,s,\text{disc}}^G(\pi) = \sum_{M \in \mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{c \in \mathcal{E}_{\text{disc}}(M, \zeta)} a_{\ell,\text{disc}}^{\varepsilon}(\pi_M \times c) r_M^G(c, b).$$

(b) If $G$ is quasisplit and $\phi \in \Phi^\varepsilon(G, V, \zeta)$, then

$$b_{r,s}^G(\phi) - b_{r,s,\text{disc}}^G(\phi) = \sum_{M \in \mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{c \in \mathcal{E}_{\text{disc}}(M, \zeta)} b_{r,s,\text{disc}}^{\varepsilon}(\phi_M \times c) s_M^G(c, b)$$

if $\phi$ lies in the subset $\Phi(G, V, \zeta)$ of $\Phi^\varepsilon(G, V, \zeta)$, and is zero otherwise.

Proof. The proof is parallel to that of Theorem 4.5 so we can be brief. Let us consider the differences of (5.14) and (5.19),

$$I_{r,s,\text{unit}}^\varepsilon(f) - I_{r,s,\text{disc}}^\varepsilon(f) = \int_{\Pi^\varepsilon(G, V, \zeta)} (a_{r,s,\varepsilon}^G(\pi) - a_{r,s,\text{disc}}^G(\pi)) f_G(\pi) d\pi,$$

and (5.16) and (5.20),

$$S_{r,s,\text{unit}}^G(f) - S_{r,s,\text{disc}}^G(f) = \int_{\Phi^\varepsilon(G, V, \zeta)} (b_{r,s}^G(\phi) - b_{r,s,\text{disc}}^G(\phi)) f_G(\phi) d\phi.$$  

Substituting the right-hand side of (5.21), we obtain the linear form

$$\sum_{M \in \mathcal{L}^0} |W_0^M||W_0^G|^{-1} \int_{\Pi_{\text{disc}}(M, V, \zeta)} \sum_{c \in \mathcal{E}_{\text{disc}}(M, \zeta)} a_{r,s,\text{disc}}^{\varepsilon}(\pi \times c) f_M(\pi) r_M^G(c, b).$$

We can rewrite this as

$$I_{r,s,\text{unit}}^\varepsilon(f) = \sum_{M \in \mathcal{L}^0} |W_0^M||W_0^G|^{-1} \int_{\Pi_{\text{disc}}(M, V, \zeta)} a_{r,s,\text{disc}}^{\varepsilon}(\hat{\pi}) f_M(\pi) r_M^G(c, b) d\hat{\pi},$$

where the integral is taken over elements $\hat{\pi} = \pi \times c$ in the product of $\Pi_{\text{disc}}^\varepsilon(M, V, \zeta)$ and $\mathcal{E}_{\text{disc}}^\varepsilon(M, \zeta)$ relative to natural measures. which we will denote by $I_{r,s,0}^\varepsilon(f)$. Hence the required identity (5.21) is equivalent to showing that

$$I_{r,s,\text{unit}}^\varepsilon(f) - I_{r,s,\text{disc}}^\varepsilon(f) = I_{r,s,0}^\varepsilon(f).$$
On the other hand, if \( G \) is quasisplit, substituting the right-hand side of (5.22), we obtain the linear form
\[
\sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Phi_{\text{disc}}(M, V, \zeta)} \sum_{c \in \mathcal{E}_{\text{disc}}(M, \zeta)} b_{r,s,\text{disc}}^M(\phi \times c) f^M(\phi)s_M^G(c, b),
\]
which again we rewrite as
\[
S_{r,s,\text{unit}}^G(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\mathcal{E}_{\text{disc}}(M, \zeta)} b_{r,s,\text{disc}}^M(\phi)f^M(\phi)s_M^G(c, b) d\phi
\]
where the integral is taken over elements \( \phi = \phi \times c \) in the product of \( \Phi_{\text{disc}}(M, V, \zeta) \) and \( \mathcal{E}_{\text{disc}}(M, \zeta) \) relative to natural measures. Then the second required identity (5.22) is equivalent to showing that
\[
S_{r,s,\text{unit}}^G(f) - S_{r,s,\text{disc}}^G(f) = S_{r,s,\text{unit}}^{G,0}(f).
\]
Now the rest of the proof follows that of Proposition 4.5 where instead of Proposition 4.1 we use Proposition 5.2. In particular, if \( \phi' = \phi' \times c' \) belongs to the product of \( \Phi_{\text{disc}}(M', V, \zeta') \) with \( \mathcal{E}_{\text{disc}}(M', \zeta') \) and has image \( \phi \times c \) in the product of \( \Phi_{\text{disc}}^e(M, V, \zeta) \) with \( \mathcal{E}_{\text{disc}}^e(M, \zeta) \) then it follows that
\[
\sum_{G' \in \mathcal{E}_{\text{disc}}^e(G)} t_{M'}(G, G') f^{M'}(\phi') s_{\mathcal{E}_{\text{disc}}^e}(c', b') = r_{M'}^G(c, b) f^G(\phi).
\]
The remainder of the argument then follows the same structure of Proposition 4.5.

The preceding proposition provides a parallel reduction of study of the global spectral coefficients to the basic discrete coefficients.

**Corollary 5.7.** Suppose that
(a) if \( G \) is arbitrary, we have
\[
a_{r,s,\text{disc}}^{G,e}(\pi) = a_{r,s,\text{ell}}^G(\pi),
\]
for any \( \pi \in \mathcal{E}_{\text{disc}}^e(G, \zeta) \) and
(b) if \( G \) is quasisplit, we have that
\[
b_{\text{disc}}^G(\phi), \quad \phi \in \Phi_{\text{disc}}^e(G, \zeta),
\]
is supported on the subset \( \Phi_{\text{disc}}^e(G, \zeta) \) of \( \Phi_{\text{disc}}^e(G, \zeta) \).

Then Theorem 5.4 holds.

Let us take stock of what we have attained so far. Propositions 4.4 and 5.5 together imply the reduction of the stable expansions to expressions
\[
S(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\delta \in \Delta(M, V, \zeta)} b_{r,s}^M(\delta)s_M(\delta, f)
\]
and
\[
S(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Phi(M, V, \zeta)} b_{r,s}^M(\phi)s_M(\phi, f)
\]
for \( f \in \mathcal{E}(G, V, \zeta) \). By the main local theorems of [Art02] and their extension to \( \mathcal{E}(G, V, \zeta) \) in Corollaries 9.3 and 9.4 the linear forms \( S_M(\delta, f) \) and \( S_M(\phi, f) \) are

\[3\text{The corresponding sets in [Art02] p.276 are mislabeled.} \]
stable. What remains is to show that the global coefficients \( b^M_r(\delta) \) and \( b^M_r(\phi) \) are stable. This will take up the rest of Part 1.

6. Global descent

6.1. Jordan decompositions. The next reduction that we shall make concerns the global descent of the elliptic coefficients. This will allow us to reduce to the case where the elements \( \gamma_S \) and \( \delta_S \) appearing in Corollary 4.6 are unipotent. The arguments of [Art01] will apply here without much modification.

The semisimple part of \( \gamma_S \in \Gamma(G_S, \zeta_S) \) is defined as a semisimple conjugacy class \( c_S \in \Gamma_{ss}(G_S) \), following [Art01 §1]. If \( c_S \) is contained in the component \( G_{\alpha,S} = G_{\alpha,S}/Z_S \), we write
\[
\tilde{G}_{c_S,+} = \prod_{v \in S} \tilde{G}_{c_v,+}
\]
for the centralizer of \( c_S \) in the component \( \tilde{G}_S \), and we write \( \tilde{G}_{c_S} \) for the connected component of the identity of \( \tilde{G}_{c_S,+} \). We shall write \( \tilde{G}_{c_S} \) for the preimage of \( \tilde{G}_{c_S} \) in \( G_{\alpha,S} \) and also let it stand for its \( F_S \)-points. The unipotent part \( \tilde{\alpha}_S \) of \( \tilde{\gamma}_S \) is then defined to be an element in the subset \( \Gamma_{unip}(G_{c_S}, \zeta_S) \) of distributions in \( \Gamma(G_{c_S}, \zeta_S) \) whose semisimple part is trivial. We note that the \( \Gamma_{unip}(G_{c_S}, \zeta_S) \) is a basis for the distributions \( D_1(G_{c_S}, \zeta_S) = D_{unip}(G_{c_S}, \zeta_S) \) supported on the conjugacy class of the identity. The elements in \( \Gamma(G_{c_S}, \zeta_S) \) then have a canonical decomposition \( \tilde{\gamma}_S = c_S\tilde{\alpha}_S \). The distribution \( \tilde{\alpha}_S \) is determined by \( \tilde{\gamma}_S \) only up to the action of the finite group
\[
\tilde{G}_{c_S,+}(F_S)/\tilde{G}_{c_S}(F_S) = \prod_{v \in S} \tilde{G}_{c_v,+}(F_v)/\tilde{G}_{c_v}(F_v)
\]
in \( \Gamma_{unip}(G_{c_S}, \zeta_S) \).

Also, we say a semisimple element \( c \) is \( F \)-elliptic in \( G \) if \( A_{G_c} = A_G \). We then define \( i^G(S,c) \) to be equal to 1 if \( c \) is an \( F \)-elliptic element in \( G \) and the \( G(A)^S \)-conjugacy class of \( c \) meets the maximal compact subgroup \( K^S \), and equal to zero otherwise. Then the following provides a descent formula for the elliptic coefficient.

**Lemma 6.1.** Suppose that \( S \) contains \( V_{\text{ram}}(G, \zeta) \), and that \( \gamma_S \in \Gamma_{\text{ell}}(G,S,\zeta) \) is admissible. Then
\[
a_{r,s,\text{ell}}^G(\gamma_S) = \sum_c \sum_{\alpha} i^G(S,c)|\tilde{G}_{c,+}(F)/\tilde{G}_{c}(F)|^{-1} a_{r,s,\text{ell}}^{G_c}(\tilde{\alpha}),
\]
where \( c \) runs over elements in \( \Gamma_{ss}(\tilde{G}) \) whose image in \( \Gamma_{ss}(G_S) \) equals \( c_S \), and \( \tilde{\alpha} \) runs over the orbit of \( \tilde{G}_{c,+}(F_S)/\tilde{G}_{c}(F_S) \) in \( \Gamma_{unip}(G_{c,S}, \zeta_S) \) determined by \( \tilde{\alpha}_S \).

**Proof.** Let \( S^\delta = \prod_{v \notin S} S_v \) and \( Z_{S^\delta} = Z(F) \cap Z_S Z(S^\delta) \). We recall that two elements \( \tilde{\gamma} \) and \( \tilde{\gamma}_1 \) in \( G(F_S) \) with standard Jordan decompositions \( \tilde{\gamma} = c\tilde{\alpha} \) and \( \tilde{\gamma}_1 = c_1\tilde{\alpha}_1 \) are said to be \((G,S)\)-equivalent if there is an element \( \tilde{\delta} \in G(F) \) such that \( \tilde{\delta}^{-1} c_1 \tilde{\delta} = c \) and \( \tilde{\delta}^{-1} c_1 \tilde{\delta} \) is conjugate to \( \alpha \) in \( G_c(F_S) \). The elliptic coefficients were then defined in [Won22 (4.10)] as
\[
a_{r,s,\text{ell}}^G(\gamma_S) = \sum_{\{\tilde{\gamma}\}} |Z(F, \tilde{\gamma})|^{-1} a_{r,s}^G(S, \tilde{\gamma})(\tilde{\gamma}/\gamma_S)
\]
where the sum over \( \{\tilde{\gamma}\} \) runs over a set of representatives of \( Z_{S^\delta}\)-orbits in the set of \((G,S)\)-equivalence classes \((G(F))_{G,S}^1 \) of \( G(F) \), and \( Z(F, \tilde{\gamma}) \) is the subset of \( z \in Z_{S^\delta} \)
such that \( z\gamma = \gamma \). Also, \( (\gamma/\gamma_S) \) is the ratio of the signed measure on \( \gamma_S \) that comes with \( \gamma \) and the invariant measure on \( \gamma_S \).

Moreover, for a general element \( \gamma = c\alpha \), we have the descent formula in \([\text{Won22} (4.9)]\) for the general coefficient
\[
d_{r,s}(S, \gamma) = i^G(S, c) |G_{c,+}(F)/G_c(F)|^{-1} d_{r,s}(S, \alpha).
\]
The function \( i^G(S, c) \) in this case will be nonzero if \( c \) is an \( F \)-elliptic element in \( G \) whose \( G(\mathbb{A}^S) \)-conjugacy class meets the maximal compact subgroup \( K^S = K^SZ(\mathbb{A}^S)/Z(\mathbb{A}^S) \). Then using the fact that
\[
|\tilde{G}_{c,+}(F)/\tilde{G}_c(F)||G_{c,+}(F)/G_c(F)|^{-1} = |Z(F, \tilde{c})|
\]
for any conjugacy class \( \tilde{c} \in \Gamma_{ss}(G) \) in the preimage of \( c \), the required formula follows by expressing the sum over \( \{\gamma\} \) as a double sum over \( c \) and \( \alpha \) as above, and comparing the expressions for \( d_{r,s,\text{ell}}(\gamma_S) \) and \( d_{r,s,\text{ell}}(\hat{\alpha}_S) \).

We shall also require Jordan decompositions of elements in \( \hat{\delta}_S \in \Delta(G_S, \zeta_S) \), where the semisimple part is given by a semisimple stable conjugacy class \( d_S \in \Delta_{ss}(G_S^\ast) \) in \( G_S^\ast = G_S^\ast/Z_S \), such that the connected centralizer \( G_{d_S}^\ast \) is quasisplit, and the unipotent part is given by an element \( \hat{\beta}_S \) in the subset \( \Delta_{\text{unip}}(G_{d_S}^\ast, \zeta_S) \) in \( \Delta(G_{d_S}^\ast, \zeta_S) \) with trivial semisimple part. The distribution \( \hat{\beta}_S \) is again determined by \( \delta_S \) only up to the action of the finite group
\[
\tilde{G}_{d_S,+}(F_S)/\tilde{G}_{d_S}(F_S) = \prod_{v \in S} \tilde{G}_{d_v,+}(F_v)/\tilde{G}_{d_v}(F_v)
\]
in \( \Delta_{\text{unip}}(G_{d_S}, \zeta_S) \).

Suppose there is an element \( d \) in the set \( \Delta_{ss}(G^\ast) \) of semisimple stable conjugacy classes in \( G^\ast(F) \) whose image in \( \Delta_{ss}(G_S^\ast) \) is \( d_S \). We then define \( i^G(S, d) \) to be equal to 1 if \( d \) is an \( F \)-elliptic element in \( G \) and bounded at each place \( v \not\in S \), and zero otherwise. We also define
\[
\tau(G^\ast)\tau(G_S^\ast)^{-1},
\]
where \( G^\ast_S \) is a quasisplit connected centralizer of an appropriate representative of the class \( d \), and \( \tau(G^\ast) \) is the absolute Tamagawa number of \( G^\ast \).

6.2. Descent. We shall first prove a descent formula for the endoscopic and stable geometric coefficients in the following special case where the derived group \( G_{\text{der}} \) of \( G \) is simply connected and \( Z \) is trivial, from which we will deduce the general result. We note that the first condition implies that \( G_{c,+} = G_c \) for any \( c \in \Gamma_{ss}(G) \), and the second implies that \( \tilde{G} = G \). The formula \( (6.1) \) then reduces to
\[
d_{r,s,\text{ell}}(\gamma_S) = \sum_{c} i^G(S, c) d_{r,s,\text{ell}}(\hat{\alpha}_S).
\]
Then we have the following descent formula.

**Proposition 6.2.** Assume that \( G_{\text{der}} \) is simply connected and that \( Z = 1 \).

(a) Suppose that \( \gamma_S \) is an admissible element in \( \Gamma_{\text{ell}}(G, S) \) with Jordan decomposition \( \gamma_S = c S \hat{\alpha}_S \). Then
\[
d_{r,s,\text{ell}}(\gamma_S) = \sum_{c} i^G(S, c) d_{r,s,\text{ell}}(\hat{\alpha}_S)
\]

(6.3)
where c runs over elements in $\Gamma_{unip}(G)$ that map to $c_S$, and $\hat{\alpha}$ is the image of $\alpha_S$ in $\Gamma_{unip}(G,c_S)$.

(b) Suppose that $G$ is quasisplit, and that $\delta_S$ is an admissible element in $\Delta_{ell}^G(G,S)$ with Jordan decomposition $\delta_S = d_S \hat{\beta}_S$. Then

$$b_{r,s,ell}(\delta_S) = \sum_d f^{G^*}(S,d) b_{r,s,ell}^G(\hat{\beta})$$

where $d$ runs over elements in $\Delta_{ss}(G^*)$ that map to $d_S$ in $\Delta_{ss}(G^*_S)$, and $\hat{\beta}$ is the image of $\beta_S$ in $\Gamma_{unip}(G^*_d,S)$. Moreover, $b_{r,s,ell}^G$ vanishes on the complement of $\Delta_{ell}(G,S)$ in the set of admissible elements in $\Delta_{ell}^G(G,S)$ whose semisimple part is not central in $G_S$.

Before we prove the proposition, we state a corollary that we shall require from our induction assumption. It is the main result of this section.

**Corollary 6.3.** Suppose that Proposition 6.2 holds for some $z$-extension $\tilde{G}$ of $G$.

(a) If $G$ is arbitrary, and $\tilde{\gamma}_S \in \Gamma_{ell}^{G^*}(G,S,\zeta)$ is an admissible element with Jordan decomposition $\tilde{\gamma}_S = c_S \tilde{\alpha}_S$. Then

$$a_{r,s,ell}^{G,\tilde{\gamma}}(\tilde{\gamma}_S) = \sum_c \sum_\alpha f^G(S,c) |\tilde{G}_{c,+}(F)/\tilde{G}_{c}(F)|^{-1} a_{r,s,ell}^{G,\tilde{\gamma}}(\tilde{\alpha}),$$

where the sums are taken as in (6.1).

(b) If $G$ is quasisplit, and $\delta_S \in \Delta_{ell}^{G^*}(G,S,\zeta)$ is an admissible element with Jordan decomposition $\delta_S = d_S \hat{\beta}_S$. Then

$$b_{r,s,ell}^{G}(\delta_S) = \sum_d \sum_{\tilde{\beta}} f^{G^*}(S,d) |(\tilde{G}_{d,+}/\tilde{G}^*_d)(F)|^{-1} b_{r,s,ell}^{G}(\tilde{\beta}),$$

where $d$ runs over elements in $\Delta_{ss}(G^*)$ that map to $d_S$, and $\tilde{\beta}$ runs over the orbit of $(\tilde{G}_{d,+}/\tilde{G}^*_d)(F)$ in $\Delta_{unip}(G,d,S,\zeta)$ determined by $\tilde{\beta}_S$.

We begin with the proof of Proposition 6.2.

**Proof.** We shall take on the induction hypothesis that Corollary 4.6 holds if $G$ is replaced by any group $H$ over $F$ such that either $\dim(H_{der}) < \dim(G_{der})$, or $\varepsilon(G) = 0$ and $H = G^*$. Our argument will follow the proof of [Art01, Theorem 1.1], where we observe that the major combinatorial problems have already been solved. Since $Z$ is trivial, we can take $G' = G'$ for any $G' \in \mathcal{E}(G)$. From the definition (4.5), it follows that the difference

$$a_{r,s,ell}^{G,\tilde{\gamma}}(\tilde{\gamma}_S) - \varepsilon(G) \sum_{\delta_S \in \Delta_{ell}^G(G,S)} b_{r,s,ell}^{G}(\delta_S) \Delta_G(\delta_S,\tilde{\gamma}_S)$$

equals

$$\sum_{G' \in \mathcal{E}_{ell}(G,S)} \iota(G,G') \sum_{\tilde{\delta}_S} b_{r,s,ell}^{G'}(\tilde{\delta}_S) \Delta_G(\tilde{\delta}_S,\tilde{\gamma}_S).$$

We also assume inductively that any $G' \in \mathcal{E}_{ell}(G,S)$ has a $z$-extension for which Proposition 6.2 and hence Corollary 6.3 holds for $G'$. Suppose $\tilde{\delta}_S = d_S \tilde{\beta}_S \in$
\[ \Delta_{\text{all}}(G', S) \text{ such that } \Delta(\delta^*_S, \gamma) \neq 0, \text{ so that } \delta^*_S \text{ is admissible. Then applying } (6.5) \text{ we have} \]

\[ b^{G'}_{r, s, \text{ell}}(\delta^*_S) = \sum_{d'} \sum_{\beta'} j^{G'}(S, d')(G_{d', +}/G_{d'})(F)|^{-1}b^{G^*}_{r, s, \text{ell}}(\beta'). \]

Substituting this into (6.7), and using the identities \[ \Delta_{G}(\delta^*_S, \gamma_S) = \Delta_{G}(d_S^*\beta_S^*, c_S\tilde{\alpha}_S) \]
and

\[ \nu(G, G')j^{G'}(S, d') = \nu(G^* (S, d)\tau(G)(G_{d'})^{-1}|\Out_{G}(G')|^{-1} \]

by (6.2) and [Kottwitz] Theorem 8.3.1, we can write (6.7) as

\[ \sum_{G'} \sum_{d'} \sum_{\beta'} |\Out_{G}(G')|^{-1} \tau(G)\tau(G_{d'})^{-1}|(G_{d', +}/G_{d'})(F)|^{-1}b^{G^*}_{r, s, \text{ell}}(\beta') \Delta_{G}(d_S^*\beta_S^*, c_S\tilde{\alpha}_S) \]

where \( G' \in \mathcal{E}^{0}_{\text{ell}}(G, S), d' \) runs over classes in \( \Delta_{\text{ass}}(G') \) that are \( F \)-elliptic and bounded at each \( v \notin S \), and \( \beta' \in \Delta_{\text{unip}}(G_{d', S}). \)

Let us assume that \( d \in \Delta_{\text{ass}}(G^*) \) is elliptic, bounded at each \( v \notin S \), and a local image of each component \( c_S \) of \( c_S \). If such a \( d \) does not exist, it follows from the argument at the end of [Artin] p.212 that the required result is trivial. We can also assume that \( d \) does not lie in the center of \( G^* \). We add to (6.8) the sum

\[ \varepsilon(G) \sum_{\beta \in \Delta_{\text{unip}}(G^*)} \tau(G^*)\tau(G_{d'})^{-1}b^{G^*}_{r, s, \text{ell}}(\beta) \Delta_{G}(d_S^*\beta_S^*, c_S\tilde{\alpha}_S), \]

which in the case \( \varepsilon(G) = 1 \), combines to make the outer sum of (6.8) into a full sum over \( \mathcal{E}_{\text{ell}}(G, S) \).

Let us write \( c_{\beta} \) for the product of \( c_S \) with \( c_v \in K_v \) for each \( v \notin S \) for which \( d_v \) is an image. Also recall that if \( G_d^* \) belongs to \( \mathcal{E}_{\text{ell}}(G^*_d, S) \), it has a canonical extension \( \tilde{G}_d^* \) by \( \check{C}_d^* \) determined by \( \check{G}_d^* \). We shall fix \( \check{\eta}_d^* \) to be an automorphic character of \( \check{C}_d^* \).

Examining the arguments of [Artin] pp.214–218], we can hence conclude that the sum of (6.7) and (6.9) is equal to

\[ \sum_{c} \sum_{G_d^* \in \mathcal{E}_{\text{ell}}(G^*_d, S)} \nu(G_c, G_d^*) \sum_{\delta_{d, S}^*} \tilde{b}^{G^*}_{\text{r, s, ell}}(\delta_{d, S}^*) \Delta_{G_c}(\check{\delta}_{d, S}^*, \check{\alpha}), \]

where \( c \) runs over classes in \( \Gamma_{\text{ass}}(G) \) that map to the \( G(\mathbb{A}) \)-conjugacy class of \( c_{\beta} \), and \( \delta_{d, S}^* \in \Delta_{\text{ell}}(G_{d^*}, S) \), \( \check{\eta}_d^* \), and \( \Delta_{G_c}(\check{\delta}_{d, S}^*, \check{\alpha}) \) is the canonical global transfer factor for \( G_c \) and \( G^*_d \) defined in p.214 and Lemma 4.2 of [Artin]. Then using the definition (4.5), the induction hypothesis, and the assumption that \( d \) is not central, it follows that the sum is equal to

\[ \sum_{c} \mathcal{a}^{G^*}_{\text{ell}}(\check{\delta}). \]

Moreover, any \( c \) that occurs in the sum is \( F \)-elliptic in \( G \) and is \( G(\mathbb{A}^S) \)-conjugate to an element in \( K^S \), so we conclude that (6.6) is equal to

\[ \sum_{c} \varepsilon^G(S, c)\mathcal{a}^{G^*}_{\text{ell}}(\check{\delta}). \]

where now \( c \) runs over the set in (6.3). Hence if \( \varepsilon(G) = 0 \), the first required identity (6.3) follows. On the other hand, suppose \( \varepsilon(G) = 1 \). Then \( \mathcal{a}^{G^*}_{\text{ell}}(\check{\delta}) = \mathcal{a}^{G^*}_{\text{ell}}(\check{\delta} \gamma) \) by definition. Any \( c \) occurring in the sum in (6.11) is not central, so \( \dim(G_{c, \text{der}}) < \)
\[ \dim(G_{\text{der}}), \text{so it follows from the induction hypothesis that } a_{\text{ell}}^{G_{\text{der}}} (\alpha) = a_{\text{ell}}^{G} (\alpha), \text{and the required identity follows from (6.1), proving part (a).} \]

For part (b), we continue to assume that \( \varepsilon(G) = 1 \). Then since the sum of \( (6.7) \) and \( (6.9) \) equals \( (6.11) \), it follows from part (a) that the coefficient of \( \varepsilon(G) \) in the latter sum is equal to zero. Moreover, since the contribution to \( (6.9) \) is nonzero only if \( d \) is an \( F \)-elliptic element in \( G \) and bounded at each place \( v \not\in S \), we can replace \( \tau(G^{\ast})\tau(G_{d}^{\ast})^{-1} \) on the lefthand side with \( j^{G_{d}^{\ast}}(S, d) \). It follows then that

\[
\sum_{\delta_S \in \Delta_{d}(G, S)} b_{r,s,\text{ell}}^G (\delta_S) \Delta_G (\delta_S, \gamma_S)
\]

is equal to

\[
\sum_{\delta_S \in \Delta_{\text{unip}}(G_{d,S}^{\ast})} j^{G_{d}^{\ast}}(S, d) b_{r,s,\text{ell}}^G (\delta_S) \Delta_G (d_S \delta_S, \gamma_S).
\]

Then using the adjoint relation in \([\text{Art02}, (5.5)]\) we may multiply by the adjoint transfer factor and sum over \( \Gamma_{\text{cl}}(G, S, \zeta) \) to invert these expressions. It follows that \( b_{r,s,\text{ell}}^G (\delta_S) \) vanishes unless it has a Jordan decomposition with \( \hat{\beta} \in \Delta_{\text{unip}}(G_{d,S}^{\ast}), \) in which case

\[
b_{r,s,\text{ell}}^G (\delta_S) = j^{G_{d}^{\ast}}(S, d) b_{r,s,\text{ell}}^G (\hat{\beta}),
\]

where \( \hat{\beta} \) is the image of \( \delta_S \) in \( \Delta_{\text{unip}}(G_{d,S}^{\ast}), \) which proves part (b). \( \square \)

Consider the commutator quotient \( G_{S}^{b} = G_{S}/G_{\text{der}, S} \). It acts by conjugation on \( \mathcal{D}_{\text{unip}}(G_{\text{der}, S}) \). We define a linear map from \( \mathcal{D}_{\text{unip}}(G_{\text{der}, S}) \) to \( \mathcal{D}_{\text{unip}}(G_{S}, \zeta_S) \) by sending any \( D \in \mathcal{D}_{\text{unip}}(G_{\text{der}, S}) \) to the distribution

\[
f \rightarrow \sum_{a \in G_{S}^{b}} (aD)(f|_{G_{\text{der}, S}}), \quad f \in \mathcal{C}^{\infty}(G_{S}, \zeta_S),
\]

where \( f|_{G_{\text{der}}} \) denotes the restriction of \( f \) to \( G_{\text{der}, S} \). We shall write \( \mathcal{D}_{\text{unip, der}}(H_{S}, \zeta_S) \) for the image of this map, and assume that the basis \( \Gamma_{\text{unip}}(G_{S}, \zeta_S) \) has been chosen such that the intersection

\[
\Gamma_{\text{unip, der}}(G_{S}, \zeta_S) = \Gamma_{\text{unip}}(G_{S}, \zeta_S) \cap \mathcal{D}_{\text{unip, der}}(G_{S}, \zeta_S)
\]

is a basis of \( \Gamma_{\text{unip, der}}(G_{S}, \zeta_S) \). Similarly, we construct stable subsets \( \mathcal{D}_{\text{unip, der}}(G_{S}, \zeta_S) \) with the basis \( \Delta_{\text{unip, der}}(G_{S}, \zeta_S) \). The unipotent global coefficients \( a_{r,s,\text{ell}}^G (\alpha) \) and \( a_{r,s,\text{ell}}^G (\hat{\beta}) \) will be supported on the subset \( \Gamma_{\text{unip, der}}(G_{S}, \zeta_S) \), and \( b_{r,s,\text{ell}}^G (\delta_S) \) on the subset \( \Delta_{\text{unip, der}}(G_{S}, \zeta_S) \) respectively. \( ^4 \)

We note that there is a canonical isomorphism between \( \mathcal{D}_{\text{unip, der}}(G_{S}, \zeta_S) \) and the space \( \mathcal{D}_{\text{unip, der}}(G_{S}) \) with trivial central data given by sending any \( D \in \mathcal{D}_{\text{unip, der}}(G_{S}, \zeta_S) \) to the linear form

\[
f \rightarrow D(f_{c}), \quad f \in \mathcal{C}^{\infty}(G_{S}, \zeta_S),
\]

where \( f_{c} \) is any function in \( C^{\infty}_{c}(G_{S}) \) that equals \( f \) on an invariant neighbourhood of \( 1 \). We now proceed to extend the preceding proposition to the general case.

**Proposition 6.4.** Suppose that Proposition 6.2 holds for some \( z \)-extension \( \tilde{G} \) of \( G \).

\( ^4 \)The term \( a_{\text{disc}}^{H} \) on [Art01, p.169] should be \( a_{\text{ell}}^{H} \).
(a) If $G$ is arbitrary, and $\gamma_S \in \Gamma_{\text{ell}}(G, S, \zeta)$ is an admissible element whose semisimple part is not central in $G_S$. Then

$$a_{r,s,\text{ell}}^G(\gamma_S) = a_{r,s,\text{ell}}^G(\tilde{\gamma}_S).$$

(b) If $G$ is quasisplit, and $\delta_S \in \Delta_{\text{ell}}(G, S, \zeta)$ is an admissible element whose semisimple part is not central in $G^*_S$. Then

$$b_{r,s,\text{ell}}^G(\tilde{\delta}_S)$$

is supported on the subset $\Delta_{\text{ell}}(G, S, \zeta)$ of $\Delta_{\text{ell}}(G, S, \zeta)$.

**Proof.** We recall that as a $z$-extension, $\tilde{G}$ is a central extension of $G$ by a central induced torus $\tilde{C}$ over $F$ such that $\tilde{G}_\text{der}$ is simply connected. We assume that $S$ is chosen such that $\tilde{G}$ and $\tilde{\zeta}$ are unramified away from $S$. Since $G_S \simeq \tilde{G}_S/\tilde{C}_S$, we therefore have a canonical isomorphism from $\mathcal{C}^0(G, S, \zeta)$ to $\mathcal{C}(G, S, \tilde{\zeta})$. We shall also assume that the bases $\Gamma(\tilde{G}, \tilde{\zeta})$, $\Delta^\mathcal{C}(\tilde{G}, \tilde{\zeta})$, and $\Delta(\tilde{G}, \tilde{\zeta})$ for $\tilde{G}$ are the images of the corresponding bases for $G$ under the canonical maps $\gamma_S \to \tilde{\gamma}_S$ and $\delta_S \to \tilde{\delta}_S$.

Let us first show that the result is valid for the pair $(G, \tilde{\zeta})$ if it is valid for the pair $(\tilde{G}, \tilde{\zeta})$. We have to show that

$$a_{r,s,\text{ell}}^G(\gamma_S) = a_{r,s,\text{ell}}^G(\tilde{\gamma}_S)$$

for any admissible element $\gamma_S \in \Gamma(G, S, \zeta)$. To do so, we consider the expansion

$$J_M^*(\tilde{f}_S) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma_S \in \Gamma_{\text{ell}}(M, S, \zeta)} a_{r,s,\text{ell}}^M(\gamma_S)J_M(\gamma_S, \tilde{f}_S)$$

in [Lan22 (4.10)], that is valid for any $\tilde{f}_S \in \mathcal{C}_{\text{adm}}^0(G, S, \zeta)$. From the map of functions $\tilde{f}_S \to f_S$ induced by the isomorphism $G_S \simeq \tilde{G}_S/\tilde{C}_S$, it follows that $J_M^*(\tilde{f}_S)$ is equal to the distribution $J_M^*(\tilde{f}_S)$ on $\tilde{G}$, and moreover the weighted orbital integrals satisfy

$$J_M^*(\gamma, \tilde{f}) = J_M^\tilde{G}(\tilde{\gamma}, \tilde{f}), \quad \gamma_S \in \Gamma(G, S, \zeta).$$

Assuming inductively that (6.13) holds if $G$ is replaced by any proper Levi subgroup $M$ of $G$, it follows that the terms with $M \neq G$ in the expansions for $J_M^*(\tilde{f}_S)$ and $J_M^*(\tilde{f}_S)$ are equal, and by varying $\tilde{f}_S$ we conclude that (6.13) holds.

Since $\tilde{G}_\text{der}$ is simply connected, there is an $L$-embedding $\tilde{G}' \to \tilde{G}$ which we can assume is unramified outside of $S$ by [Art02, Lemma 7.1]. The composition of $L$-embeddings $\mathcal{C}' \to L\tilde{G} \to L\tilde{G}$ maps $\mathcal{C}'$ into the image of $\ell(\tilde{G})$ in $L\tilde{G}$, giving an $L$-emebdding $\xi' : \mathcal{C}' \to L\tilde{G}'$ which makes up an auxiliary datum $(\tilde{G}', \xi')$ for $\mathcal{C}'$. We thus have a bijection from $\mathcal{C}_{\text{ell}}(G, S)$ to $\mathcal{C}_{\text{ell}}(\tilde{G}, S)$, together with an equality of transfer maps

$$f'_S = (\tilde{f}_S)', \quad \tilde{f}_S \in \mathcal{C}_{\text{adm}}^0(G, S, \zeta)$$

by [LS77, §4.2]. Then if we consider the elliptic parts (4.12), it follows that the difference

$$I_{r,s,\text{ell}}^G(\tilde{f}_S) - \varepsilon(G)S_{r,s,\text{ell}}^G(\tilde{f}_S) = \sum_{G' \in \mathcal{E}_{\text{ell}}^0(G, S)} \iota(G, G')S_{r,s,\text{ell}}^{G'}(f'_S)$$

equals

$$I_{r,s,\text{ell}}^G(\tilde{f}_S) - \varepsilon(G)S_{r,s,\text{ell}}^G(\tilde{f}_S) = \sum_{\tilde{G}' \in \mathcal{E}_{\text{ell}}^0(\tilde{G}, S)} \iota(\tilde{G}, \tilde{G}')S_{r,s,\text{ell}}^{\tilde{G}'}(\tilde{f}_S).$$
where we have used the fact that \( \iota(G, G') = \iota(\hat{G}, \hat{G}') \) [Art01 (2.2)]. On the other hand, it follows from [6.13] that the elliptic part [4.9] satisfies
\[
I_{r,s,\text{ell}}(\hat{f}^1_S) = I_{r,s,\text{ell}}(\hat{\tilde{f}}_S).
\]
Hence if \( \varepsilon(G) = 1 \), we conclude from [4.14] and [4.16] that
\[
I_{r,s,\text{ell}}^\varepsilon(\hat{f}^1_S) = I_{r,s,\text{ell}}^\varepsilon(\hat{\tilde{f}}_S), \quad S_G^{\ell,\text{ad}}(\hat{f}^1_S) = S_{r,s,\text{ell}}^{\ell,\text{ad}}(\hat{\tilde{f}}_S).
\]
The general induction hypothesis concerning Corollary [6.16] then allows us to compare the expansions of each one to deduce that
\[
a_{r,s,\text{ell}}^G,\varepsilon(\gamma_S) = a_{r,s,\text{ell}}^G,\varepsilon(\tilde{\gamma}_S)
\]
in general for any admissible element \( \gamma_S \in \Gamma_{\text{ell}}^G(G, S, \zeta) \), and
\[
(6.14) \quad b_{r,s,\text{ell}}^G(\delta_S) = b_{r,s,\text{ell}}^G(\tilde{\delta}_S)
\]
if \( G \) is quasisplit for any admissible element \( \delta_S \in \Delta^\varepsilon_{\text{ell}}(G, S, \zeta) \). It follows then that the proposition is valid for the pair \((G, \zeta)\) if it is valid for the pair \((\hat{G}, \tilde{\zeta})\). This reduces the result to the case where \( G = \hat{G} \).

We next want to show that the result is valid for \( G \) with central datum \((Z, \zeta)\) if it holds for \((Z, \zeta)\) trivial. The map [2.1] provides a projection from the subspace of admissible functions \( \mathcal{C}_{\text{adm}}^\circ(G, S) \) onto \( \mathcal{C}^\circ(G, S, \zeta) \). We then have
\[
I_{r,s,\text{ell}}(\hat{f}^1_S) = \int_{Z_{S,1}} \iota_{r,s,\text{ell}}(\hat{\tilde{f}}_S) \zeta(z) dz, \quad \hat{f}^1_S \in \mathcal{C}_{\text{adm}}^\circ(G, S, \zeta)
\]
where \( \hat{\tilde{f}}_S \) is any function in \( \mathcal{C}_{\text{adm}}(G, S) \) that maps to \( \hat{f}^1_S \), and \( \hat{\tilde{f}}_S \) is its translate by \( z \). Moreover, the map [2.1] commutes with endoscopic transfer by [Won22 Lemma 6.1]. To see this, observe that \((\hat{f}^1_S)\)' lies in \( S\mathcal{C}^\circ(G', S, \tilde{\eta}') \), where \( \tilde{\eta}' \) is the automorphic character of the central induced torus \( \hat{G}' \) defining the \( z \)-extension \( \hat{G}' \), and [2.1] induces a map from \((\hat{f}^1_S)'\) to a function in \( S\mathcal{C}^\circ(G', S, \hat{\zeta}') \) where \( \hat{\zeta}' = \tilde{\zeta}' \). It follows then from [3.2] and [2.12], and [LS87] Lemma 4.4.A that
\[
(\hat{\tilde{f}}_S)' = (\hat{f}^1_S)' \tilde{\eta}'(z), \quad z \in Z_{S,1},
\]
and integrating against \( \zeta(z) \), the claim follows. Then as before, we have that
\[
I_{r,s,\text{ell}}^\varepsilon(\hat{f}^1_{S,2}) - \varepsilon(G) S_{r,s,\text{ell}}^G(\hat{\tilde{f}}_S) = \sum_{G' \in \mathcal{C}_{\text{ell}}^0(G, S)} \iota(G, G') S_{r,s,\text{ell}}^{G'}((\hat{f}^1_S)' \tilde{\eta}'(z))
\]
for any \( z \in Z_{S,1} \). Assume inductively that for any \( G' \in \mathcal{C}_{\text{ell}}^0(G, S) \), the integrand in
\[
\int_{Z_{S,1}} \hat{S}_{r,s,\text{ell}}^{G'}((\hat{f}^1_S)' \tilde{\eta}'(z)) dz
\]
is invariant under translation by \( Z_{S,1} \), and that the integral is equal to \( \hat{\tilde{S}}_{r,s,\text{ell}}^{G'}(\hat{\tilde{f}}_S) \). It follows then that the integral
\[
\int_{Z_{S,1}} (I_{r,s,\text{ell}}^\varepsilon(\hat{\tilde{f}}_S) - \varepsilon(G) S_{r,s,\text{ell}}^{G'}(\hat{f}^1_S)) \zeta(z) dz
\]
equals
\[
\sum_{G' \in \mathcal{C}_{\text{ell}}^0(G, S)} \iota(G, G') S_{r,s,\text{ell}}^{G'}(\hat{f}^1_S),
\]
which in turn is equal to $I_{r,s,\text{ell}}^G(\hat{f}_S) - \varepsilon(G)S_{r,s,\text{ell}}^G(\hat{f}_S)$. Then arguing as before, we conclude that

$$I_{r,s,\text{ell}}^G(\hat{f}_S) = \int_{Z_{r,s} \setminus Z^G_{r,s}} I_{r,s,\text{ell}}^G(\hat{f}^G_{S,z})\zeta(z)dz,$$

in general, and

$$S_{r,s,\text{ell}}^G(\hat{f}_S) = \int_{Z_{r,s} \setminus Z^G_{r,s}} S_{r,s,\text{ell}}^G(\hat{f}^G_{S,z})\zeta(z)dz,$$

if $G$ is quasisplit. Finally, choose $\hat{f}_S \in \mathcal{C}_{\text{adm}}(G,S,\zeta)$ such that $\hat{f}_{S,G}(\hat{g})$ vanishes for any $\hat{g} \in \Gamma(G_S,\zeta)$ whose semisimple part is not central, and moreover such that $\hat{f}_{S,G}^\gamma = 0$ if $G$ is quasisplit. We choose the associated function $\hat{f}^G_{S,z}$ similarly. Then assuming that the proposition holds for $(Z,\zeta)$ trivial, it follows that $I_{r,s,\text{ell}}^G(\hat{f}^G_{S,z}) = I_{r,s,\text{ell}}(\hat{f}^G_{S,z})$ and $S_{r,s,\text{ell}}^G(\hat{f}_S) = 0$. The expansions (6.14) and (6.16) then imply that the desired results holds for admissible elements $\hat{g}_S \in \Gamma^G_{\text{ell}}(G,S,\zeta)$ and $\hat{\delta}_S \in \Delta^G_{\text{ell}}(G,S,\zeta)$ whose semisimple parts are not central, and hence for arbitrary $(G,\zeta)$.

We can now complete the extension to the general case, which is Corollary 6.3.

**Proof of Corollary 6.3.** We shall prove (b), and the proof of (a) will be similar to it. We assume therefore that $G$ is quasisplit. We claim that (6.5) holds for $(G,\zeta)$ if it holds for $(G,\zeta)$. By (6.14) it follows that the right hand side of (6.5) for $(G,\zeta)$ is equal to the corresponding expression for $(\hat{G},\hat{\zeta})$. Then since $\hat{G} = G$, the sums over $d$ of the two expansions run over the same set, and there is a canonical bijection from $\hat{\beta}$ to $\beta$ where by (6.14) again we have

$$b^G_{r,s,\text{ell}}(\hat{\beta}_S) = b^G_{r,s,\text{ell}}(\beta_S),$$

whence the claim follows. It suffices then to treat the case $\hat{G} = G$.

We next show that if $G_{\text{der}}$ is simply connected, and if (6.5) holds for $(Z,\zeta)$ trivial, then it holds for arbitrary $(Z,\zeta)$ as well. Let $\hat{f}_S \in \mathcal{C}_{\text{adm}}(G,S,\zeta)$ such that $\hat{f}_{S,G}^\gamma$ is supported on $\Delta(G_S,\zeta)$. Then (6.16) can be written as

$$S_{r,s,\text{ell}}^G(\hat{f}_S) = \sum_{\hat{\delta}_S \in \Delta_{\text{ell}}(G,S,\zeta)} b^G_{r,s,\text{ell}}(\hat{\delta}_S)\hat{f}_{S,G}(\hat{\delta}_S)$$

and from (6.15) we have also

$$S_{r,s,\text{ell}}^G(\hat{f}_S) = \int_{Z_{r,s} \setminus Z^G_{r,s}} \sum_{\hat{\delta}_S \in \Delta_{\text{ell}}(G,S)} b^G_{r,s,\text{ell}}(\hat{\delta}_S)\hat{f}_{S,G}(z\hat{\delta}_S)\zeta(z)dz.$$

We are assuming the coefficients $b^G_{r,s,\text{ell}}(\hat{\delta}_S)$ satisfy (6.5), which reduce to the (6.4). Substituting, we obtain

$$\int_{Z_{r,s} \setminus Z^G_{r,s}} \sum_{\hat{\beta}} \sum_d j^{G^*}(S,d^1)b^G_{r,s,\text{ell}}(\hat{\beta}_S)\hat{f}_{S,G}^\gamma(zd^1\hat{\beta}_S)\zeta(z)dz$$

where the sum over $d^1$ runs over classes in $\Delta_{\text{ss}}(G^*)$ that are bounded at each $v \not\in S$, and $\hat{\beta}_S$ runs over $\Delta_{\text{unip}}(G_{d^1}.S)$.

Our goal then is to compare the expansions (6.5) and (6.16). First, recall that $b^G_{r,s,\text{ell}}(\hat{\beta}_S)$ is supported on the subset $\Delta_{\text{unip,der}}(G_{d^1,S})$ of $\Delta_{\text{unip}}(G_{d^1,S})$, which we
can choose to be in bijection with \( \Delta_{\text{unip,der}}(G_{d^1}, S, \zeta_S) \) by the map (6.12). Second, observe that there is a surjection \( d^1 \to d \) from \( \Delta_{ss}(G^*) \) to \( \Delta_{ss}(G^*) \) of classes that are bounded away from \( S \). Then by [Art02, p.177], the coefficients \( j^{G^*}(S, d^1) \) and \( j^{G^*}(S, d) \) are equal. Fourth, assume inductively that if \( d^1 \) is not central in \( G \), then

\[
b_{r, s, \text{ell}}(\hat{\beta}^1) = b_{r, s, \text{ell}}(\hat{\beta}).
\]

Third, by [Art02, p.177], it is enough to assume that the analogue holds for some \( z \)-extension of the group \( G^*_{d^1} = G^*_{d} \). Finally, we note that

\[
\int_{Z_S^1} \hat{f}(zx) \zeta(z) dz = \hat{f}(x), \quad x \in G_S^1,
\]

and assume first that \( \hat{f} \) vanishes on an invariant neighborhood of the center of \( G_S \), then \( \hat{f}^{G^*}(z \hat{\beta}^1) \) vanishes if \( d^1 \) is central in \( G \), then by varying \( \hat{f} \) appropriately it follows that (6.5) holds for any \( \hat{\delta} \) whose semisimple part is not central. Next, removing the vanishing condition on \( \hat{f} \), by a comparison of terms we conclude that (6.5) also holds for \( \hat{\delta} \) whose semisimple part is central, and moreover, that

\[
b_{r, s, \text{ell}}(\hat{\beta}^1) = b_{r, s, \text{ell}}(\hat{\beta})
\]

holds for any \( \hat{\beta} \) in the subset \( \Delta_{\text{unip,der}}(G^*_{d^1}, S, \zeta_S) \) of \( \Delta_{\text{unip}}(G^*_{d^1}, S, \zeta_S) \) on which \( b_{r, s, \text{ell}}^{G^*} \) is supported. This completes the induction argument. \( \square \)

6.3. The unipotent part. We now consider the unipotent analogues of the elliptic part of the trace formula. We shall introduce objects that will be parallel to those considered in 6.1. Let \( \Gamma_{\text{unip}}(G, V, \zeta) \) denote the subset of classes in \( \Gamma_{\text{ell}}(G, V, \zeta) \) whose semisimple parts are trivial, and similarly we let \( \mathcal{K}_{\text{unip}}'(G, S) \) be the corresponding unipotent subset of \( \mathcal{K}_{\text{ell}}'(G, S) \). More generally, we shall let the subscript ‘unip’ denote the unipotent variant of the elliptic objects analogously. We then define the unipotent part of the invariant trace formula

\[
I_{r, s, \text{unip}}(f, S) = \sum_{\alpha \in \Gamma_{\text{unip}}(G, V, \zeta)} a_{r, s, \text{unip}}^G(\alpha, S)f_G(\alpha)
\]

with unipotent coefficients

\[
a_{r, s, \text{unip}}^G(\alpha, S) = \sum_{k \in \mathcal{K}_{\text{unip}}'(G, S)} a_{r, s, \text{unip}}^G(\alpha \times k)r_G(k, b), \quad \alpha \in \Gamma_{\text{unip}}(G, V, \zeta).
\]

We shall also define endoscopic and stable analogues inductively by the formula

\[
I_{r, s, \text{unip}}^E(f, S) = \sum_{G' \in \mathcal{E}_{\text{unip}}(G)} \iota(G, G')S_{r, s, \text{unip}}^{G'}(f', S) + \varepsilon(G)S_{r, s, \text{unip}}^G(f, S)
\]

with the requirement that \( I_{r, s, \text{unip}}^E(f, S) = I_{r, s, \text{unip}}(f, S) \) in the case that \( G \) is quasisplit. Using the natural variation of the argument [Art02, Lemma 7.2] we obtain expansions

\[
I_{r, s, \text{unip}}^E(f) = \sum_{\alpha \in \Gamma_{\text{unip}}^E(G, V, \zeta)} a_{r, s, \text{unip}}^{G, E}(\alpha)f_G(\alpha),
\]

where \( \Gamma_{\text{unip}}^E(G, V, \zeta) \) denotes the subset of classes in \( \Gamma_{\text{unip}}(G, V, \zeta) \) whose semisimple part is central, and moreover, that

\[
b_{r, s, \text{ell}}(\hat{\beta}^1) = b_{r, s, \text{ell}}(\hat{\beta})
\]

holds for any \( \hat{\beta} \) in the subset \( \Delta_{\text{unip,der}}(G^*_{d^1}, S, \zeta_S) \) of \( \Delta_{\text{unip}}(G^*_{d^1}, S, \zeta_S) \) on which \( b_{r, s, \text{ell}}^{G^*} \) is supported. This completes the induction argument. \( \square \)

6.3. The unipotent part. We now consider the unipotent analogues of the elliptic part of the trace formula. We shall introduce objects that will be parallel to those considered in 6.1. Let \( \Gamma_{\text{unip}}(G, V, \zeta) \) denote the subset of classes in \( \Gamma_{\text{ell}}(G, V, \zeta) \) whose semisimple parts are trivial, and similarly we let \( \mathcal{K}_{\text{unip}}'(G, S) \) be the corresponding unipotent subset of \( \mathcal{K}_{\text{ell}}'(G, S) \). More generally, we shall let the subscript ‘unip’ denote the unipotent variant of the elliptic objects analogously. We then define the unipotent part of the invariant trace formula

\[
I_{r, s, \text{unip}}(f, S) = \sum_{\alpha \in \Gamma_{\text{unip}}(G, V, \zeta)} a_{r, s, \text{unip}}^G(\alpha, S)f_G(\alpha)
\]

with unipotent coefficients

\[
a_{r, s, \text{unip}}^G(\alpha, S) = \sum_{k \in \mathcal{K}_{\text{unip}}'(G, S)} a_{r, s, \text{unip}}^G(\alpha \times k)r_G(k, b), \quad \alpha \in \Gamma_{\text{unip}}(G, V, \zeta).
\]

We shall also define endoscopic and stable analogues inductively by the formula

\[
I_{r, s, \text{unip}}^E(f, S) = \sum_{G' \in \mathcal{E}_{\text{unip}}(G)} \iota(G, G')S_{r, s, \text{unip}}^{G'}(f', S) + \varepsilon(G)S_{r, s, \text{unip}}^G(f, S)
\]

with the requirement that \( I_{r, s, \text{unip}}^E(f, S) = I_{r, s, \text{unip}}(f, S) \) in the case that \( G \) is quasisplit. Using the natural variation of the argument [Art02, Lemma 7.2] we obtain expansions

\[
I_{r, s, \text{unip}}^E(f) = \sum_{\alpha \in \Gamma_{\text{unip}}^E(G, V, \zeta)} a_{r, s, \text{unip}}^{G, E}(\alpha)f_G(\alpha),
\]

where \( \Gamma_{\text{unip}}^E(G, V, \zeta) \) denotes the subset of classes in \( \Gamma_{\text{unip}}(G, V, \zeta) \) whose semisimple part is central, and moreover, that

\[
b_{r, s, \text{ell}}(\hat{\beta}^1) = b_{r, s, \text{ell}}(\hat{\beta})
\]

holds for any \( \hat{\beta} \) in the subset \( \Delta_{\text{unip,der}}(G^*_{d^1}, S, \zeta_S) \) of \( \Delta_{\text{unip}}(G^*_{d^1}, S, \zeta_S) \) on which \( b_{r, s, \text{ell}}^{G^*} \) is supported. This completes the induction argument. \( \square \)
if \( G \) is arbitrary, and
\[
S_{r,s,\text{unip}}^{G}(f) = \sum_{\beta \in \Delta_{\text{unip}}^{\varepsilon}(G,V,\zeta)} b_{r,s,\text{unip}}^{G}(\beta \varepsilon f_{G}^{\beta}),
\]
if \( G \) is quasisplit, where
\[
a_{r,s,\text{unip}}^{G,\varepsilon}(\alpha,S) = \sum_{k \in K_{\text{unip}}^{V,\varepsilon}(G,S)} a_{r,s,\text{unip}}^{G,\varepsilon}(\alpha \times k) r_{G}(k,b), \quad \alpha \in \Gamma_{\text{unip}}^{\varepsilon}(G,V,\zeta)
\]
and
\[
b_{r,s,\text{unip}}^{G}(\beta,S) = \sum_{\ell \in \mathcal{Z}_{\text{unip}}^{V,\varepsilon}(G,S)} b_{r,s,\beta}^{G}(\beta \times \ell) r_{G}(k,b), \quad \beta \in \Delta_{\text{unip}}^{\varepsilon}(G,V,\zeta)
\]
respectively.

Let \( Z(G)_{V,\alpha} \) be the subgroup of elements \( z \) in \( Z(G)(F) \) such that for every \( v \notin V \), \( z_{v} \) is bounded in \( Z(G)(F_{v}) \). It acts discontinuously on \( G_{V} \), and so does the quotient \( Z(G)_{V,\alpha} = Z(G)_{V,\alpha}Z_{V}/Z_{V} \) on \( G_{V} \). For any \( z \in Z(G)_{V,\alpha} \), we set
\[
I_{z,r,s,\text{unip}}(f,S) = I_{r,s,\text{unip}}(f_{z},S),
\]
and similarly
\[
I_{z,r,s,\text{unip}}^{\varepsilon}(f,S) = I_{r,s,\text{unip}}^{\varepsilon}(f_{z},S),
\]
and
\[
S_{z,r,s,\text{unip}}^{G}(f,S) = S_{r,s,\text{unip}}^{G}(f_{z},S).
\]
Also, following Section 9.4, we define \( \mathcal{C}_{\text{uns}}^{\varepsilon}(G_{V},\zeta_{V}) \) be the subset of functions \( f \in \mathcal{C}_{\text{uns}}^{\varepsilon}(G_{V},\zeta_{V}) \) whose stable orbital integrals vanish, that is, \( f_{G}^{\varepsilon} = 0 \). We call such functions unstable.

**Corollary 6.5.**

(a) If \( G \) is arbitrary and \( f \in \mathcal{C}_{\text{uns}}^{\varepsilon}(G_{V},\zeta_{V}) \), we have
\[
I_{r,s,\text{ell}}^{\varepsilon}(f,S) - I_{r,s,\text{ell}}(f,S) = \sum_{z \in Z(G)_{V,\alpha}} (I_{z,r,s,\text{unip}}^{\varepsilon}(f,S) - I_{z,r,s,\text{unip}}(f,S)).
\]

(b) If \( G \) is quasisplit and \( f \in \mathcal{C}_{\text{uns}}^{\varepsilon}(G_{V},\zeta_{V}) \), we have
\[
S_{r,s,\text{ell}}^{G}(f,S) = \sum_{z \in Z(G)_{V,\alpha}} S_{z,r,s,\text{unip}}^{G}(f,S).
\]

**Proof.** We begin with (a). Using the expansions (4.19) and (4.11), it follows that
\[
I_{r,s,\text{ell}}^{\varepsilon}(f,S) - I_{r,s,\text{ell}}(f,S) = \sum_{\gamma \in \Gamma^{\varepsilon}_{\text{unip}}(G,S,\zeta)} (a_{r,s,\text{ell}}^{G,\varepsilon}(\gamma,S) - a_{r,s,\text{ell}}^{G}(\gamma,S)) f_{G}(\gamma),
\]
and by (4.6) and (4.3), we have that
\[
a_{r,s,\text{ell}}^{G,\varepsilon}(\gamma,S) - a_{r,s,\text{ell}}^{G}(\gamma,S) = \sum_{k \in K_{\text{ell}}^{V,\varepsilon}(G,S)} (a_{r,s,\text{ell}}^{G,\varepsilon}(\gamma \times k) - a_{r,s,\text{ell}}^{G}(\gamma \times k)) r_{G}(k,b).
\]
By Proposition 6.4(a), the coefficients \( a_{r,s,\text{ell}}^{G,\varepsilon}(\gamma \times k) \) and \( a_{r,s,\text{ell}}^{G}(\gamma \times k) \) are equal if the semisimple part of \( \gamma \times k \) is not central in \( G \), hence \( a_{r,s,\text{ell}}^{G,\varepsilon}(\gamma,S) = a_{r,s,\text{ell}}^{G}(\gamma,S) \) for such \( \gamma \). On the other hand, if the semisimple part of \( \gamma \) is central, it has the Jordan decomposition
\[
\gamma = z \alpha, \quad z \in Z(G)_{V,\alpha}, \alpha \in \Gamma_{\text{unip}}^{\varepsilon}(G,V,\zeta),
\]
we can therefore break the sum over $\gamma$ into a double sum over $z$ and $\alpha$. In this case, Corollary 6.3(a) implies that

$$a_{G,\text{ell}}^{G,\varepsilon}(\gamma, S) - a_{G,\text{ell}}^{G,\varepsilon}(\gamma, S) = a_{G,\text{unip}}^{G,\varepsilon}(\alpha, S) - a_{G,\text{unip}}^{G,\varepsilon}(\alpha, S),$$

and using the definition

$$I_{z, r, s, \text{unip}}(f, S) - I_{z, r, s, \text{unip}}(f, S) = \sum_{\alpha \in \Gamma(G, V, \zeta)} (a_{G,\text{unip}}^{G,\varepsilon}(\alpha) - a_{G,\text{unip}}^{G,\varepsilon}(\alpha)) f_G(z\alpha),$$

the required formula (a) follows.

The case of (b) is similar. Recall the expansions

$$S_{r, s, \text{ell}}^G(f, S) = \sum_{\delta \in \Delta_{\text{ell}}(G, S, \zeta)} b_{r, s, \text{ell}}^G(\delta, S) f_G^\varepsilon(\delta)$$

and

$$b_{r, s, \text{ell}}^G(\delta, S) = \sum_{\ell \in \mathcal{L}_V(\bar{G})} b_{r, s, \text{ell}}^G(\delta \times \ell) r_G(k, b)$$

from (4.20) and (4.7) respectively. Since $f$ is unstable, it follows that $f_G^\varepsilon(\delta)$ vanishes on the subset $\Delta_{\text{ell}}(G, S, \zeta)$ of $\Delta_{\text{ell}}^\varepsilon(G, S, \zeta)$. On the other hand, suppose $\delta$ belongs to the complement of $\Delta_{\text{ell}}(G, S, \zeta)$ in $\Delta_{\text{ell}}^\varepsilon(G, S, \zeta)$. If its semisimple part is not central in $G$, Proposition 6.13(b) implies that $b_{r, s, \text{ell}}^G(\delta, S)$ vanishes. If the semisimple part of $\delta$ is central, it again has the Jordan decomposition

$$\delta = z\beta, \quad z \in Z(\bar{G}), \beta \in \Delta_{\text{unip}}^\varepsilon(G, V, \zeta),$$

we can therefore break the sum over $\delta$ into a double sum over $z$ and $\beta$. In this case, Corollary 6.3(b) implies that

$$b_{r, s, \text{ell}}^G(\gamma, S) = b_{r, s, \text{unip}}^G(\beta, S),$$

and by the definition of $S_{z, r, s, \text{unip}}^G(f, S)$ the required formula (b) follows.

This completes the primary application of the descent formula for the global geometric coefficients, which has allowed us to reduce the study of the elliptic part of the trace formula to the unipotent terms.

7. Stabilization

7.1. The parabolic parts. We recall that our goal is to prove Theorems (4.3) and (5.4). The remainder of the argument, following [Art03], is now a largely formal process, having the preceding results in place, hence we allow ourselves to be brief. Our induction will be based on an integer $d_{\text{der}}$. We assume that the theorems hold if $\dim(G_{\text{der}}) < d_{\text{der}}$, and if $G$ is not quasisplit we shall assume that $\dim(G_{\text{der}}) = d_{\text{der}}$ and the relevant theorems hold for the quasisplit inner $K$-form of $G$. This, together with the assumption of Conjecture 3.3 on the geometric side are the hypotheses that we will carry.

We shall first analyze the parabolic parts of the geometric expansions, which are defined to be the terms associated to $M \neq G$,

$$I_{r, s, \text{par}}(f) = \sum_{M \in \mathcal{F}_0} |W_0^M| W_0^G |^{-1} \sum_{\gamma \in \Gamma(M, V, \zeta)} a_{r, s}^M(\gamma) I_M(\gamma, f),$$

This completes the primary application of the descent formula for the global geometric coefficients, which has allowed us to reduce the study of the elliptic part of the trace formula to the unipotent terms.
Proof.
By (4.8), (4.10) and Proposition 4.4(a), we have by definition
and

and similarly

and by (4.10) and (4.15), we have

whence from (4.19) and (4.11) we have

follows that

Then applying the global induction hypothesis to (4.2) and Proposition 4.5(a), it

Lemma 7.1.
(a) If \( G \) is arbitrary and \( f \in \mathcal{C}^0(G_V, \zeta_V) \),

\[
I_{r,s,\text{par}}^G(f) - I_{r,s,\text{par}}(f) = (I_{r,s,\text{disc}}^G(f) - I_{r,s,\text{disc}}(f)) - \sum_{z \in Z(G) \cap V} (I_{r,s,z,\text{unip}}^G(f, S) - I_{r,s,z,\text{unip}}(f, S)).
\]

(b) If \( G \) is quasisplit and \( f \in \mathcal{C}_\text{uns}^0(G_V, \zeta_V) \),

\[
S_{r,s,\text{par}}^G(f) = S_{r,s,\text{disc}}^G(f) = \sum_{z \in Z(G) \cap V} S_{r,s,z,\text{unip}}^G(f, S).
\]

Proof. By (4.8), (4.10) and Proposition 4.4(a), we have by definition

\[
I_{r,s,\text{par}}^G(f) - I_{r,s,\text{par}}(f) = (I_{r,s,\text{disc}}^G(f) - I_{r,s,\text{disc}}(f)) - (I_{r,s,\text{orb}}^G(f) - I_{r,s,\text{orb}}(f))
\]

and by (4.10) and (4.15), we have

\[
I_{r,s,\text{orb}}^G(f) - I_{r,s,\text{orb}}(f) = \sum_{\gamma \in \Gamma_\ell^G(G_V, \zeta)} (a_{r,s,\text{ell}}^G(\gamma) - a_{r,s,\text{ell}}^G(\gamma)) f_G(\gamma).
\]

Then applying the global induction hypothesis to (4.2) and Proposition 4.5(a), it follows that

\[
a_{r,s,\text{ell}}^G(\gamma) - a_{r,s,\text{ell}}^G(\gamma) = a_{r,s,\text{ell}}^G(\gamma, S) - a_{r,s,\text{ell}}^G(\gamma, S),
\]

whence from (4.19) and (4.11) we have

\[
I_{r,s,\text{orb}}^G(f) - I_{r,s,\text{orb}}(f) = I_{r,s,\text{ell}}^G(f, S) - I_{r,s,\text{ell}}(f, S).
\]

Then by Corollary 6.5(a) we see that \( I_{r,s,\text{par}}^G(f) - I_{r,s,\text{par}}(f) \) equals

\[
(I_{r,s}^G(f) - I_{r,s}(f)) - \sum_{z \in Z(G) \cap V} (I_{r,s,z,\text{unip}}^G(f, S) - I_{r,s,z,\text{unip}}(f, S)).
\]

Moreover, by (5.7) and Proposition 5.5(a) we can express the spectral expansion of \( I_{r,s}^G(f) - I_{r,s}(f) \) as the sum of \( I_{r,s,\text{unit}}^G(f) - I_{r,s,\text{unit}}(f) \) and

\[
\int_{\Pi_\ell^G(M_V, \zeta)} (a_{r,s,\text{ell}}^G(\pi) I_{M}^G(\pi, f) - a_{r,s,\text{ell}}^G(\pi) I_{M}(\pi, f)) \pi d\pi.
\]
By the global induction hypothesis again the coefficients $a_{r,s}^M(\pi)$ and $a_{r,s}^M(\pi)$ are equal, and by Corollary 9.4(a), so too are the local distributions $I_{r,s}^M(\pi,f)$ and $I_{r,s}(\pi,f)$. On the other hand, by 5.9 and 5.14 we have

$$I_{r,s,\text{unit}}(f) - I_{r,s,\text{unit}}(f) = \int_{\Pi^\xi_{\text{disc}}(G,V,\zeta)} (a_{r,s}^G(\pi) - a_{r,s}G(\pi)) f_G(\pi) d\pi.$$ 

Then by applying the global induction hypothesis to (5.4) and Theorem 5.6(a), it follows that

$$a_{r,s}^G(\pi) - a_{r,s}G(\pi) = a_{r,s,\text{disc}}^G(\pi) - a_{r,s,\text{disc}}(\pi),$$

whence from (5.10) and (5.19) we have

$$I_{r,s}(f) - I_{r,s}(f) = I_{r,s,\text{disc}}(f) - I_{r,s,\text{disc}}(f).$$

This gives the required formula (a).

The proof of (b) follows in a similar fashion. In this case, it follows from Proposition 4.4 that $S_{r,s,\text{par}}^G(f)$ equals $S_{r,s}(f) - S_{r,s,\text{orb}}^G(f)$, then using the expansion (4.17) of the orbital part and the fact that $f$ is unstable, it follows from (4.20) and Proposition (4.4)(b) that

$$S_{r,s,\text{orb}}^G(f) = \sum_{\delta \in \Delta^G(G,V,\zeta)} b_{G}^{r,s}(\delta,S) f_{G}^S(\delta) = S_{r,s}^G(f).$$

Then by Corollary 6.5(b), we see that $S_{r,s,\text{par}}^G(f)$ equals

$$S_{r,s}^G(f) - \sum_{z \in Z(G,V,s)} S_{z,r,s,\text{unip}}^G(f,S).$$

Moreover by (5.16) and Proposition 5.5(b) we can express the spectral expansion of $S_{r,s}^G(f)$ as the sum of

$$S_{r,s,\text{unit}}^G(f) = \int_{\Phi^G_{\text{unit}}(G,V,\zeta)} b_{r,s}^G(\phi)f_{G}^S(\phi)d\phi.$$ 

and

$$\sum_{M \in \mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{M' \in \mathcal{E}_{\text{disc}}(M,V)} \iota(M,M') \int_{\Phi(M',V,\zeta')} b_{r,s}^{M'}(\phi') S_{M}^G(M',\phi',f)d\phi.'$$

The second term again vanishes by Corollary 9.4(b) and the fact that $f$ is unstable. Finally, from Theorem 5.6(b) and (5.20) that

$$S_{r,s,\text{unit}}(f) = \sum_{\phi \in \Phi_{\text{unit}}^G(G,V,\zeta)} b_{r,s,\text{disc}}^G(\phi)f_{G}^S(\phi) = S_{r,s,\text{disc}}^G(f).$$

This gives the required formula (b).
for any \( f \in \mathcal{C}^\circ(G_V, \zeta_V) \) and \( G \) arbitrary, and \( S^G_{r,s,\text{par}}(f) \) equals
\[
\sum_{M \in \mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{M' \in \mathcal{E} \mathcal{M}(M,V)} \iota(M, M') \sum_{\delta' \in \Delta(M', \zeta')} b_{r,s}^M(\delta') S_M(M', \delta', f) = 0
\]
for any \( f \in \mathcal{C}^\circ(\text{uns})(G_V, \zeta_V) \) and \( G \) quasisplit.

### 7.2. Proofs of the global theorems

We have shown that the left hand side of Lemma 7.1 vanishes. We shall next establish that the discrete parts on the right hand side also vanish.

**Proposition 7.2.**

(a) If \( G \) is arbitrary and \( f \in \mathcal{C}^\circ(G_V, \zeta_V) \), we have
\[
I^\varepsilon_{r,s,\text{disc}}(f) - I_{r,s,\text{disc}}(f) = 0.
\]

(b) If \( G \) is quasisplit and \( f \in \mathcal{C}^\circ(\text{uns})(G_V, \zeta_V) \), we have
\[
S^G_{r,s,\text{disc}}(f) = 0.
\]

**Proof.** The proof relies on a difficult local argument of Arthur that culminates in [Art03, Corollary 5.2]. As in §9.4, for any \( v \) in \( V \), we recall that \( f_v \in \mathcal{C}^\circ(G_V, \zeta_v) \) is called \( M \)-cuspidal if \( f_v|L_v = 0 \) for any \( L_v \in \mathcal{L}_v \) that does not contain a \( G_v \)-conjugate of \( M_v \), and we define \( \mathcal{C}^\circ_M(G_V, \zeta_V) \) to be the subspace of \( \mathcal{C}^\circ(G_V, \zeta_V) \) spanned by functions
\[
f = f_v f^w = f_v \prod_{w \neq v} f_w
\]
such that \( f_v \) is \( M \)-cuspidal at two places \( v \in V \). If \( G \) is quasisplit, we set
\[
\mathcal{C}^\circ_{M, \text{uns}}(G_V, \zeta_V) = \mathcal{C}^\circ_M(G_V, \zeta_V) \cap \mathcal{C}^\circ_{\text{uns}}(G_V, \zeta_V).
\]

If \( v \) is a nonarchimedean place, we define \( \mathcal{C}(G_v, \zeta_v)^0 \) to be the subspace of functions \( f \in \mathcal{C}(G_v, \zeta_v) \) such that \( f_v|G_v(z_v, a_v) = 0 \) for any \( z_v \) in the center of \( G_v = G_v/Z_v \), and \( a_v \) in the parabolic subset \( R_{\text{unip,par}}(G_v, \zeta_v) \) of the basis \( R_{\text{unip}}(G_v, \zeta_v) \) of unipotent orbital integrals in [Art03 §3]. We also write \( \mathcal{C}(G, V, \zeta)^0 \) for the product of functions \( f_v \in \mathcal{C}(G_v, \zeta_v)^0 \) for \( v \in V \), and similarly for \( \mathcal{C}(G, V, \zeta)^{00} \). We also denote by the intersections of these various spaces by using overlapping notation, for example, we write \( \mathcal{C}_M^\circ(G_v, \zeta_v)^0 = \mathcal{C}_M^\circ(G_v, \zeta_v) \cap \mathcal{C}(G_v, \zeta)^0 \). We also write \( W(M) \) for the Weyl group of \((G, M)\).

Following the argument of [Art03 Lemma 2.3], it follows that if \( G \) is arbitrary, then \( I^\varepsilon_{r,s,\text{par}}(f) - I_{r,s,\text{par}}(f) \) is equal to
\[
|W(M)|^{-1} \sum_{v \in \text{Im}(G, M)} \sum_{\gamma \in \Gamma(M, V, \zeta)} a^M_{r,s}(\gamma) I^\varepsilon_M(\gamma_v, f_v) - I_M(\gamma_v, f_v) f^v_M(\gamma_v)
\]
for any \( f \in \mathcal{C}_M(G_V, \zeta_V) \), and if \( G \) is quasisplit, then \( S^G_{r,s,\text{par}}(f) \) is equal to
\[
|W(M)|^{-1} \sum_{M' \in \mathcal{E} \mathcal{M}(M,V)} \sum_{\nu \in \text{Im}(G, M)} \sum_{\delta' \in \Delta(M', \zeta')} b^M_{r,s}(\delta') S^G_M(M'_v, \delta'_v, f_v)(f^v)(M'_v((\delta')^v)
\]
be careful about the construction of two families of mappings
for any \( f \) functions, namely Proposition 3.1 whose domains need to be extended to noncompactly supported functions, namely

\[
\varepsilon_M : \mathcal{C}^0(G_v, \zeta_v) \to I\mathcal{C}^0_{ac}(M_v, \zeta_v)
\]

such that

\[
\varepsilon_M(f_v, \gamma_v) = I^c_M(\gamma_v, f_v) - I_M(\gamma_v, f_v), \quad \delta_v \in \Gamma(M_v, \zeta_v)
\]

if \( G \) is arbitrary, and

\[
\varepsilon^M : \mathcal{C}^0_{uns}(G_v, \zeta_v) \to S\mathcal{C}^0_{ac}(M_v, \zeta_v)
\]

such that \( \varepsilon^M(f_v, \delta_v) = S_{\delta_v}^M(f_v, \delta_v) \) for \( \delta_v \in \Delta(M_v, \zeta_v) \), and

\[
\varepsilon^{M'} : \mathcal{C}^0_{uns}(G_v, \zeta_v) \to S\mathcal{C}^0_{ac}(\tilde{M}'_v, \tilde{\zeta}_v)
\]

such that \( \varepsilon^{M'}(f_v, \delta'_v) = S_{\delta'_v}^{\tilde{M}'_v}(f_v, \delta'_v) \) for \( \delta'_v \in \Delta(\tilde{M}'_v, \tilde{\zeta}_v) \) if \( G \) is quasisplit. Here \( I\mathcal{C}^0_{ac} \) and \( S\mathcal{C}^0_{ac} \) are defined as natural analogues of \( \mathcal{F}_{ac} \) and \( \mathcal{F}_{ac} \). These maps extend naturally to the larger spaces, as they are defined in terms of local tempered distributions. While the construction of these functions require endoscopic and stable variants of the functions \( \theta \) and \( \theta' \) in the unpublished reference [A12] of [Art03], they have now also been constructed in the more general twisted setting for smooth compactly supported test functions in [MW16b], VIII, IX. The constructions also rely on germ expansions of endoscopic and stable orbital integrals in the unpublished reference [A11] of [Art03], which are also described in Chapter 11 and also in [MW16b] in a slightly different formulation. More importantly, the variants \( c\varepsilon_M \) and \( c\varepsilon^M \) are no longer compactly supported, but can be shown to be rapidly decreasing, which is sufficient for applications. We also note that the finite partitions of unity required in [Art03], pp.793,796 need only be replaced by smooth partitions of unity obtained from an open covering of the support of \( f \). Then arguing as in [Art03 Corollary 3.3] it follows that if \( G \) is arbitrary,

\[
I^c_{r,s,\text{par}}(f) - I_{r,s,\text{par}}(f) = |W(M)|^{-1} I^M_{r,s}(\varepsilon_M(f)),
\]

and if \( G \) is quasisplit,

\[
S^G_{r,s,\text{par}}(f) = |W(M)|^{-1} \sum_{M' \in \mathcal{E}_{\text{nil}}(M,V)} \iota(M, M') \hat{S}'_{r,s}(\varepsilon^{M'}(f)).
\]

The remainder of the proof thus follows in the same way, and is moreover simplified by the absolute convergence of the spectral side [PLM11].

We then combine the preceding result with Lemma 7.1 and equations (7.1) and (7.2) to deduce the following vanishing result.

**Corollary 7.3.**

(a) If \( G \) is arbitrary and \( f \in \mathcal{C}^0(G_V, \zeta_V) \), we have

\[
\sum_{z \in \mathcal{Z}(G)_V} (I^c_{r,s,z,\text{unip}}(f, S) - I_{r,s,z,\text{unip}}(f, S)) = 0.
\]

(b) If \( G \) is quasisplit and \( f \in \mathcal{C}^0_{uns}(G_V, \zeta_V) \), we have

\[
\sum_{z \in \mathcal{Z}(G)_V} S^G_{r,s,z,\text{unip}}(f, S).
\]
We are now ready to complete the main global Theorems 4.3 and 5.4. Recall that we have reduced them to Corollaries 4.6 and 5.7 respectively.

**Proof of Theorem 4.3.** We recall that by the descent formula Proposition 6.3, we can assume that $\hat{\gamma}_S$ and $\hat{\delta}_S$ are admissible elements in $\Gamma_{\text{unip}}^G(G, S, \zeta)$ and $\Delta_{\text{unip}}^G(G, S, \zeta)$ respectively. We shall take $S = V \supset V_{\text{ram}}(G, \zeta)$ and $f = \hat{f}_S$ to be an admissible function in $\mathcal{E}(G_S, \zeta_S)$. Using (6.19) and (6.17) we express Corollary 7.3(a) as

$$
\sum_{z \in Z(G)_{S,a}} \sum_{\hat{\alpha}_S \in \Gamma_{\text{unip}}^G(G, S, \zeta)} (a_{r,s,\text{ell}}^{G, \hat{e}}(\hat{\alpha}_S) - a_{r,s,\text{ell}}^{G}(\hat{\alpha}_S)) \hat{f}_{S,G}(z\hat{\alpha}_S) = 0
$$

where we have used the fact that $a_{r,s,\text{ell}}^{G, \hat{e}}(\hat{\alpha}_S) = a_{\text{unip}}^{G}(\hat{\alpha}_S, S)$ by Corollary 6.3(a) and (6.21), and $a_{r,s,\text{ell}}^{G}(\hat{\alpha}_S) = a_{\text{unip}}^{G}(\hat{\alpha}_S, S)$ by Lemma 6.1 and (6.18). On the other hand since the linear forms

$$
\hat{f}_S \to \hat{f}_{S,G}(z\hat{\alpha}_S), \quad z \in Z(G)_{S,a}, \hat{\alpha}_S \in \Gamma_{\text{unip}}^G(G, S, \zeta)
$$

are linearly independent on $\mathcal{E}_{\text{adm}}(G_S, \zeta_S)$, and it follows then that

$$a_{r,s,\text{ell}}^{G, \hat{e}}(\hat{\alpha}_S) - a_{r,s,\text{ell}}^{G}(\hat{\alpha}_S) = 0$$

for any $\hat{\alpha}_S \in \Gamma_{\text{unip}}^G(G, S, \zeta)$. This proves Theorem 4.3(a) for unipotent $\hat{\gamma}_S$, hence in general.

Next suppose that $G$ is quasisplit, and write $\Delta_{\text{unip}}^{G,0}(G, S, \zeta)$ for the complement of $\Delta_{\text{unip}}^G(G, S, \zeta)$ in $\Delta_{\text{unip}}^G(G, S, \zeta)$. Using (6.20) we express Corollary 7.3(b) as

$$
\sum_{z \in Z(G)_{S,a}} \sum_{\hat{\beta}_S \in \Delta_{\text{unip}}^{G,0}(G, S, \zeta)} b_{r,s,\text{ell}}^{G}(\hat{\beta}_S) = 0
$$

for unstable functions $\hat{f}_S$, and where we have used the fact that $b_{\text{ell}}^{G}(\hat{\beta}_S) = b_{\text{unip}}^{G}(\hat{\beta}_S, S)$ by Corollary 5.3(b) and (6.22). On the other hand since, the linear forms

$$
\hat{f}_S \to \hat{f}_{S,G}(z\hat{\beta}_S), \quad z \in Z(G)_{S,a}, \hat{\beta}_S \in \Delta_{\text{unip}}^{G,0}(G, S, \zeta)
$$

are linearly independent on the intersection of $\mathcal{E}_{\text{adm}}(G_S, \zeta_S)$ and $\mathcal{E}_{\text{uns}}(G, S, \zeta_S)$, and it follows then that

$$b_{r,s,\text{ell}}^{G}(\hat{\beta}_S) = 0$$

for any $\hat{\beta}_S \in \Gamma_{\text{unip}}^{G,0}(G, S, \zeta)$. This proves Theorem 4.3(a) for unipotent $\hat{\beta}_S$, hence in general.

**Proof of Theorem 5.4.** We shall take $V \supset V_{\text{ram}}(G, \zeta)$ and $f = f \times b_{\text{ell}}$ for $f \in \mathcal{E}(G, \zeta_V)$ and $b_{\text{ell}} \in \mathcal{E}_{\text{uns}}(G, \zeta_V)$. Using (5.8), (5.13), and (5.17) we express Proposition 7.2(a) as

$$
\sum_{\hat{\pi} \in \Pi_{\text{disc}}^G(G, \zeta)} (a_{r,s,\text{disc}}^{G, \hat{e}}(\hat{\pi}) - a_{r,s,\text{disc}}^{G}(\hat{\pi})) \hat{f}_{G}(\hat{\pi}) = 0.
$$

On the other hand since, the linear forms

$$
\hat{f} \to \hat{f}_{G}(\hat{\pi}), \quad \hat{\pi} \in \Pi_{\text{disc}}^G(G, \zeta)
$$

are linearly independent, and it follows then that

$$a_{r,s,\text{disc}}^{G, \hat{e}}(\hat{\pi}) - a_{r,s,\text{disc}}^{G}(\hat{\pi}) = 0$$

for any $\hat{\pi} \in \Pi_{\text{disc}}(G, \zeta)$. This proves Theorem 5.4(a).
Next suppose that $G$ is quasisplit, and write $\Phi_{\text{disc}}^E(G, \zeta)$ for the complement of $\Phi_{\text{disc}}(G, \zeta)$. Using (5.15) and (5.18) we express Proposition 7.2(b) as
\[
\sum_{\phi \in \Phi_{\text{disc}}^E(G, \zeta)} b_{G, r,s, \text{disc}}^{G, \zeta} (\phi) f_{G}^{E} (\phi) = 0
\]
for unstable functions $f$, whereby $\tilde{f}_{G}^{E}$ vanishes on $\Phi_{\text{disc}}(G, \zeta)$. On the other hand since, the linear forms
\[
\tilde{f} \rightarrow \tilde{f}_{G}^{E} (\phi), \quad \phi \in \Phi_{\text{disc}}^E(G, \zeta)
\]
are linearly independent, and it follows then that
\[
b_{G, r,s, \text{disc}}^{G, \zeta} (\phi) = 0
\]
for any $\phi \in \Phi_{\text{disc}}^E(G, \zeta)$. This proves Theorem 5.4(b).

Having now established the main global theorems regarding the geometric and spectral coefficients, the derivation of the endoscopic and stable trace formulas in Theorem 1 in the introduction then follows by defining inductively
\[
I_{r,s}^{E}(f) = \sum_{G' \in \mathcal{E}(G,V)} \iota(G', G') S_{r,s}^{G'}(f) + \varepsilon(G) S_{r,s}^{G}(f)
\]
with the requirement that $I_{r,s}^{E}(f) = I_{r,s}(f)$ in the case that $G$ is quasisplit. We see that the form $S_{r,s}^{G}(f)$ is stable, since the terms defining it in the spectral and geometric expansions are stable, and $I_{r,s}^{E}(f) = I_{r,s}(f)$ in general. We therefore have the decomposition
\[
I_{r,s}(f) = \sum_{G' \in \mathcal{E}(G,V)} \iota(G', G') S_{r,s}^{G'}(f)
\]
of the weighted invariant trace formula into weighted stable trace formulas for endoscopic groups.

8. Application to $L$-functions

The principal aim of this chapter is to show how the weighted trace formula can be used to prove results about automorphic $L$-functions. But first, we shall prove an auxiliary result about the absolute convergence of the noninvariant trace formula which is of independent interest.

8.1. The coarse geometric expansion. Let $F$ be a global field, and fix an algebraic closure $\bar{F}$ of $F$. Using the Jordan decomposition, we can write any $\gamma \in G(F)$ as the product of a semisimple element $c$ and a unipotent element $\alpha$ in $G(F)$ that commutes with $c$. We define the relation $\gamma \sim \gamma'$ if $\gamma' = g \gamma g^{-1}$ for some $g$ in $G(F)G_{c}(\bar{F})$. Then the equivalence classes in the coarse class of a semisimple element $\sigma \in G(F)$ are indexed by geometric unipotent classes of $G_{\sigma}$ containing a rational point. If $G_{\text{der}}$ is simply connected, the relation coincides with Arthur’s coarse equivalence relation where $\gamma$ and $\gamma'$ are called $\mathcal{O}$-equivalent if their semisimple parts $c$ and $c'$ are conjugate in $G(F)$, and we write $\mathcal{O}$ for the set of $\mathcal{O}$-equivalence classes. Let us also recall briefly the induction of conjugacy classes following Lusztig and Spaltenstein. Let $P$ be a parabolic subgroup of $G$ over $F$, and $\gamma \in M_{P}$. Then there exists a unique conjugacy class $I_{P}(\gamma)$ of $G$ which intersects $\gamma N_{P}$ in a Zariski dense open set. Let $I_{P}(\gamma) = I_{P}(\gamma) \cap o_{\gamma}$, where $o_{\gamma}$ is the $\mathcal{O}$-equivalence class of $\gamma$ in
$G(F)$. It is a non-empty union of conjugacy classes of $G(F)$. The coarse expansion of the trace formula will be stated in terms of these induced conjugacy classes.

Given any standard parabolic $P$ of $G$, let $A_P$ be the maximal $F$-split torus in $M_P$ and let $\Delta_0$ be the set of simple roots attached to the pair $(P_0, A_{P_0})$, which forms a basis of the real vector space $(a_0^P)\ast = (a_{P_0}^P)\ast$. As usual, we can form the set of simple coroots $\Delta_0^\vee = \{\alpha^\vee : \alpha \in \Delta_0\}$ which forms a basis of $a_0^P$, the set of simple weights $\Delta_0 = \{\varpi_\alpha : \alpha \in \Delta_0\}$ which forms a basis of $(a_0^P)^\ast$ dual to $\Delta_0^\vee$, and the set of simple coweights $\hat{\Delta}_0 = \{\varpi^\vee : \varpi \in \hat{\Delta}_0\}$ dual to $\Delta_0$. Let $\Delta_0^P$ be a basis of $a_0^P$ such that

$$\mathfrak{a}_P = \{H \in a_0 : \alpha(H) = 0, \alpha \in \Delta_0^P\}$$

and $\hat{\Delta}_P = \{\varpi_\alpha : \alpha \in \Delta_0 \setminus \Delta_0^P\}$, which forms a basis of $(a_0^P)^\ast$. More generally, given any $P, Q \in \mathcal{F}$ with $P \subset Q$, we write $a_0^Q = a_{MQ}^P$. We then define $\Delta_0^Q$ to be the set of linear forms on the subspace $a_0^Q$ of $a_P$ obtained by restricting elements in $\Delta_0^Q \setminus \Delta_0^P$, and similarly we define $\hat{\Delta}_0^Q$ to be the set of linear forms on $a_0^Q$ obtained by restricting elements in $\Delta_0^Q \setminus \Delta_0^P$. Finally, we define $\tau_P^G$ and $\tau_P^Q$ to be characteristic functions of the subsets

$$\{H \in a_0 : \alpha(H) > 0, \alpha \in \Delta_0^P\}$$

and

$$\{H \in a_0 : \varpi(H) > 0, \varpi \in \hat{\Delta}_0^Q\}$$

respectively. As is typical, we shall denote $\tau_P^G = \tau_P$ and $\tau_P^Q = \tau_P^Q$, and analogously for $\hat{\tau}_P^Q$.

Now let $T$ be a point in $a_0$. We recall the coarse expansion for the geometric side of the trace formula given by

$$J^T(f) = \sum_{\sigma \in \mathcal{C}} J_{\sigma}^T(f), \quad f \in \mathcal{C}(G),$$

where

$$J_{\sigma}^T(f) = \int_{G(F) \setminus G(\mathbb{A})} k_{\sigma}^T(x) dx,$$

$$k_{\sigma}^T(x) = \sum_{P > P_0} (-1)^{\dim(A_P/A_{G_2})} \sum_{\delta \in P(\mathbb{F}) \setminus G(\mathbb{F})} k_{P, \sigma}(\delta x) \hat{\tau}_P(H_P(\delta x) - T),$$

and

$$k_{P, \sigma}(x) = \sum_{\gamma \in M_P(\mathbb{F})} \int_{N_P(\mathbb{A})} f(x^{-1} \gamma nx) dn.$$

It follows from [FL16, Theorem 7.1] that $J^T(f)$ and $J_{\sigma}^T(f)$ are absolutely convergent for all $T$ such that

$$d(T) = \min_{\sigma \in \Delta_0} (\alpha(T)) > d_0,$$

for some constant $d_0$ independent of $f$, and are polynomials in $T$ of degree at most $\dim a_0$. Moreover, it converges absolutely for all $T$ if for example $f$ is compactly supported. (In fact, this is proved for more general functions in [Hof08].)

Let $A_P^\circ$ be the identity component of $A_P(\mathbb{R})$. Recall that any element $x \in G(\mathbb{A})$ can be written as the product $x = n mak$ with $n \in N_P(\mathbb{A}), m \in M_P(\mathbb{A})^1, a \in 58 TIAN AN WONG
projection of \( T \) over, we form the truncated Siegel set \( S \) and is invariant under translation by \( k = 8.1 \)

\[
\sum_{P_1 \subset P_2} \sum_{\delta \in P_1(F) \setminus G(F)} \hat{F}_1(\delta x, T) \sigma_{\hat{P}_1}^P(H_{\hat{P}_1}(\delta x) - T) k_{P_1, P_2, \sigma}(\delta x),
\]

where the sum is taken over pairs of parabolic subgroups \( P_1 \subset P_2 \),

\[
\sigma_{\hat{P}_1}^P(H) = \sum_{P_1 \subset P_2} (-1)^{\dim \hat{P}_1} \delta x_{\hat{P}_1} \delta \hat{P}_1 \sigma_{\hat{P}_1}^P(H) \hat{p}_{P_1}(H),
\]

and

\[
k_{P_1, P_2, \sigma}(x) = \sum_{P_1 \subset P_2} (-1)^{\dim \delta_P} k_{P_1, \sigma}(x).
\]

We also set \( \chi^T(x) = \chi^T_{P_1, P_2}(x) = F^{P_2}(x, T) \sigma_{\hat{P}_1}^P(H_{\hat{P}_1}(x) - T) \). Furthermore, we note that \( k_{P_1, P_2, \sigma}(x) \) can be written as

\[
\sum_{P_1 \subset P} (-1)^{\dim \delta_P} \sum_{\eta \in \hat{P}_1(F) \setminus N_{P_1}(F)} \sum_{I_{P}(\eta \nu) = \sigma} \int_{N_{P}(\hat{P})} f(x^{-1} \eta \nu n x) d\eta,
\]

where \( N_{P_1} = N_{P_1} \cap M_P \).

8.2. Absolute convergence. The absolute convergence of the coarse geometric expansion of \( J^T(f) \) was proved by Finis and Lapid for functions in \( \mathcal{C}^\infty(G) \) with \( T \) sufficiently regular. To extend the domain of convergence, we recall Hoffman’s extension of the function \( F^{P_2}(x, T) \) for any \( P_2 \supset P_1 \) given by

\[
F^{P_2}(x, T) = \sum_{P_1 \subset P_2} (-1)^{\dim \delta_P} \sum_{\delta \in P(F) \setminus P_2(F)} \hat{p}_{P_1}(H_{\delta x} - T_P),
\]

where \( T_P \) is the projection of \( T \) onto \( a_P \). By [Hof08, Lemma 1], it satisfies the analogue of Arthur’s partition lemma

\[
\sum_{P_1 \subset P_2} \sum_{\delta \in P(F) \setminus P_2(F)} F^{P}(x, T) \hat{p}_{P_1}(H_{\delta x} - T) = 1,
\]

for any \( T \in a_0 \), and coincides with the characteristic function defined above for sufficiently regular \( T \) in the positive chamber \( a_0^+ \) of \( a_0 \). It also follows from [Hof08, Theorem 5] that the identity (8.1) continues to hold for arbitrary \( T \), as long as the inner sum is absolutely convergent.

Given a compact open subgroup \( K \) of \( G(\mathbb{A}_f) \) where \( \mathbb{A}_f \) denotes the finite adelic of \( F \), we let \( \mathcal{C}^\infty(G; K) \) be the space of smooth right \( K \)-invariant functions on \( G(\mathbb{A}_f) \) that belong to \( L^1(G(\mathbb{A}_f)) \) along with all its derivatives. Then \( \mathcal{C}^\infty(G) \) is defined to be the union of all \( \mathcal{C}^\infty(G, K) \) over all compact open subgroups \( K \), and endowed with the inductive limit topology.
Lemma 8.1. The coarse geometric expansion of \( J^T(f) \) is absolutely convergent for any \( f \in \mathcal{C}^\circ(G) \) and \( T \in \mathfrak{a}_0 \). For any \( K \), there exist \( r \geq 0 \) and a continuous seminorm \( \mu \) on \( \mathcal{C}^\circ(G; K) \) such that

\[
\sum_{\tau \in \mathcal{O}} \int_{G(F) \backslash G(\mathbb{A}')} \left| F^\tau(x, T) k_\tau(x) - k_\tau^T(x) \right| \mu dx \leq \mu(f)(1 + ||T||)^{r} e^{-d(T)},
\]

for any \( f \in \mathcal{C}^\circ(G; K) \) and \( T \in \mathfrak{a}_0 \).

Proof. We indicate the necessary modifications to the proof of [FL16, Theorem 5.1]. The properties of the extended function \( F^\tau(x, T) \) above allows for its use for general \( T \) as indicated above. Define \( G(F)^\prime_p \) to be the complement in \( G(F) \) of all proper parabolics \( P'(F) \) containing \( P(F) \), and set

\[
\xi_P = \sum_{\alpha \in \Delta \setminus \Delta^0} \alpha.
\]

It then suffices to observe that in specializing [FL16, Theorem 3.1] to the case \( P = P_0, l = 0 \), and \( T = T_1 \), we see that from the estimate

\[
\int_{P(F) \backslash G(\mathbb{A}')} F^\tau(x, T) \tau_p(H_p(g) - T) \sum_{\gamma \in G(F)^\prime_p} |f(g^{-1}(\tau_g))| dg \ll (1 + ||T||)^{r} e^{-\langle \xi_P, T \rangle} \mu(f),
\]

which a priori holds for any \( T \in \mathfrak{a}_0 \) and fixed \( K \), it now follows from the extension of \( F^\tau(x, T) \) that

\[
\sup_{T \in \mathfrak{a}_0} (1 + ||T||)^{-r} \int_{P(F) \backslash G(\mathbb{A}')} F^\tau(x, T) \tau_p(H_p(g) - T) \sum_{\gamma \in \mathcal{O}} |f(g^{-1}(\tau_g))| dg
\]

is a continuous seminorm on \( \mathcal{C}^\circ(G) \), replacing the required bound [FL16 (8)]. \( \square \)

8.3. The unipotent contribution. Let \( F \) be a global field, and let \( \mathcal{U}_G \) be the Zariski closure in \( G \) of the unipotent set in \( G(F) \). Then \( \mathcal{U}_G(F) \) is a closed algebraic subvariety of \( G \) over \( F \) made up of a finite union of geometric unipotent conjugacy classes in \( G(F) \). Following Arthur, we denote by

\[
J^T_{\text{unip}}(f) = J^T_{\mathfrak{o}}(f)
\]

the contribution of the unique unipotent class \( \mathfrak{o} = \mathcal{U}_G(F) \). We write \( (\mathcal{U}_G) \) for the set of \( \text{Gal}(\bar{F}/F) \)-orbits in \( \mathcal{U}_G(F) \). Any \( U \in (\mathcal{U}_G) \) is a locally closed subset of \( G \), defined over \( F \), and is a finite union of unipotent conjugacy classes of \( G \). From [Won22, Lemma 4.1], there is a family of distributions \( J^T_U \) indexed by \( U \in (\mathcal{U}_G) \) such that

\[
J^T_{\text{unip}}(f) = \sum_{U \in (\mathcal{U}_G)} J^T_U(f), \quad f \in \mathcal{C}^\circ(G),
\]

which corresponds to the decomposition of the kernel

\[
k_{\text{unip}}(x, x) = \sum_{U \in (\mathcal{U}_G)} k_U(x, x) = \sum_{U \in (\mathcal{U}_G)} \sum_{u \in U(F)} f(x^{-1}u x),
\]
and similarly \( k_{P,\text{unip}}(x) \). The sum over \( U \) being finite, it again follows from [Won22, Lemma 4.1] and [FL16, Theorem 7.1] that the distributions \( J^T_U(f) \) converge absolutely. We then define \( J^T_U(f, \lambda) \) analogously by the associated kernel

\[
k_{P,U}(x) = \sum_{u \in U} \int_{\mathbb{H}(\mathfrak{a})} f(x^{-1}ux)dn,
\]

where \( U^G_u \) is the induced unipotent conjugacy class in \( G \) that coincides with \( I_P(u) \) for any \( P \in \mathcal{P}(M) \). This yields the decomposition

\[
J^T_{\text{unip}}(f) = \sum_{U \in (U^G_u)} J^T_U(f), \quad f \in \mathcal{C}_c(G).
\]

Its value at a suitably regular point \( T = T_0 \) is given by the fine expansion

\[
J_U(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_G^1|^{-1} \sum_{u \in (U^G_u)_{M,S}} a^M(S,u)J_M(u,f),
\]

which follows from the proof of [Art85, Corollary 8.4] applied to [Won22, Proposition 4.2]. Here \( J_M(u,f) \) are unipotent weighted orbital integrals that we shall recall later below, and \( (U^G_u)_{M,S} \) denotes the set of \( (M,S) \)-equivalent classes in \( U^G_u = U_G \cap M(F) \), where \( u, u' \in U^G_u \) are said to be \( (M,S) \)-equivalent if

\[
u_S = \prod_{v \in S} u_v, \quad u'_S = \prod_{v \in S} u'_v
\]

are conjugate in \( G(F_S) \).

8.4. Global coefficients. We now turn to the global coefficients. We shall show how knowledge of the unipotent contribution implies strong results about automorphic \( L \)-functions. Whereas the preceding sections focused on the coarse expansion of the noninvariant trace formula, here we shall also consider the weighted invariant and stable trace formulas. We begin with the geometric side. Our weighted noninvariant trace formula takes the form

\[
J^*_s(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_G^1|^{-1} \sum_{\gamma \in \Gamma(M,V,\zeta)} a^M_{r,s}(\gamma)J_M(\gamma,f),
\]

where \( a^M_{r,s}(\pi) \) and \( a^M_{r,s}(\gamma) \) play the role of the global coefficients as in the unweighted trace formula. In particular, the dependence on \( s \) is solely determined by these coefficients. Also, note that in the weighted invariant form

\[
I^*_s(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_G^1|^{-1} \sum_{\gamma \in \Gamma(M,V,\zeta)} a^M_{r,s}(\gamma)I_M(\gamma,f),
\]

the coefficients remain unchanged.

**Theorem 8.2.** If \( J^T_{\text{unip}}(f^*_s) \) has meromorphic continuation to all \( s \in \mathbb{C} \), then so does the coefficient \( b^M_{r,s}(\delta) \) for all \( \delta \).

**Proof.** We first make a series of standard reductions. We shall assume for simplicity that \( G \) is a connected reductive group, as the disconnected case is largely formal here. In general one chooses a central character datum \((Z, \zeta)\) attached to \( G \), which again without loss of generality we can assume to be trivial. We may also assume, though it is not important, that the derived group \( G_{\text{der}} \) of \( G \) is simply connected,
as the general case follows by the usual techniques of descent for global geometric coefficients, for example in [Art01 §2]. More importantly, the coefficients $b^M_{r,s}(\delta)$ are inductively constructed from the original $a^M_{r,s}(\gamma)$ obtained in [Won22], and furthermore by descent (Chapter 6), our considerations are finally reduced to the case of coefficients $a^M_{r,s}(\gamma)$ attached to unipotent elements.

By assumption, it follows from (8.3) that

$$J^T_{\text{unip}}(f^T_s) = \sum_{U \in \mathcal{U}(G)} J^T_U(f^T_s)$$

is meromorphic in $s$. Then following the proof of [Won22 Proposition 4.2], using the linear independence of unipotent weighted orbital integrals we have that for any $S$, there are uniquely determined numbers

$$a^M_{r,s}(S,u), \quad M \in \mathcal{L}, \; u \in (\mathcal{W}(F))_{M,S},$$

such that

$$J^T_{\text{unip}}(f^T_s) = \sum_{M \in \mathcal{L}} \frac{|W^M_0||W^L_0|^{-1}}{2} \sum_{u \in (\mathcal{W}(F))_{M,S}} a^M_{r,s}(S,u)J_M(u,f^T_s)$$

for any $L \in \mathcal{L}$ and $f^T_s = f \times b^S_{r,s}$ with $f \in C^\infty_c(G^M_\mathfrak{L})$. In particular, we have that the global coefficients $a^M_{r,s}(S,u)$ have meromorphic continuation to all $s \in \mathbb{C}$.

From this, the main result follows quickly. It follows from the definitions (4.10) and (4.13) in [Won22], namely,

$$a^G_{r,s,\text{ell}}(\gamma_S) = \sum_{\gamma} |Z(F,\hat{\gamma})|^{-1} a^G_{r,s}(\gamma_S)\gamma_S(\gamma_S)$$

and

$$a^G_{r,s}(\gamma) = \sum_{M \in \mathcal{L}} \frac{|W^M_0||W^G_0|^{-1}}{2} \sum_{k \in \mathcal{K}(M)} a^M_{r,s,\text{ell}}(\gamma_M \times k)\rho^G_M(k,b),$$

together with the global descent formula to unipotent elements of Lemma 6.1

$$a^G_{r,s,\text{ell}}(\gamma_S) = \sum_{c} \sum_{\alpha} \tilde{G}(S,c)|\tilde{G}_{c,+}(F)/\tilde{G}_{c}(F)|^{-1} a^G_{r,s,\text{ell}}(\tilde{\alpha}),$$

it follows that $a^M_{r,s}(\gamma)$ has meromorphic continuation for general $\gamma$. Finally, by the inductive definition of the stable coefficient $b^M_{r,s}(\gamma)$, we have the meromorphic continuation of $b^M_{r,s}(\gamma)$ also.

Applying this to the geometric side of the stable trace formula, we conclude the following.

**Corollary 8.3.** Let $G$ be quasisplit over $F$. For $f \in C^\infty_c(G_\mathfrak{V})$, the distribution

$$S^G_{r,s}(f) = \sum_{M \in \mathcal{L}} \frac{|W^M_0||W^G_0|^{-1}}{2} \sum_{\delta \in \Delta(M,V,\zeta)} b^M_{r,s}(\delta)S_M(\delta,f)$$

has meromorphic continuation to all $s \in \mathbb{C}$.

**Proof.** By the inductive definition of the stable coefficient $b^M_{r,s}(\delta)$, we have the meromorphic continuation of $b^M_{r,s}(\delta)$ also. Then the result follows from the fact that the sum over semisimple elements has only finitely many nonzero terms, and the meromorphy of the unipotent contribution. □
Of course, the same thus holds for the geometric expansions of \( J_s^r(f) \) and \( I_r^s(f) \).

On the spectral side, we can conclude that \( a_{r,s}^M(\pi) \) and therefore also \( b_{r,s}^M(\phi) \) have meromorphic continuation by linear independence of characters and induction.

**Theorem 8.4.** If for all \( \delta \), the coefficient \( b_{r,s}^M(\delta) \) has meromorphic continuation to all \( s \in \mathbb{C} \), then so does the coefficient \( b_{r,s}^M(\phi) \) for all \( \phi \).

**Proof.** We say that a function \( \dot{f} \) on \( G(\mathbb{A}) \) is cuspidal at a valuation \( w \) of \( F \) if \( \dot{f} \) is a finite sum of functions \( \prod_v f_v \) on \( G(F_v) \) for each \( v \) such that \( f_{w,M_w} = 0 \) for any \( M \in \mathcal{L} \). This means that the function

\[
 f_{w,M_w}(\pi_w) = f_{w,G}(\pi^G)
\]

vanishes for any tempered representation \( \pi_w \) of \( M_w = M(F_w) \), and also that the orbital integral \( J_G(\gamma,f) \) vanishes for any \( G \)-regular element \( \gamma \in G(F_w) \) that is not \( F_w \)-elliptic. Here \( \pi_w^G \) is the representation of \( G_w \) induced from \( \pi_w \). Then if we assume that \( \dot{f} = f_r^s \) is cuspidal at two places \( w_1 \) and \( w_2 \), we can apply the simple trace formula as in [Art883 Theorem 7.1] to our weighted trace formula [Won22 Theorem 1], and thereby deduce the identity

\[
 \sum_{\pi \in \Pi_{\text{disc}}(G)} a_{\text{disc}}^G(\pi) J_G(\pi, f_r^s) = \sum_{\gamma \in G(F) \backslash G, S} a_{\gamma}^G(\gamma) J_G(\gamma, f),
\]

where we note that the linear forms \( (f_s)_G(\pi) = J_G(\dot{\pi}, f_r^s) \) and \( f_C(\gamma) = J_G(\gamma, f) \) are in fact invariant, and we assume that \( S \) contains \( w_1 \) and \( w_2 \). The lefthand sum is meromorphic because the righthand side is, by Theorem 3.2, and the same follows for the individual summands by the linear independence of characters.

It remains to express the lefthand side in terms of the global spectral coefficients. Since \( f_r^s \) is unramified outside \( S \), the sum vanishes unless \( \pi \) is unramified outside of \( S \). We can replace the sum over \( \pi \) with a sum over \( \pi \) in \( \Pi_{\text{disc}}(M,S,\zeta) \) and \( c \in C_{\text{disc}}^S(M,\zeta) \), thus writing \( \dot{\pi} = \pi \times c \). Then, we write

\[
 a_{\text{disc}}^G(\dot{\pi}) J_G(\dot{\pi}, f_r^s) = a_{\text{disc}}^G(\dot{\pi})(b_{r,s}^S(c)) f_G(\pi),
\]

and note that

\[
 (b_{r,s}^S(c))^G = b_{G,c}^S(c) = L^S(c, r, s).
\]

Recall from the definitions and notation of [Won22 (5.3)] that the global spectral coefficient is given by

\[
 a_r^G(\pi) = \sum_{L \in \mathcal{L}} |W_0^L||W_0^G|^{-1} \sum_{c \in C^V_{\text{disc}}(M,\zeta)} a_{\text{disc}}^M(\pi_M \times c) r_M^G(c, b),
\]

where \( \pi_M \times c \) is a finite sum of representations \( \dot{\pi} \) in \( \Pi_{\text{unit}}(M(\mathbb{A}), \zeta) \), and \( a_{\text{disc}}^M(\pi_M \times c) \) is the sum of corresponding values \( a_{\text{disc}}^G(\dot{\pi}) \). Also

\[
 r_M^L(c, b) = r_M^L(c, \zeta) b_M^S(c)
\]

are coefficients weighting the spectral terms, for which it is only important to us that \( r_M^G(c, b) = b_G(c) \). The term corresponding to \( M = G \) is

\[
 a_{r,s}^G(\dot{\pi}) = \sum_{c \in C^V_{\text{disc}}(G,\zeta)} a_{\text{disc}}^G(\pi \times c) r_G^G(c, b),
\]

so that

\[
 \sum_{\dot{\pi} \in \Pi_{\text{disc}}(G)} a_{\text{disc}}^G(\dot{\pi}) J_G(\dot{\pi}, f_r^s) = \sum_{\pi \in \Pi_{\text{disc}}(G,S,\zeta)} a_{r,s}^G(\pi) f_G(\pi).
\]
This proves the meromorphy of \( a_{r,s,\text{disc}}^G(\pi) \), and the result follows for general \( a_{r,s}^G(\pi) \) by induction on the rank of \( G \).

The following is then an immediate consequence.

**Corollary 8.5.** Let \( \tilde{\pi} \) be a cuspidal automorphic form of \( G(\mathbb{A}) \) unramified outside of \( S \). Then \( L^S(s, \pi, r) \) has meromorphic continuation to all \( s \in \mathbb{C} \).

**Corollary 8.6.** Let \( G \) be quasisplit over \( F \). For \( f \in C_c^\infty(G_\mathbb{S}) \), the distribution

\[
S_{r,s}^G(f) = \sum_{M \in \mathcal{E}} |W_0^M| |W_0^G|^{-1} \int_{\Phi(M,V,\zeta)} b_{r,s}^M(\phi) S_M(\phi, f) d\phi
\]

has meromorphic continuation for all \( s \in \mathbb{C} \).

**Proof.** By the inductive definition of the stable coefficient \( b_{r,s}^M(\phi) \), we have the meromorphic continuation of \( b_{r,s}(\phi) \) also. The result then follows from the fact that the sum over \( \Pi_{\text{disc}}(G, S, \zeta) \) has finitely many nonzero terms, and induction on the rank of \( G \).

Again, the analogous statements thus hold for the spectral expansions of \( J'_r(f) \) and \( I'_r(f) \). Note, on the other hand, that though we have focused on the statements for the stable trace formula and stable coefficients, it is enough to prove the analytic properties for the noninvariant trace formula, which then easily propagate to the invariant and stable trace formulas, as we have seen above.

**Part 2. Properties of the stable trace formula**

**9. Continuity of the stable trace formula**

The goal of this chapter is twofold: the first is to prove a mild extension of the Langlands-Shelstad transfer mapping to Schwartz functions in the nonarchimedean case. In the archimedean case, the transfer is known by work of Shelstad. The second goal is to prove the continuity of the stable trace formula for the test function \( \hat{f} = f \times u^V \). This builds upon the continuity of the invariant trace formula \( I \) proved in [Won22, Theorem 2]. Our strategy is to stabilise the linear form \( I \) that we have shown to be valid on \( \mathcal{E}^\mathbb{S}(G,V,\zeta) \).

**9.1. Smooth transfer.** We refer to [3] and [Art02] for definitions of the objects that we consider here. In particular, we will require noncompactly supported variants of the many objects involved in the stabilization of the trace formula. Let \( \mathcal{E}(G_v) \) be the set of endoscopic data \((G'_v, G'_v, s'_v, \xi'_v)\) for \( G \) over \( F_v \), represented by \( G'_v \). We shall also assume that the auxiliary data \( \tilde{G}' \to G' \) and \( \tilde{\xi}' : G' \to \tilde{G}' \) to be chosen according to [Art02, Lemma 7.1]. The Langlands-Shelstad transfer conjecture states that for any \( G'_v \in \mathcal{E}(G_v) \), the map that sends \( f \in \mathcal{H}(G_v, \zeta_v) \) to the function

\[
f'_{\gamma}(\delta') = f''_{\gamma}(\delta') = \sum_{\gamma \in \Gamma_{\text{reg}}(G_v, \zeta_v)} \Delta(\delta', \gamma) f_G(\gamma)
\]

on \( \Delta_{\text{reg}}(\tilde{G}'_v, \tilde{\xi}'_v) \) exists, and maps \( \mathcal{H}(G_v, \zeta_v) \) continuously to the space \( \mathcal{F}(\tilde{G}'_v, \tilde{\xi}'_v) \). In the nonarchimedean case, the Langlands-Shelstad transfer was proved for smooth functions \( f \in C_c^\infty(G_v) \) as a consequence of [Wal97] and the solution of the Fundamental Lemma [Ngo10], and thus holds also for \( \mathcal{F}^{-1} \)-equivariant space \( C_c^\infty(G_v, \zeta_v) \). Moreover, since the orbital integrals are tempered distributions, it makes sense to
formulate the smooth transfer for the larger Schwartz space $\mathcal{E}(G_v, \zeta_v)$, in which case the transfer would lie in the corresponding space of stable orbital integrals $S\mathcal{E}(\hat{G}_v', \hat{\zeta}_v')$ of functions in $\mathcal{E}(\hat{G}_v', \hat{\zeta}_v')$. Recall that we are taking $G$ to be a $K$-group, so if $f$ equals $\oplus_a f_a$, then

$$f' = \sum_{\alpha \in \pi_0(G)} f'_\alpha.$$  

The Langlands-Shelstad transfer for Schwartz functions is then a simple consequence of the smooth transfer. We note that in the archimedean case, the result for Schwartz functions follows from work of Shelstad (c.f. [She08]). The proof relies on the fact that $C^\infty_v(G_v, \zeta_v)$ is a dense subspace of $\mathcal{E}(G_v, \zeta_v)$, topologized by the family of seminorms used to define the Harish-Chandra Schwartz space. Moreover, the spaces $I\mathcal{E}(G_v, \zeta_v)$ and $S\mathcal{E}(G_v, \zeta_v)$ are topologized in a manner such that the maps $f \to f_G$ and $f \to f^G$ respectively are continuous. That is, in [Art02, p.187] Arthur defines the spaces

$$\mathcal{J}(G^H_V, \zeta_V) = I\mathcal{H}(G^H_V, \zeta_V) = \{f_G \in \mathcal{H}(G^H_V, \zeta_V)\}$$

and

$$I\mathcal{E}(G^H_V, \zeta_V) = \{f_G \in \mathcal{E}(G^H_V, \zeta_V)\},$$

both topologized such that the surjective map $f \to f_G$ is open and continuous. This describes the topologies of the source and targets, and in particular it follows that $I\mathcal{H}(G^H_V, \zeta_V)$ is dense in $I\mathcal{E}(G^H_V, \zeta_V)$. We also define the stably invariant subspaces

$$\mathcal{J}(G^H_V, \zeta_V) = S I\mathcal{H}(G^H_V, \zeta_V) = \{f^G \in \mathcal{H}(G^H_V, \zeta_V)\}$$

and

$$S\mathcal{E}(G^H_V, \zeta_V) = S I\mathcal{E}(G^H_V, \zeta_V) = \{f^G \in \mathcal{E}(G^H_V, \zeta_V)\},$$

spanned by strongly regular, stable orbital integrals.

**Lemma 9.1.** Let $F_v$ be a nonarchimedean local field. Then for $f \in \mathcal{E}(G_v, \zeta_v)$, the map from $f$ to the function

$$f'(\delta') = f^G(\delta') = \sum_{\gamma \in \Gamma_{\text{reg}}(G_v, \zeta_v)} \Delta(\delta', \gamma) f_G(\gamma)$$

on $\Delta_{\text{reg}}(\hat{G}_v', \hat{\zeta}_v')$ exists, and maps $\mathcal{E}(G_v, \zeta_v)$ continuously to $S\mathcal{E}(G_v, \zeta_v)$.

**Proof.** Given $f \in \mathcal{E}(G_v, \zeta_v)$, we may choose a sequence $(f_n)$ in $C^\infty_v(G_v, \zeta_v)$ converging to $f$ as $n$ tends to infinity, namely,

$$\nu(f - f_n) \to 0,$$

where $\nu$ is any seminorm used to define the Schwartz space. Applying the Langlands-Shelstad transfer, it follows then that there is a family of transfers $(f'_n)$ in $I\mathcal{E}(\hat{G}_v', \hat{\zeta}_v')$ such that for any $\delta' \in \Delta_{\text{reg}}(\hat{G}_v', \hat{\zeta}_v')$, we have

$$f'_n(\delta') = \sum_{\gamma \in \Gamma_{\text{reg}}(G_v, \zeta_v)} \Delta(\delta', \gamma) f_{n,G}(\gamma)$$

in $\mathcal{J}(\hat{G}_v', \hat{\zeta}_v')$.

Estimating then the difference

$$|f'_n(\delta') - f'_{n+1}(\delta')| \leq \sum_{\gamma \in \Gamma_{\text{reg}}(G_v, \zeta_v)} |\Delta(\delta', \gamma)||f_{n,G}(\gamma) - f_{n+1,G}(\gamma)|$$
for any fixed \( \delta' \), where we note that the sums are finite since the orbital integral of \( f_n \) is compactly supported on the regular set for any \( n \), we see that the difference

\[
|f_G(\gamma) - f_{n,G}(\gamma)| \ll_n \nu(f - f_n)
\]

vanishes for \( f_G \) in \( I\mathcal{C}(G_v, \zeta_v) \) and \( f_{n,G} \) in \( I\mathcal{H}(G_v, \zeta_v) \). Here we have used the standard estimate on orbital integrals,

\[
|f_G(\gamma)| \leq \nu(f)(1 + |\log|D(\gamma)||) p(1 + ||H_G(\gamma)||)^{-n}
\]

for any integer \( n > 0 \), \( p \in \mathbb{R} \), and \( f \in \mathcal{C}(G_v, \zeta_v) \) (c.f. [Art94 (5.7)]). It follows that \( f_n(\delta') \) converges in \( \mathcal{F}(G'_v, \zeta'_v) \), and by continuity in \( S'\mathcal{C}(G'_v, \zeta'_v) \). By completeness, we denote by \( f' \) the function in \( \mathcal{C}(G'_v, \zeta'_v) \) such that \( f'_n \) converges to \( f' \). We note that the choice of \( f' \) is unique only up to stable conjugacy, and satisfies the identity

\[
f'(\delta') = \sum_{\gamma \in \Gamma_{reg}(G_v, \zeta_v)} \Delta(\delta', \gamma)f_G(\gamma).
\]

as required. \( \square \)

The stabilization of the trace formula relies on the local results of Arthur on orbital integrals such as in [Art96, Art99b, Art06, Art08, Art16]. In order to stabilize the invariant linear form \( I(f) \) for \( f \times u^V \) with \( f \in \mathcal{C}(G, \zeta) \), we note that the local results of Arthur above hold for more general Schwartz functions \( f \in \mathcal{C}(G, V, \zeta) \) either as explicitly stated, or otherwise can be shown using the fact that the linear forms \( I_M(\gamma, f) \) extend to tempered distributions on \( G \). These local transfer mappings are required to construct the stable basis \( \Delta(G^Z_V, \zeta_V) \) that is used to index the geometric side of the stable trace formula.

We have to show that this construction holds in our case also. While Arthur’s stabilization is carried out for functions \( f \) belonging to \( \mathcal{H}(G, V, \zeta) \), his construction of these spaces holds generally for functions in \( \mathcal{C}(G, V, \zeta) \), as long as the transfer mappings exist. We shall follow Arthur’s construction here, extended to the slightly more general setting of \( \mathcal{C}(G, V, \zeta) \). In the local setting, we can often work with the full Schwartz space \( \mathcal{C}(G, V, \zeta) \).

9.2. Geometric transfer factors. As usual, if \( S' \) is a stable, tempered \( \tilde{\zeta}' \)-equivariant distribution on \( \tilde{G}'(F_v) \), then we write \( \tilde{S}' \) for the corresponding continuous linear form on \( S'\mathcal{C}(G'_v, \zeta'_v) \). Applying the transfer to each of the components \( G_{\alpha_v} \) of \( G_v \), we have a mapping

\[
f_v \to f'_v = f'_v \quad \text{from } \mathcal{C}(G_v, \zeta_v) \text{ to } S'\mathcal{C}(G'_v, \zeta'_v),
\]

which can be identified with a mapping

\[
a_v \to a'_v
\]

from \( I\mathcal{C}(G_v, \zeta_v) \) to \( S'\mathcal{C}(G'_v, \zeta'_v) \). It follows that the product mapping from \( \prod_v a_v \) to \( \prod_v a'_v \) gives a linear mapping from \( I\mathcal{C}(G_V, \zeta_V) \) to \( S'\mathcal{C}(G'_V, \zeta'_V) \). This mapping is attached to the product \( G'_V \) of the data \( G'_v \), which we can think of as the endoscopic data of \( G \) over \( F_V \). Letting \( G'_V \) vary, we obtain a mapping

\[
I\mathcal{C}(G_V, \zeta_V) \to \prod_{G'_V} S'\mathcal{C}(G'_V, \zeta'_V)
\]
by putting together the individual images of \( \alpha' \). The image \( I\mathcal{E}^\delta(G_V, \zeta_V) \) of \( I\mathcal{E}(G_V, \zeta_V) \) fits into a sequence of inclusions

\[
I\mathcal{E}^\delta(G_V, \zeta_V) \subset \bigoplus_{(G'_V)} I\mathcal{E}^\delta(G'_V, G_V, \zeta_V) \subset \prod_{\Delta_V} S\mathcal{E}(\tilde{G}'_V, \tilde{\zeta}_V)
\]

in which the summand \( I\mathcal{E}^\delta(G'_V, G_V, \zeta_V) \) is a vector space of families of functions on \( \tilde{G}' \) parametrized by transfer factors for \( G \) and \( \tilde{G}' \), depending only on the \( F_V \)-isomorphism class of \( G'_V \).

The mappings of functions have dual analogues for distributions. Given \( G'_V \) with auxiliary data \( \tilde{G}'_V \) and \( \tilde{\xi}'_V \), assume that \( \delta' \) belongs to the space of stable distributions \( S\mathcal{D}((G'_V)\bar{\mathbb{Z}}', \tilde{\zeta}'_V) \). By Lemma 9.1, we may evaluate the transfer \( f' \) of any function \( f \in \mathcal{E}(G, V, \zeta) \) at \( \delta' \). Since the distribution \( f \to f'(\delta') \) belongs to \( \mathcal{D}(G'_V, \tilde{\zeta}'_V) \), we can construct the extended geometric transfer factors at each local place

\[
\Delta(\delta', \gamma), \quad G' \in \mathcal{E}(G), \delta' \in \Delta(\tilde{G}'_V, \tilde{\zeta}'_V), \gamma \in \Gamma(G, \zeta).
\]

defined for fixed bases \( \Delta(\tilde{G}'_V, \tilde{\zeta}'_V) \) of the spaces \( S\mathcal{D}(\tilde{G}'_V, \tilde{\zeta}'_V) \) such that

\[
f'(\delta') = \sum_{\gamma \in \Gamma(G, \zeta)} \Delta(\delta', \gamma) f_G(\gamma)
\]

holds for \( \delta' \in \Delta(\tilde{G}'_V, \tilde{\zeta}'_V) \) and \( f \in \mathcal{E}(G, \zeta) \). We can see that the extended local transfer factor, as a function on \( \Delta(\tilde{G}'_V, \tilde{\zeta}'_V) \times \Gamma(G, \zeta) \) is defined in the exact same manner as [Art02 §4], and depends linearly on \( \delta' \). We can then define the global transfer factor as the corresponding product

\[
\Delta(\delta, \gamma) = \prod_{v \in V} \Delta(\delta_v, \gamma_v)
\]

for \( \delta \in \Delta^\delta(G_V, \zeta_V) \) and \( \gamma \in \Gamma(G_V, \zeta_V) \). The sequence of inclusions (9.1) is dual to a sequence of surjective linear mappings

\[
\prod_{G'_V} S\mathcal{D}((G'_V)\bar{\mathbb{Z}}', \tilde{\zeta}'_V) \to \bigoplus_{(G'_V)} \mathcal{D}^\delta(G'_V, G_V, \zeta_V) \to \mathcal{D}^\delta(G'_V, \zeta_V)
\]

between spaces of distributions. Since \( f' \) is the image of the function \( f_G \) in \( I\mathcal{E}(G, V, \zeta) \), it follows that \( f'(\delta') \) depends only on the image \( \delta \) of \( \delta' \) in \( \Delta^\delta(G'_V, \zeta_V) \).

In other words,

\[
f'(\delta') = f^G_G(\delta),
\]

where \( f^G_G \) is the image of \( f_G \) in \( I\mathcal{E}(G, V, \zeta) \), so that by the adjoint relations satisfied by the geometric transfer factor [Art02 §5] the map \( f_G \to f^G_G \) is an isomorphism. The same is true therefore of the coefficients \( \Delta_G(\delta', \gamma) \), so we may write

\[
\Delta_G(\delta, \gamma) = \Delta_G(\delta', \gamma)
\]

for \( \gamma \in \Gamma(G'_V, \zeta_V) \) and complex numbers \( \Delta_G(\delta, \gamma) \) that depend linearly on \( \delta \in \Delta^\delta(G'_V, \zeta_V) \). The image in \( \Delta^\delta(G'_V, \zeta_V) \) of the subspace

\[
S\mathcal{D}((G'_V)\bar{\mathbb{Z}}', \tilde{\zeta}'_V) \simeq S\mathcal{D}(G'_V, G_V, \zeta_V)
\]

can be identified with the subspace \( S\mathcal{D}(G'_V, \zeta_V) \) of stable distributions in \( \mathcal{D}(G'_V, \zeta_V) \).

The coefficients in the geometric expansion should really be regarded as elements in the appropriate completion of \( \mathcal{D}(M'_V, \zeta_V) \) and \( S\mathcal{D}(M'_V, \zeta_V) \), which we shall identify with the dual space of \( \mathcal{D}(M'_V, \zeta_V) \) by fixing suitable bases \( \Gamma(M'_V, \zeta_V) \) and \( \Delta(M'_V, \zeta_V) \) of the relevant spaces of distributions. In particular, we shall fix a
basis $\Delta((\mathcal{G}_V)^{2'}, \zeta_V')$ of $S\mathcal{D}((\mathcal{G}_V)^{2'}, \zeta_V')$ for any $F_V$-endoscopic datum $G'_V$ with auxiliary data $\mathcal{G}_V'$ and $\xi'_V$. We also fix a basis $\Delta^G(G'_V, \zeta_V)$ of $\mathcal{D}^G(G'_V, \zeta_V)$ such that

$$\Delta(G'_V, \zeta_V) = \Delta^G(G'_V, \zeta_V) \cap S\mathcal{D}(G'_V, \zeta_V)$$

forms a basis of $S\mathcal{D}(G'_V, \zeta_V)$, and in the case that $G$ is quasisplit, that $\Delta(G'_V, \zeta_V)$ is isomorphic to the image of the basis $\Delta((G'_V)^{2'}, \zeta_V)$.

9.3. Spectral transfer factors. The construction on the spectral side is parallel. In place of the spaces of distributions described by (9.3), we have the spectral analogue $\mathcal{F}(G'_V, \zeta_V)$ of $\mathcal{D}(G'_V, \zeta_V)$, and the sequence of maps

$$\prod_{G'_V} S\mathcal{F}((\mathcal{G}_V)^{2'}, \zeta_V') \to \bigoplus_{(G'_V)} \mathcal{F}^G(G'_V, G'_V, \zeta_V') \to \mathcal{F}^G(G'_V, \zeta_V).$$

In place of the basis $\Gamma(G'_V, \zeta_V)$ of $\mathcal{D}(G'_V, \zeta_V)$, we have the basis

$$\Pi(G'_V, \zeta_V) = \bigoplus_{\ell \geq 0} \Pi_{\ell}(G'_V, \zeta_V)$$

of $\mathcal{F}(G'_V, \zeta_V)$ consisting of irreducible characters. If $\phi'$ belongs to $S\mathcal{F}((\mathcal{G}_V)^{2'}, \zeta_V')$, then the distribution $f \to f'(\phi')$ belongs to $\mathcal{F}(G'_V, \zeta_V)$, we can construct the spectral transfer factors at each local place

$$\Delta(\phi', \pi), \quad G' \in \mathbb{D}(G), \phi' \in \Phi(\mathcal{G}', \zeta'), \pi \in \Pi(G, \zeta)$$

defined for fixed bases $\Phi(\mathcal{G}', \zeta')$ of the spaces $S\mathcal{F}(\mathcal{G}', \zeta')$ such that

$$f'(\phi') = \sum_{\pi \in \Pi(G, \zeta)} \Delta(\phi', \pi) f_G(\pi)$$

holds for $\phi' \in \Phi(\mathcal{G}', \zeta')$ and $f \in \mathcal{E}(G, \zeta)$, parallel to (9.2). We also define the corresponding product

$$\Delta(\phi, \pi) = \prod_{v \in V} \Delta(\phi_v, \pi_v)$$

for $\phi \in \mathcal{E}(G, \zeta_V)$ and $\pi \in \Pi(G, \zeta_V)$. Given an element $\phi'$ in $S\mathcal{F}((\mathcal{G}_V)^{2'}, \zeta_V')$, We have that $f'(\phi')$ depends only on the image $\phi$ of $\phi'$ in $\mathcal{F}^G(G'_V, \zeta_V)$, that is,

$$f'(\phi') = f_G^G(\phi),$$

and the spectral coefficients satisfy the relation

$$\Delta_G(\phi, \pi) = \Delta_G(\phi', \pi)$$

for $\pi \in \Pi(G'_V, \zeta_V)$ and complex numbers $\Delta_G(\phi, \pi)$ that depend linearly on $\phi \in \mathcal{F}^G(G'_V, \zeta_V)$. They satisfy adjoint relations parallel to the geometric transfer factors. Here as in [Art02, §5] we shall fix an endoscopic basis $\Phi^G(G'_V, \zeta_V)$ of $\mathcal{F}(G'_V, \zeta_V)$, and a subset

$$\Phi(G'_V, \zeta_V) = \Phi^G(G'_V, \zeta_V) \cap S\mathcal{F}(G'_V, \zeta_V)$$

that forms a basis of $S\mathcal{F}(G'_V, \zeta_V)$, and in the case that $G$ is quasisplit, such that $\Phi(G'_V, \zeta_V)$ is isomorphic to the image of the basis $\Phi((\mathcal{G}_V)^{2'}, \zeta_V')$. If $v$ is archimedean, we can identify $\Phi(\mathcal{G}_V', \zeta_V')$ with the relevant set of Langlands parameters. If $v$ is nonarchimedean, we construct $\Phi(\mathcal{G}_V', \zeta_V')$ in terms of abstract bases.
\( \Phi_{\text{ell}}(G_v, \zeta_v) \) of the cuspidal subspaces \( \mathcal{J}_{\text{cusp}}(M_v, \zeta_v) \), and similar objects for endoscopic groups \( M' \) of \( M \), where we observe that the relevant constructions of \cite{Art03} extend readily to \( \mathcal{H}(G_v, \zeta_v) \) (see \cite{Art03} p.825).

9.4. The stable and endoscopic expansions. Having defined the relevant objects, we now turn to the continuity of the stable trace formula. As before, our attention will be on extending the arguments in \cite{Art02} \cite{Art01} \cite{Art03}, which will essentially follow from properly constructing the natural generalizations of the required objects. As the stabilization of the trace formula involves a much more intricate argument than that needed for the invariant trace formula, we are forced to follow the same path here. We note that a similar argument is provided in \cite{MW16a} \cite{MW16b} for the stabilization of the twisted trace formula.

**Theorem 9.2.** The linear forms \( I^\delta \) and \( S \) extend continuously from \( \mathcal{H}(G, V, \zeta) \) to \( \mathcal{E}(G, V, \zeta) \).

**Proof.** We first observe that Global Theorem 1’ in \cite{Art02} §7 states that the global geometric coefficients satisfy

\[ a^{G, \delta}(\gamma) = a^G(\gamma), \quad \gamma \in \Gamma^\delta(G, V, \zeta) \]

for any \( G \), and that

\[ b^G(\delta), \quad \delta \in \Delta^\delta(G, V, \zeta) \]

vanishes on the complement of \( \Delta(G, V, \zeta) \) if \( G \) is quasisplit. Notice that \( \Gamma^\delta(G, V, \zeta) \) and \( \Delta(G, V, \zeta) \) are constructed as subsets of bases \( \Gamma(G_v^\nu, \zeta_v) \) and \( \Delta(G_v^\nu, \zeta_v) \) of the spaces \( \mathcal{J}(G_v^\nu, \zeta_v) \) and \( S\mathcal{J}(G_v^\nu, \zeta_v) \) respectively. In particular, we see that this space contains the orbital integrals, and also derivatives of orbital integrals in the archimedean cases, of functions \( f \) in \( \mathcal{E}(G, V, \zeta) \). Similarly, Global Theorem 2’ states that the global geometric coefficients satisfy

\[ a^{G, \delta}(\pi) = a^G(\pi), \quad \pi \in \Pi^\delta(G, V, \zeta) \]

for any \( G \), and that

\[ b^G(\phi), \quad \delta \in \Phi^\delta(G, V, \zeta) \]

vanishes on the complement of \( \Phi(G, V, \zeta) \) if \( G \) is quasisplit. Here the spaces

\[ \Pi^\delta(G, V, \zeta), \quad \Phi^\delta(G, V, \zeta), \quad \Phi(G, V, \zeta) \]

are the subset of elements in

\[ \Pi^\delta(G, V, \zeta), \quad \Phi^\delta(G, V, \zeta), \quad \Phi(G, V, \zeta) \]

respectively whose archimedean infinitesimal characters \( \nu \) have norms \( t = ||\text{Im}(\nu)|| \). Notice that \( \Phi_t(G, V, \zeta) \) and \( \Pi_t^\delta(G, V, \zeta) \) are constructed as discrete subsets of the bases \( \Pi_t^\delta(G_v^\nu, \zeta_v) \) and \( \Phi_t(G_v^\nu, \zeta_v) \) of the spaces \( \mathcal{J}(G_v^\nu, \zeta_v) \) and \( S\mathcal{J}(G_v^\nu, \zeta_v) \) respectively. As we have indicated above, in both the geometric and spectral cases, the construction of the endoscopic spaces implicitly rely on the Langlands-Shelstad transfer, hence by Lemma 9.1 these spaces exist unconditionally.

Let \( S \) be a finite set of valuations containing \( V \). There is a natural map

\[ f \mapsto f_S = f \times u_S^V \]

from \( \mathcal{E}(G, V, \zeta) \) to \( \mathcal{E}(G, S, \zeta) \). We have an admissible subspace \( \mathcal{E}^\text{adm}(G, S, \zeta) \) of \( \mathcal{E}(G, S, \zeta) \), using the same notion of admissibility in \cite{Art02} §1. The polynomial

\[ \text{det}(1 + t - \text{Ad}(x)) = \sum_k D_k(x)t^k \]

for \( x \in G \) defines a morphism

\[ \mathcal{D} = (D_0, \ldots, D_d) : G \to G_{d+1}^{\text{adm}} \]
where \( d = \dim G \). If \( X \) is a nonzero point in \( G^{d+1}_n \), we shall denote \( X_{\text{min}} = X_k \) where \( k \) is the smallest integer such that \( X_k \) is nonzero. Let \( \mathcal{O}_v \) be product of all \( v \) not in \( S \) of \( \mathcal{O}_v \), the ring of integers of \( F_v \). We call a subset \( C_S \) of \( F_S^{d+1} \setminus \{0\} \) admissible if any point \( X \) in the intersection

\[
F^{d+1} \cap (C_S \times (\mathcal{O}^S)^{d+1})
\]
satisfies \( |X_{\text{min}}|_v = 1 \) for all \( v \notin S \). Assume moreover that \( S \) contains the places over which \( G \) and \( \zeta \) are ramified, and that \( Z(A) = Z(F)Z_SZ(\mathcal{O}^S) \). Then we call a subset \( \Delta_S \) of \( G_S \) admissible if \( \mathcal{D}(\Delta_S) \) is admissible in \( F_S^{d+1} \). This implies that

\[
|D(\bar{f})|_v = 1
\]

for all \( \gamma \in G(F) \cap (\Delta_S \times K^S) \) and \( v \notin S \). Also, \( \Delta_S \) is admissible if and only if its projection onto \( G_S = G_S/Z_S \) is admissible. Finally, we recall that \( \mathcal{E}_G^G(G, S, \zeta) \) is the subspace of functions in \( \mathcal{E}_G(G, S, \zeta) \) whose support is admissible. Also, we shall say a subset \( \Delta \) of \( G(A) \) is \( S \)-admissible if for some finite set \( S \) there is an admissible subset \( C_S \) of \( F_S^{d+1} \) such that \( \mathcal{D}(\Delta) \) is contained in \( C_S \times (\mathcal{O}^S)^{d+1} \). We note that it is this condition of admissibility and \( S \)-admissibility that the reductions of \[\text{Art02 Art01 Art03}\] are based upon, rather than the compact support of the test functions \( f \).

Having made these preliminary remarks, we now proceed as follows. Let \( I \) be the invariant linear form on \( \mathcal{E}_G^G(V, \zeta) \) obtained in [Won22 Theorem 2]. If \( G \) is arbitrary, we define an endoscopic linear form inductively by setting

\[
I^E(f) = \sum_{G' \in \mathcal{E}_G^G(G, V)} \iota(G, G')\hat{S}(f')
\]

for stable linear forms \( \hat{S}' = \hat{S}^G \) on \( \mathcal{E}_G^G(G, V, \zeta) \). In the case that \( G \) is quasisplit, we define a linear form

\[
S^G(f) = I(f) - \sum_{G' \in \mathcal{E}_G^G(G, V)} \iota(G, G')\hat{S}(f')
\]

and also the endoscopic linear form by the trivial relation

\[
I^E(f) = I(f).
\]

We assume inductively that if \( G \) is replaced by a quasisplit inner \( K \)-form of \( \bar{G} \), the corresponding analogue of \( S^G \) is defined and stable. At this stage, the reductions of \[\text{Art02 Art01}\] can now be applied without difficulty. In particular, if on the geometric side, we define \( I_{\text{orb}}^E(f) \) and \( S^G_{\text{orb}}(f) \) to be the summands corresponding to \( M = G \) in \( I^E(f) \) and \( S^G(f) \) respectively, we see from the proof of [Art02 Theorem 10.1] that if \( G \) is arbitrary,

\[
I^E(f) - I_{\text{orb}}^E(f) = \sum_{M \in \mathcal{E}_G^G} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M, V, \zeta)} a^M,\gamma(\gamma)I_M^E(\gamma, f)
\]

and if \( G \) is quasisplit, we have that \( S^G(f) - S^G_{\text{orb}}(f) \) is equal to

\[
\sum_{M \in \mathcal{E}_G^G} |W_0^M||W_0^G|^{-1} \sum_{M' \in \mathcal{E}_G^G(M, V)} \iota(M, M') \sum_{\delta' \in \Delta(M', V, \zeta)} b^{\tilde{M}}(\delta')S^G_M(M', \delta', f).
\]

Here \( I_M^E(\gamma, f) \) and \( S^G_M(M', \delta', f) \) are the local geometric distributions defined in \[\text{Art02 §6}\]. While on the spectral side, we define \( I_{\text{unit}}^E(f) \) and \( S^G_{\text{unit}}(f) \) using the
decomposition according to the norm of the archimedean infinitesimal character,

\[ I^ε(f) = \sum_{i \geq 0} I^ε_i(f) \]

and

\[ S^G(f) = \sum_{i \geq 0} S^G_i(f). \]

It follows then from the proof of [Art02, Theorem 10.6] that if \( G \) is arbitrary,

\[ I^ε_i(f) - I^ε_{i,\text{unit}}(f) = \sum_{M \in \mathcal{L}^0} |W_0^M||W_0^G|^{-1} \int_{\Pi^ε(M,V,ζ)} a^{M,ε}(π)I^ε_{i}(π, f)dπ \]

and if \( G \) is quasisplit, we have that \( S^G_i(f) - S^G_{i,\text{unit}}(f) \) is equal to

\[ \sum_{M \in \mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{M' \in \mathcal{L}^0} \iota(M, M') \int_{\Phi_π(M', V, ζ)} b^{M'}(φ')S^G_{i}(M', φ', f)dφ'. \]

Here again \( I^ε_{M}(π, f) \) and \( S^G_{M}(M', φ', f) \) are the local spectral distributions defined in [Art02 §6]. These identities reduce the study of the global geometric coefficients \( a^{G,ε}(γ), b^{G}(δ) \) and global spectral coefficients \( a^{G,ε}(π), b^{G}(φ) \) to the terms \( M = G \) in their expansion, namely \( a^{G,ε}_{\text{ell}}(γ), b^{G}_{\text{ell}}(δ) \) and \( a^{G,ε}_{\text{disc}}(π), b^{G}_{\text{disc}}(φ) \) respectively by the arguments of Propositions 10.3 and 10.7 of [Art02]. Moreover, the global descent formula of [Art01 Corollary 2.2] further reduces the study of the global geometric coefficients to unipotent elements. (Our extension of the notion of admissibility is crucial here for the extension of this result, which is a long but straightforward verification.) More precisely, given an admissible element \( ˆγ_S \) in \( Γ^ε(G, S, ζ) \) with Jordan decomposition \( ˆγ_S = c_S ˆδ_S \), we have

\[ a^{G,ε}_{\text{ell}}(γ_S) = \sum_c \sum_α i^{G}(S, c)|\bar{G}_{c,+}(F)|\bar{G}_{c}(F)|^{-1} a^{G,ε}_{\text{ell}}(α) \]

and if \( G \) is quasisplit, given an admissible element \( ˆδ_S \) in \( Δ_{\text{ell}}(G, S, ζ) \) with Jordan decomposition \( ˆδ_S = δ_S ˆβ_S \), we have

\[ b^{G}_{\text{ell}}(δ_S) = \sum_δ \sum_β j^{G}(S, d)|\bar{G}_{d,+}(F)|\bar{G}_{d}(F)|^{-1} b^{G}_{\text{ell}}(β), \]

where \( G_{c,+} \) denotes the centralizer of \( c \) in \( G \), and \( G_c \) is the identity component of \( G_{c,+} \). We refer the reader to [Art01] for complete definitions of these expressions.

Turning to the local setting, the analogues of Local Theorems 1 and 2 of [Art02 §6] for \( f \in \mathcal{C}^0(G, V, ζ) \) follow from the analogues of Local Theorems 1’ and 2’, which concern the compound linear forms \( I^ε_{M}(γ, f), S^G_{M}(M', β', f) \) and \( I^ε_{M}(π, f), S^G_{M}(M', φ', f) \) as a consequence of the geometric and spectral splitting and descent formulae respectively [Art02 Propositions 6.1 and 6.3]. The required geometric formulae are given in Chapter [10] whereas the spectral formulae can be deduced from [MW16b X.4].

To apply the arguments of [Art03], we recall the constructions of various subspaces of the Hecke space \( \mathcal{H}(G, V, ζ) \) used in Chapter [7]. If \( G \) is quasisplit, we define the unstable subspace

\[ \mathcal{C}_{\text{uns}}^0(G, V, ζ) \]
of functions \( f \in \mathcal{C}^0(G, V, \zeta) \) such that \( f^G = 0 \). It is spanned by functions \( f = \prod_{v \in V} f_v \) such that for some \( v \in V \), \( f_v \) satisfies the property that \( f_v^G = 0 \). We shall also define the subspace
\[
\mathcal{C}^0_M(G, V, \zeta)
\]
of functions \( f \in \mathcal{C}(G, V, \zeta) \) such that \( f_v \) is \( M \)-cuspidal at two places \( v \in V \). Recall that \( f_v \in \mathcal{C}^0(G_v, \zeta_v) \) is said to be \( M \)-cuspidal if \( f_v(L_v) = 0 \) for any element \( L_v \in \mathcal{L}_v \) that does not contain a \( G_v \)-conjugate of \( M_v \). If \( v \) is a nonarchimedean place, we define
\[
\mathcal{C}(G_v, \zeta_v)^{00}
\]
to be the subspace of functions \( f \in \mathcal{C}^0(G_v, \zeta_v) \) such that \( f_vG(z_v\alpha_v) = 0 \) for any \( z_v \) in the center of \( G_v = G_v/Z_v \) and \( \alpha_v \) in the basis \( R_{\text{unip}}(G_v, \zeta_v) \) of unipotent orbital integrals in [Art03] §3. We lastly define
\[
\mathcal{C}(G_v, \zeta_v)^{0}
\]
analogously, with \( \alpha_v \) ranging over the parabolic subset \( R_{\text{unip,par}}(G_v, \zeta_v) \). We also write \( \mathcal{C}(G, V, \zeta)^{0} \) for the product of functions \( f_v \in \mathcal{C}(G_v, \zeta_v)^{0} \) for \( v \in V \), and similarly for \( \mathcal{C}(G, V, \zeta)^{00} \). We shall denote by the intersections of these various spaces by using overlapping notation, for example, we write \( \mathcal{C}^0_M(G_v, \zeta_v) = \mathcal{C}^0_M(G_v, \zeta_v) \cap \mathcal{C}(G_v, \zeta_v)^{0} \).

The remainder of the proof proceeds by a double induction on integers \( r_{\text{der}} \) and \( d_{\text{der}} \) such that
\[
0 < r_{\text{der}} < d_{\text{der}}.
\]
Namely, we assume inductively that Local Theorem 1 holds if \( \dim(G_{\text{der}}) < d_{\text{der}} \) and if
\[
\dim(G_{\text{der}}) = d_{\text{der}}, \quad \dim(A_M \cap G_{\text{der}}) < d_{\text{der}}
\]
for a local non-archimedean field; the archimedean transfer for \( f \in \mathcal{C}(G, \zeta) \) follows from [Art08] Theorem 1.1. We assume inductively that Global Theorems 1 and 2 hold if \( \dim(G_{\text{der}}) < r_{\text{der}} \). In both local and global cases, we assume that if \( G \) is not quasisplit and \( \dim(G_{\text{der}}) = d_{\text{der}} \), the relevant theorems hold for the quasisplit inner \( K \)-form of \( G \).

We will be content with recapitulating the broad strokes of the arguments in [Art03], indicating the points in which we use the new spaces that we have defined above instead of the compactly supported ones. We shall use the subscripts ‘unip’ to denote the unipotent variant of objects with subscript ‘ell,’ and ‘par’ with the objects corresponding to terms \( M \neq G \). For example, we write
\[
I_{\text{unip}}(f, S) = \sum_{\alpha \in \Gamma_{\text{unip}}(G, V, \zeta)} a_{\text{unip}}^G(\alpha, S) f_G(\alpha)
\]
where
\[
a_{\text{unip}}^G(\alpha, S) = \sum_{k \in \mathcal{K}^0_{\text{unip}}(G, S)} a_{\text{unip}}^G(\alpha \times k) r_G(k)
\]
for \( \alpha \in \Gamma_{\text{unip}}(G, V, \zeta) \). By the inductive definitions we obtain \( I_{\text{unip}}^G \) and \( S_{\text{unip}}^G \) analogously. The global induction hypothesis then implies that
\[
I_{\text{par}}^G(f) - I_{\text{par}}(f) = \sum_t (I_{t, \text{disc}}^G(f) - I_{t, \text{disc}}(f)) - \sum_z (I_{\text{unip}}^G(f, S) - I_{t, \text{unip}}(f, S))
\]
and if $G$ is quasisplit and $f$ belongs to $\mathcal{E}_{\text{uns}}^0(G,V,\zeta)$, then

$$S^G_{\text{par}}(f) = \sum_t S^G_{t,\text{disc}}(f) - \sum_z S^G_{t,\text{unip}}(f,S),$$

where $z$ belongs to the quotient $Z(G)_{V,\sigma}Z_{V}/Z_{V}$, and $Z(G)_{V,\sigma}$ is the subgroup of elements in $(Z(G))(F)$ such that for every $v \not\in V$, the element $z_v$ is bounded in $(Z(G))(F_v)$. The induction hypotheses further lead to a cancellation of $p$-adic singularities, allowing us to express

$$I^G_{\text{par}}(f) - I_{\text{par}}(f) = |W(M)|^{-1} \hat{I}^M(\varepsilon_M(f))$$

for $f$ in the intersection $\mathcal{E}_M(G,V,\zeta)^0$. and if $G$ is quasisplit,

$$S^G_{\text{par}}(f) = |W(M)|^{-1} \sum_{M' \in \text{der}(M,v)} \iota(M,M') \hat{S}^M(\varepsilon_M'(f))$$

for $f$ in the intersection $\mathcal{E}_M^\text{uns}(G,V,\zeta)^0$ as in [Art03 Corollary 3.3]. Here $\varepsilon_M$ is a map from $\mathcal{E}_M^\text{uns}(G_v,\zeta_v)^0$ to the subspace of cuspidal functions in $I^\mathcal{E}_M^\text{uns}(M_v,\zeta_v)$ such that

$$\varepsilon_M(f_v,\gamma_v) = I^G_M(\gamma_v, f_v) - I_M(\gamma_v, f_v)$$

for any $\gamma_v \in \Gamma(M_v,\zeta_v)$, and in the case that $G_v$ is quasisplit, $\varepsilon_M$ is a map from $\mathcal{E}_M^\text{uns}(G_v,\zeta_v)^0$ to $S^\mathcal{E}_M^\text{uns}(M_v,\zeta_v)$ such that

$$\varepsilon_M^\text{unip}(f_v,\delta_v) = S^G_M(\delta_v, f_v)$$

for any $\delta_v \in \Delta(M_v,\zeta_v)$. These maps are given in [Art03 Proposition 3.1], also studied in Chapters VIII and IX of [MW16b].

The separation of the spectral sides according to infinitesimal character follows from [Art03 §4–5] and the properties of the function spaces we have defined, but is not strictly necessary given the absolute convergence of the spectral side. On the other hand, the stabilization of the invariant local trace formula in [Art02 §10] and [Art03 §6] extends to our setting following Lemma 9.1 and together with the global results above lead to the proof of Local Theorem 1 in the nonarchimedean case, again using the local and global induction hypotheses. We note that this implies Local Theorem 2 according to an unpublished work of Arthur, and we may also refer to sections X.5 and X.7 of [MW16b] for a variant argument that can be used instead.

To complete the global theorems, we apply the local theorems to conclude that

$$I^G_{\text{par}}(f) - I_{\text{par}}(f) = \sum_{M \in \mathcal{L}_0} |W^G_0||W^G_0|^{-1} \sum_{\gamma \in \Gamma(M,V,\zeta)} a^M(\gamma)(I^G_M(\gamma,f) - I_M(\gamma,f))$$

vanishes for $\mathcal{E}(G,V,\zeta)$, and if $G$ is quasisplit, that

$$S^G_{\text{par}}(f) = \sum_{M \in \mathcal{L}_0} |W^G_0||W^G_0|^{-1} \sum_{\delta^* \in \Delta(M^*,V,\zeta^*)} b^{M^*}(\delta^*) S^G_M(M^*,\delta^*,f)$$

vanishes for $f \in \mathcal{E}_{\text{uns}}(G,V,\zeta)$. The induction argument on $r_{\text{der}}$ implies that the terms $I^G_{t,\text{disc}}(f) - I_{t,\text{disc}}(f)$ and $S^G_{t,\text{disc}}(f)$ vanish for $f$ in $\mathcal{E}(G,V,\zeta)$ and $\mathcal{E}_{\text{uns}}(G,V,\zeta)$ respectively, so that

$$\sum_z \left( I^G_{t,\text{unip}}(f,S) - I_{t,\text{unip}}(f,S) \right) = 0, \quad f \in \mathcal{E}(G,V,\zeta)$$
and
\[ \sum_z S^G_{z, \text{unip}}(f, S) = 0, \quad f \in \mathcal{C}^\circ_{\text{uns}}(G, V, \zeta) \]
in the case that $G$ is quasisplit. Choosing $V = S$, and using the property that the linear forms
\[ \hat{f}_S \rightarrow \hat{f}_{S,G}(z\hat{\alpha}_S), \quad z \in Z(G)_{S,P}, \quad \hat{\alpha}_S \in \Gamma_{\text{unip}}^\varnothing(G, S, \zeta) \]
on the subspace of admissible functions in $\mathcal{C}^\circ_{\text{unip}}(G, S, \zeta)$, we conclude from the definitions of $I^\varnothing_{\text{unip}}$ and $I^\varnothing_{t, \text{unip}}$ that
\[ a^G_{\text{all}}(\hat{\alpha}_S) - a^G_{\text{all}}(\hat{\alpha}_S) = 0 \]
for $\hat{\alpha}_S \in \Gamma_{\text{unip}}^\varnothing(G, S, \zeta)$, and similarly
\[ \hat{f}_S \rightarrow \hat{f}_{S,G}(z\hat{\beta}_S), \quad z \in Z(G)_{S,P}, \quad \hat{\beta}_S \in \Delta_{\text{unip}}^\varnothing(G, S, \zeta) \setminus \Delta_{\text{unip}}(G, S, \zeta) \]
on the subspace of admissible functions in $\mathcal{C}^\circ_{\text{uns}}(G, S, \zeta)$ are linearly independent, whence we conclude that
\[ b^G_{\text{all}}(\hat{\beta}_S) = 0 \]
for $\hat{\alpha}_S$ in the complement of $\Delta_{\text{unip}}(G, S, \zeta)$ in $\Delta_{\text{unip}}^\varnothing(G, S, \zeta)$. Applying the global descent formula to the coefficients then yields the geometric Global Theorem 1. The spectral Global Theorem 2 follows similarly, using the vanishing of
\[ \sum_t \left( I^\varnothing_{t, \text{disc}}(f) - I^\varnothing_{t, \text{disc}}(f) \right) = 0, \quad \hat{f} \in \mathcal{C}^\circ(G, \zeta) \]
and
\[ \sum_t S^G_{t, \text{disc}}(f) = 0, \quad \hat{f} \in \mathcal{C}^\circ_{\text{uns}}(G, \zeta). \]
Arguing as in the geometric case we conclude that
\[ a^G_{\text{disc}}(\hat{\pi}) - a^G_{\text{disc}}(\hat{\pi}) = 0 \]
for any $\hat{\pi} \in \Pi_{t, \text{disc}}(G, \zeta)$, and in the case that $G$ is quasisplit,
\[ b^G_{\text{disc}}(\hat{\phi}) = 0 \]
for any $\hat{\phi}$ in the complement of $\Phi_{t, \text{disc}}(G, \zeta)$ in $\Phi_{t, \text{disc}}^\varnothing(G, \zeta)$, and the desired result follows.

Finally, we can conclude from these general remarks the extension of the endoscopic and stable trace formulae, with the required expansions
\[ I^\varnothing(f) = \sum_{M \in \mathcal{L}} |W^M_0||W^G_0|^{-1} \sum_{\gamma \in \Gamma^\varnothing(M, V, \zeta)} a^M_{\gamma^\varnothing}(\gamma) I_M^\varnothing(\gamma, f) \]
\[ = \sum_t \sum_{M \in \mathcal{L}} |W^M_0||W^G_0|^{-1} \int_{\Gamma^\varnothing_1(M, V, \zeta)} a^M_{\gamma^\varnothing}(\pi) I_M^\varnothing(\pi, f) d\pi \]
and
\[ S(f) = \sum_{M \in \mathcal{L}} |W^M_0||W^G_0|^{-1} \sum_{\delta \in \Delta(M, V, \zeta)} b^M(\delta) S_M(\delta, f) \]
\[ = \sum_t \sum_{M \in \mathcal{L}} |W^M_0||W^G_0|^{-1} \int_{\Phi_1(M, V, \zeta)} b^M(\phi) S_M(\phi, f) d\phi \]
for $f \in \mathcal{C}^\circ(G, V, \zeta)$. \qed
We state the local consequences explicitly, which we shall require in the stabilization of the form $I^r_\ast(f)$. We record them separately according to the geometric and spectral sides, respectively.

**Corollary 9.3.** Let $F$ be a number field, and let $V$ be a finite set of valuations $V_{\text{ram}}(G,\zeta)$.

(a) If $G$ is arbitrary,

$$I^\xi_M(\gamma, f) = I_M(\gamma, f), \quad \gamma \in \Gamma(M_V^2, \zeta_V), f \in \mathcal{C}_\infty(G, V, \zeta).$$

(b) Suppose that $G$ is quasisplit, and that $\delta'$ belong to $\Delta((\tilde{M}_V')^2, \tilde{\zeta}_V')$ for some $M' \in \mathcal{E}_{\text{ell}}(M, V)$. Then the linear form

$$f \to S^G_M(M', \delta', f), \quad f \in \mathcal{C}_\infty(G, V, \zeta)$$

vanishes unless $M = M^*$, in which case it is stable.

**Corollary 9.4.** Let $F$ be a number field, and let $V$ be a finite set of valuations $V_{\text{ram}}(G,\zeta)$.

(a) If $G$ is arbitrary,

$$I^\xi_M(\pi, f) = I_M(\pi, f), \quad \gamma \in \Pi(M_V^2, \zeta_V), f \in \mathcal{C}_\infty(G, V, \zeta).$$

(b) Suppose that $G$ is quasisplit, and that $\phi'$ belong to $\Phi((\tilde{M}_V')^2, \tilde{\zeta}_V')$ for some $M' \in \mathcal{E}_{\text{ell}}(M, V)$. Then the linear form

$$f \to S^G_M(M', \phi', f), \quad f \in \mathcal{C}_\infty(G, V, \zeta)$$

vanishes unless $M = M^*$, in which case it is stable. The desired result follows from the splitting and descent formulas for these distributions, which are proved in this setting in Chapter 10, and are also stated in a different form in [MW16b].

**Proof.** The statements concern the compound linear forms that arise on either side of the endoscopic and stable trace formulas. In the proof of Theorem 9.2 we established the analogues of the local theorems in [Art02] which can be stated for a local field $F$ and test functions $f \in \mathcal{C}_\infty(G, \zeta)$. The result will follow from the necessary splitting and descent formulas in the same manner as for functions $f \in \mathcal{H}(G, V, \zeta)$ described in [Art02, §6]. On the geometric side, we conclude that if $G$ is arbitrary, then

$$I^\xi_M(\gamma, f) = I_M(\gamma, f), \quad \gamma \in \Gamma_{G-\text{reg},\text{ell}}(M, \zeta), f \in \mathcal{C}_\infty(G, \zeta)$$

whereas if $G$ is quasisplit and $\delta'$ belongs to $\Delta_{G-\text{reg}}(\tilde{M}_V', \tilde{\zeta}_V')$ for some $M' \in \mathcal{E}_{\text{ell}}(M)$, then the linear form

$$f \to S^G_M(M', \delta', f), \quad f \in \mathcal{C}_\infty(G, \zeta)$$

vanishes unless $M' = M^*$, in which case it is stable. The desired result follows from the splitting and descent formulas for these distributions, which are proved in this setting in Chapter 10 and are also stated in a different form in [MW16b].

Similarly, on the spectral side, we conclude from the proof of Theorem 9.2 that if $G$ is arbitrary, then

$$I^\xi_M(\pi, f) = I_M(\pi, f), \quad \pi \in \Pi_{G-\text{reg},\text{ell}}(M, \zeta), f \in \mathcal{C}_\infty(G, \zeta)$$

whereas if $G$ is quasisplit and $\phi'$ belongs to $\Phi_{G-\text{reg}}(\tilde{M}_V', \tilde{\zeta}_V')$ for some $M' \in \mathcal{E}_{\text{ell}}(M)$, then the linear form

$$f \to S^G_M(M', \phi', f), \quad f \in \mathcal{C}_\infty(G, \zeta)$$
vanishes unless $M' = M^*$, in which case it is stable. The desired result follows from
the splitting and descent formulas for these distributions, which second of which is
stated in this setting in Chapter 10, and are also stated in a different form in
[MTW16].

\[ \square \]

10. Splitting and descent formulas

Recall the family of stable distributions $S_{G}^{M}(M', \delta', f)$ for any $\delta' \in \Delta^G(\widetilde{M}', \widetilde{\zeta}')$ defined as in [Art02] inductively by the formula

\[ I_{M}^{G}(\delta', f) = \sum_{G' \in E_{M}^{G}(G)} \iota_{M'}(G, G')S_{M'}^{G'}(\delta', f) + \varepsilon(G)S_{M}^{G}(M', \delta', f'), \]

where $\varepsilon(G) = 1$ if $G$ is quasisplit and 0 otherwise. We also require that $I_{M}^{G}(\delta', f) = I_{M}(\delta, f)$ if $G$ is quasisplit and $\delta'$ maps to $\delta \in \Delta^G(M, \zeta)$. If $G$ is quasisplit and $M' = M^*$, we set $S_{G}^{M}(\delta, f) = S_{G}^{M}(M^*, \delta', f)$. The distributions $S_{G}^{M}(\delta, f)$ are first constructed as stable linear forms on a connected reductive group, and then extended to a product of several copies of $G$. It is these compound forms that occur in the stable trace formula. They satisfy splitting formulas, and if $\delta$ belongs to a proper Levi subgroup, descent formulas that reduce the study of the compound distributions to the special case of the simple ones, in which $\delta$ is elliptic in the corresponding Levi subgroup. To state the result, we first require some definitions. Let $V = V_{1} \sqcup V_{2}$ be a finite set of places of $F$, and fix a function $f_{V} \in \mathcal{C}(G_{V}, \zeta_{V})$ where

$\hat{f}_{V} = \hat{f}_{V_{1}} \times \hat{f}_{V_{2}}, \quad f_{V_{i}} \in \mathcal{C}(G_{V_{i}}, \zeta_{V_{i}})$.

Given Levi subgroups $L_{1}, L_{2} \in \mathcal{L}(M)$, we define

$e_{M}^{G}(L_{1}, L_{2}) = d_{M}^{G}(L_{1}, L_{2})|Z(\hat{L}_{1})^{F} \cap Z(\hat{L}_{2})^{F}/Z(\hat{G})^{F}|^{-1}$.

If $d_{M}^{G}(L_{1}, L_{2})$ is nonzero, then $a_{L_{1}} \cap a_{L_{2}} = a_{G}$, so the identity component of $Z(\hat{L}_{1})^{F} \cap Z(\hat{L}_{2})^{F}$ is equal to that of $Z(\hat{G})^{F}$, therefore $e_{M}^{G}(L_{1}, L_{2})$ is also nonzero. The same also holds true for $e_{M}^{G}(M, L_{2})$.

Moreover, let $M_{1}$ be a proper Levi subgroup of $M$, with a fixed dual subgroup $\tilde{M}_{1} \subset M$. Any Levi subgroup $L_{1} \in \mathcal{L}(M_{1})$ comes with a dual Levi subgroup $\tilde{L}_{1} \subset \tilde{G}$ containing $\tilde{M}_{1}$, by which we define the stable coefficient

\[ e_{M_{1}}^{G}(M, L_{1}) = \sum_{L_{1} \in \mathcal{L}(M_{1})} d_{M}^{G}(M, L_{1})|Z(\hat{M})^{F} \cap Z(\hat{L_{1}})^{F}/Z(\hat{G})^{F}|^{-1} \]

where the coefficient $e_{M_{1}}^{G}(M, L_{1})$ is defined as in [Art99b].

The formulas are a direct consequence of the splitting and descent formulas for $S_{G}^{M}(\delta, f)$. For $\delta$ strongly $G$-regular, they are given by Theorem 6.1 and Theorem 7.1 of [Art99b], and provide a decomposition of the compound distributions over $\Delta((\tilde{M}_{1})^{Z'}, \tilde{\zeta}_{1})$ into simple linear forms over $\Delta_{G-reg}(\tilde{M}', \tilde{\zeta}')$. For the general case, the formal structure of the arguments are similar to that for strongly $G$-regular elements. We provide the argument here for the singular distributions $S_{G}^{M}(\delta, f)$. The general descent formulas are described in unpublished work of Arthur, and we shall provide some of the arguments here.

**Proposition 10.1.** (a) Suppose $G$ is arbitrary. Then we have the descent formula

\[ I_{M}^{G}(\delta', f) = \sum_{L_{1} \in \mathcal{L}(M_{1})} d_{M_{1}}^{G}(M, L_{1})\hat{I}_{M_{1}}^{L_{1}}(\delta, f^{L_{1}}).\]
(b) Suppose that $G$ is quasisplit, and that $M' = M^*$. Then

\begin{equation}
S_M^G(\delta, f) = \sum_{L_1 \in \mathcal{L}(M_1)} e_{M_1}^G(M, L_1) \hat{S}_{M_1}^L(\delta_1, f^{L_1}).
\end{equation}

for $\delta = \delta', \delta_1 = \delta_1'$. If $M' \neq M^*$, then the distribution $S_M^G(M', \delta', f)$ vanishes.

Proof. For any $\delta' \in \Delta(\hat{M}', \hat{\xi}', \hat{\zeta}', M' \in \mathcal{E}_{\ell}(M, V), f \in \mathcal{H}(G, V, \zeta)$, we view the stable linear forms $S_M^G(M', \delta', f)$ as an element of $\mathcal{A}(G, V, \zeta)$. We shall first show the descent formula \[10.3\]. We begin with the difference

\begin{equation}
I_M^G(\delta', f) - \varepsilon(G) S_M^G(M', \delta', f') = \sum_{G' \in \mathcal{E}_{M'}(G)} \iota_M(G, G') \hat{S}_{M}^{G'}(\delta', f).
\end{equation}

Applying \[10.3\] inductively to the terms in the right-hand sum, we write

\begin{equation}
\hat{S}_{M_1}(\delta', f') = \sum_{L_1' \in \mathcal{L}(M_1')} e_{M_1}^{G'}(M', L_1') \hat{S}_{M_1}^L(\delta_1', f_1').
\end{equation}

where $f_1' = (f')^{L_1}$. We can assume that the endoscopic datum $G'$ for $G$ is elliptic. There is a canonical Levi subgroup $G' \in \mathcal{L}(M_1)$ for which $L_1'$ is an elliptic endoscopic datum. On the other hand, if $L_1$ is given, then $L_1'$ is uniquely determined by $G'$ and there is a mapping $G' \to L_1'$ from $\mathcal{E}_{M'}(G)$ to $\mathcal{E}_{M_1}(L_1)$, sending $s' \in s_M^G Z(\hat{M})^F / Z(\hat{G})^F$ to the point $s_{M_1}^G Z(\hat{M}_1)^F / Z(\hat{L}_1)^F$. We may therefore write \[10.4\] as the sum

\begin{equation}
\sum_{L_1 \in \mathcal{L}(M_1)} \sum_{G' \in \mathcal{E}_{M'}(G)} \iota_M(G, L_1') e_{M_1}^{G'}(M', L_1') \hat{S}_{M_1}^L(\delta_1', f_1).
\end{equation}

If we let $\varepsilon_M(G) = 1$ if $G$ is quasisplit and $M' = M^*$, and 0 otherwise, then we can replace the inner sum above with the complete set $\mathcal{E}_{M'}(G)$ if we also subtract the contribution

\begin{equation}
\varepsilon_M(G) \sum_{L_1 \in \mathcal{L}(M_1)} e_{M_1}^{G}(M, L_1) \hat{S}_{M_1}^L(\delta_1, f^{L_1}).
\end{equation}

Then substituting \[10.2\], and using the fact that $e_{M_1}^{G'}(M', L_1') = e_{M_1}^{G}(M, L_1)$, we conclude that \[10.4\] is equal to the difference of the sum

\begin{equation}
\sum_{L_1 \in \mathcal{L}(M_1)} d_{M_1}^{G}(M, L_1) \sum_{G' \in \mathcal{E}_{M'}(G)} \frac{|Z(\hat{M})^F / Z(\hat{G})^F|}{|Z(\hat{M})^F \cap Z(\hat{L})^F / Z(\hat{G})^F|} \hat{S}_{M_1}^L(\delta_1, f_1')
\end{equation}

and the expression \[10.5\].

The sum over $G' \in \mathcal{E}_{M'}(G)$ depends only on the image of $G'$ in $\mathcal{E}_{M_1}(G)$, so we may replace the sum if we multiply by the order of the preimage of $L_1' \in \mathcal{E}_{M_1}(G)$ in $\mathcal{E}_{M'}(G)$. Also, by [Art99b] Lemma 1.1, we have $Z(\hat{M}_1) = Z(\hat{M})^F Z(\hat{L}_1)^F$. It then follows that $\mathcal{E}_{M'}(G)$ maps surjectively onto $\mathcal{E}_{M_1}(L_1)$, and the finite group $Z(\hat{M})^F \cap Z(\hat{L})^F / Z(\hat{G})^F$ acts simply transitively on the fibres. Therefore the order of this preimage is equal to the order of the finite group. For the resulting product of coefficients, observe that there is a surjective map

\[
(Z(\hat{M})^F / Z(\hat{G})^F) \times (Z(\hat{L}_1)^F / Z(\hat{G})^F) \to Z(\hat{M}_1)^F / Z(\hat{M}_1)^F
\]
whose kernel is isomorphic to the quotient
\[
(Z(M')^\Gamma \cap Z(\hat{L}_i^\Gamma))/(Z(M)^\Gamma \cap Z(\hat{L}_i^\Gamma)).
\]

It follows then that
\[
|Z(M')^\Gamma / Z(M)^\Gamma||Z(\hat{L}_i^\Gamma)\cap Z(\hat{G})^\Gamma|^{-1}|Z(M)^\Gamma \cap Z(\hat{L}_i^\Gamma) / Z(\hat{G})^\Gamma|
= |Z(\hat{L}_i^\Gamma) / Z(\hat{L}_i^\Gamma)|^{-1}|Z(M')^\Gamma / Z(M)^\Gamma| = \iota_{M'_i}(L_1, M_1).
\]

We can therefore write (10.6) as
\[
\sum_{L_1 \in Z(M_1)} d_{M_1}^G(M, L_1) \sum_{L'_1 \in E_{M_1}(G)} \iota_{M'_i}(L_1, M_1) \hat{S}_{M'_i}^{L'_i} (\delta'_1, f'_1),
\]
which is equal to
\[
\sum_{L_1 \in Z(M_1)} d_{M_1}^G(M, L_1) \hat{I}_{M'_i}^{L'_i} (\delta'_1, f_{L_1})
\]
by the inductive definition (10.1). In the case \(\varepsilon(G) = 1\), it follows then that if we let \(\delta\) and \(\delta_1\) be the images of \(\delta'\) and \(\delta'_1\) in \(E(G, \zeta)\) and \(E(M, \zeta)\) respectively, we obtain
\[
I_M(\delta, f) = \sum_{L_1 \in Z(M_1)} d_{M_1}^G(M, L_1) \hat{I}_{M'_i}^{L'_i} (\delta_1, f_{L_1}),
\]
and the descent formula (10.3) follows from the identity between \(S_{M}^G(M', \delta', f')\) in (10.4) and (10.5). \(\Box\)

We now turn to the splitting formula.

**Proposition 10.2.** (a) Suppose that \(G\) is arbitrary, and that \(\gamma_V = (\gamma_{V_1}, \gamma_{V_2}) \in \Gamma_{G_V}(M_V)\). Then
\[
I_M^{G, \varepsilon}(\gamma_V, f_{V_1}) = \sum_{L_1, L_2 \in Z(M)} d_{L_1}^G(L_1, L_2) \hat{I}_{M'_i}^{L'_i} (\gamma_{V_1}, f_{V_1}) \hat{I}_{M'_i}^{L'_i} (\gamma_{V_2}, f_{V_2}).
\]
(b) Suppose that \(G\) is quasisplit, and that \(\gamma_V = (\delta_{V_1}, \delta_{V_2}) \in \Delta_{G_V}(M_V)\). Then
\[
S_M^{G}(\delta_V, f_{V_2}) = \sum_{L_1, L_2 \in Z(M)} e_{L_1}^G(L_1, L_2) \hat{S}_{M'_i}^{L'_i} (\delta_{V_1}, f_{V_1}) \hat{S}_{M'_i}^{L'_i} (\delta_{V_2}, f_{V_2}).
\]

**Proof.** We shall proceed as in the proof of the descent formula. Beginning again with the difference
\[
I_M^{G, \varepsilon}(\delta_V', f_{V_1}) - \varepsilon(G) S_M^{G}(M'_V, \delta_V', f_{V_2}') = \sum_{G'_V \in \mathcal{E}_{M'_V}(G_V)} \iota_{M'_V}(G'_V, \delta_V', f_{V_2}'),
\]
we apply (10.7) inductively to the right-hand sum to get
\[
\hat{S}_{M'_i}^{G'}(\delta'_V, f) = \sum_{L'_1, L'_2 \in Z(M'_V)} e_{L'_1}^{G'}(L'_1, L'_2) \hat{S}_{M'}^{L'_i} (\delta'_V, f_{V_1}') \hat{S}_{M'}^{L'_i} (\delta'_V, f_{V_2}'),
\]
where \(f_{V_i}' = (f')_{V_i}'\), \(i = 1, 2\). We may assume as before that \(G'\) is elliptic. As Levi subgroups of \(G'\), there are a canonical Levi subgroups \(L_i \in Z(M), i = 1, 2\) for which \(L'_i\) is an elliptic endoscopic datum. To switch the sum over \(G'_V\) with the sums over \(L'_1, L'_2\), we observe that there is a natural map of endoscopic data
\[
\mathcal{E}_{M'_i}(G_V) \to \mathcal{E}_{M'_V}(L_{1, V_1}) \times \mathcal{E}_{M'_V}(L_{2, V_2})
\]
given by \( G' \rightarrow (L'_1, L'_2) \). If \( G' \) corresponds to \( s'_{L_i} \in S'_{M, V} Z(\hat{M})^\Gamma / Z(\hat{G})^\Gamma \), then \((L'_1, L'_2)\) corresponds to the element
\[
(s'_{L_1}, s'_{L_2}), \quad s'_{L_i} \in S'_{M, V} Z(\hat{M})^\Gamma / Z(\hat{L}_i)^\Gamma, \quad i = 1, 2,
\]
by projecting \( s'_{L_i} \) onto each factor \( S'_{M, V} Z(\hat{M})^\Gamma / Z(\hat{L}_i)^\Gamma \). It follows then that we may write the sum in (10.8) as
\[
(10.10) \quad \sum_{L_1, L_2 \in \mathcal{L}(M)} \sum_{G' \in \mathcal{E}_{M'}(G_V)} \epsilon_{M'}(L_1, L_2) \delta_{\Gamma, f_{V_1}} \delta_{\Gamma, f_{V_2}}.
\]
Moreover, we may take the inner sum to be over \( \mathcal{E}_{M'}(G) \) instead of \( \mathcal{E}_{M'}(G) \) if we subtract the contribution
\[
\delta_{M'}(G) \quad \sum_{L_1, L_2 \in \mathcal{L}(M)} \epsilon_{M}(L_1, L_2) \delta_{\Gamma, f_{V_1}} \delta_{\Gamma, f_{V_2}}
\]
of \( G' = G^* \), in the case \( G \) is quasisplit.

We can assume that the coefficient \( \epsilon_{M'}(L_1, L_2) \) is nonzero, otherwise the summand indexed by \( L_1, L_2 \) would vanish. Therefore \( d_{M'}^G(L_1, L_2) = d_{M}^G(L_1, L_2) \) is nonzero, and it follows that the connected component \( (Z(\hat{M})^\Gamma)^0 \) is equal to the product of \( (Z(\hat{L}_1)^\Gamma)^0 \) with \( (Z(\hat{L}_2)^\Gamma)^0 \). On the other hand, \( Z(\hat{M})^\Gamma = (Z(\hat{M})^\Gamma)^0 Z(\hat{G})^\Gamma \), so by \cite{Art99b} Lemma 1.1, we have again \( Z(\hat{M})^\Gamma = Z(\hat{L}_1)^\Gamma Z(\hat{L}_2)^\Gamma \). It follows that the map (10.9) is surjective. Moreover, the finite group
\[
(10.11) \quad Z(\hat{L}_1)^\Gamma \cap Z(\hat{L}_2)^\Gamma / Z(\hat{G})^\Gamma
\]
acts simply transitively on the fibres. The sum over \( G' \) in (10.10) depends only on \((L'_1, L'_2)\), therefore we can replace the sum over \( \mathcal{E}_{M'}(G_V) \) with a sum over \( \mathcal{E}_{M'_1}(L_1, V_1) \times \mathcal{E}_{M'_2}(L_2, V_2) \) if we multiply the summand by the order of the group (10.11). To compute the resulting product of coefficients, we observe that
\[
(Z(\hat{L}_1)^\Gamma / Z(\hat{L}_1)^\Gamma) \times (Z(\hat{L}_2)^\Gamma / Z(\hat{L}_2)^\Gamma) \rightarrow Z(\hat{M})^\Gamma / Z(\hat{M})^\Gamma
\]
is a surjective map with kernel isomorphic to the quotient of \( Z(\hat{L}_1)^\Gamma \cap Z(\hat{L}_2)^\Gamma \) by \( Z(\hat{L}_1)^\Gamma \cap Z(\hat{L}_2)^\Gamma \). It follows that the product is given by
\[
|Z(\hat{M})^\Gamma / Z(\hat{M})^\Gamma| |Z(\hat{L}_1)^\Gamma \cap Z(\hat{L}_2)^\Gamma / Z(\hat{G})^\Gamma| = |Z(\hat{M})^\Gamma / Z(\hat{M})^\Gamma| (Z(\hat{L}_1)^\Gamma / Z(\hat{L}_1)^\Gamma|^{-1} Z(\hat{M})^\Gamma / Z(\hat{M})^\Gamma| Z(\hat{L}_2)^\Gamma / Z(\hat{L}_2)^\Gamma|^{-1})
\]
We can therefore write (10.10) as the sum over \( L_1, L_2 \in \mathcal{L}(M) \) of
\[
d_{M}(L_1, L_2) \prod_{i=1,2} \left( \sum_{L'_i} \epsilon_{M'}(L_i, L'_i) \delta_{\Gamma, f_{V_i}} \right)
\]
where the sum over \( L'_i \) runs over \( \mathcal{E}_{M'_i}(L_1, V_1) \), which is equal to
\[
d_{M}(L_1, L_2) \hat{l}_{M}(L_1, L_2) \delta_{V_1, f_{V_1}, L_1} \delta_{V_2, f_{V_2}, L_2}
\]
by definition.
11. Stable germ expansions

We now turn to the stable germ expansions for orbital integrals developed. Particular attention will be paid to the archimedean setting, since in that case the distributions are more complicated and the expansions are asymptotic rather than exact.

11.1. Invariant germ expansions. We first recall the ordinary germ expansions. Fix a conjugacy class \( c \in \Gamma_{\text{ss}}(\bar{G}) \), and let \( \mathcal{U}_c(G) \) the union of conjugacy classes \( \Gamma_c(G) \) in \( G(F) \) whose semisimple part maps to the conjugacy class of \( c \). Then define

\[
D_c(G, \zeta) = \bigoplus_{c' \in \Gamma_{\text{ss}}(\bar{G})} D_{c'}(G, \zeta)
\]

as the space of singular invariant distributions on \( G \), with a suitably chosen basis \( \Gamma(G, \zeta) = \coprod_{c' \in \Gamma_{\text{ss}}(\bar{G})} \Gamma_{c'}(G, \zeta) \).

The semisimple component of a distribution \( \gamma \in D_c(G, \zeta) \) is the underlying conjugacy class \( c \), whereas the unipotent component \( \alpha \) is a distribution in the space \( D_{\text{unip}}(G_c, \zeta) = D_1(G_c, \zeta) \) and is determined only up to the action of the finite group \( \bar{G}_c(F)/\bar{G}_c(F) \). Note that because of the central datum \((Z, \zeta)\), the Jordan decomposition \( \gamma = c\alpha \) is not canonical, and depends the choice of a suitable section \( c \mapsto \tilde{c} \) from \( \Gamma_{\text{ss}}(\bar{G}) \) to \( G(F) \). We shall assume that such a section has been fixed.

11.1.1. Nonarchimedean case. If \( F \) is nonarchimedean, \( D_c(G, \zeta) \) is finite dimensional and has a basis of singular invariant orbital integrals

\[
f \mapsto f_G(\gamma), \quad \rho \in \Gamma_c(G, \zeta)
\]

taken over classes in \( \Gamma_c(G, \zeta) = \Gamma_c(G) \). We have the ordinary Shalika germ expansion for the invariant orbital integral, decomposing it into a finite linear combination of functions parametrised by conjugacy classes

\[
f_G(\gamma) = \sum_{\rho \in \Gamma_c(G, \zeta)} \rho^\vee(\gamma) f_G(\rho), \quad f \in C_c^\infty(G(F)),
\]

where \( \gamma \) is a strongly regular point close to \( c \), in a sense that depends on \( f \). The terms \( \rho^\vee(\gamma) = g_G^c(\gamma, \rho) \) for each \( \rho \in \Gamma_c(G, \zeta) \) are known as Shalika germs, but are in fact homogeneous functions defined on a fixed neighbourhood of \( c \).

For weighted orbital integrals

\[
J_M(\gamma, f) = |D(\gamma)|^{\frac{1}{2}} \int_{G(\gamma) \backslash G(F)} f(x^{-1}\gamma x) dx,
\]

there is a finite expansion

\[
J_M(\gamma, f) = \sum_{L \in \mathcal{L}(M)} \sum_{\rho \in \Gamma_c(L, \zeta)} g_M^L(\gamma, \rho) J_L(\rho, f)
\]

which follows from [Art88]. The terms \( g_M^L(\gamma, \rho) \) are defined as germs of functions of \( \gamma \) in \( M(F) \cap G_{\text{reg}}(F) \) near to \( c \), and the coefficients \( J_L(\rho, f) \) are singular weighted orbital integrals.
More generally, the functions \( g_M^f \) will belong the space \( \mathcal{H}_c(M, G, \zeta) \), consisting of germs of smooth \( W(M)M(F_v) \)-invariant, \( \zeta^{-1} \)-equivariant functions on invariant neighbourhoods of \( c \) in \( M_{G_{\text{reg}}}(F_v) \). The analogous expansion then also holds for invariant distributions \([\text{Art88a}, (2.5)]\)

\[
I_M(\gamma, f) = \sum_{L \in \mathcal{Z}(M)} \sum_{\rho \in \Gamma_L} g_M^f(\gamma, \rho) I_L(\rho, f).
\]

11.1.2. Archimedean case. If \( F = \mathbb{R} \), then \( \mathcal{H}_c(G, \zeta) \) is infinite dimensional, containing normal derivatives of orbital integrals, as well as more general distributions associated to harmonic differential operators, and has a basis \( R_c(G) \) described in \([\text{Art16} \ \S 1]\). Denote by \( R_c(G, \zeta) \) by the subset of \( \zeta \)-equivariant distributions. We describe the elements of \( \mathcal{H}_c(G, \zeta) \) in further detail. Let \( \mathcal{J}_c(G) \) be a fixed set of representatives of \( G_{c,+}(\mathbb{R}) \)-orbits of maximal tori in \( G_c \) over \( \mathbb{R} \), or equivalently, a fixed set of representatives of the \( G(\mathbb{R}) \)-orbits of maximal tori in \( G \) over \( \mathbb{R} \) that contain \( c \). Denote by \( S_c(G) \) the set of triplets \( \sigma = (T, \Omega, X) \) where \( T \in \mathcal{J}_c(G) \), \( \Omega \in \pi_{0,c}(T_{\text{reg}}(\mathbb{R})) \) the set of connected components of \( T_{\text{reg}}(\mathbb{R}) \) whose closure contains \( c \), and \( X \) is an invariant differential operator on \( T(\mathbb{R}) \). Given and \( f \) in the Schwartz space \( \mathcal{E}(G) \) of \( G(\mathbb{R}) \), by Harish-Chandra we know that the orbital integral

\[
f_G(\gamma), \quad f \in \mathcal{E}(G), \gamma \in \Omega
\]

extends to a continuous linear map from \( \mathcal{E}(G) \) to the space of smooth functions on the closure of \( \Omega \). It follows from this that the limit

\[
f_G(\sigma) = \lim_{\gamma \to c} (Xf_G)(\gamma)
\]

exists and is continuous in \( f \). If \( f \) is compactly supported and vanishes on a neighbourhood of \( \mathcal{H}_c(G) \), then \( f_G(\sigma) = 0 \), therefore the linear form \( f \mapsto f_G(\sigma) \) belongs to \( \mathcal{H}_c(G, \zeta) \), and in fact spans \( \mathcal{H}_c(G, \zeta) \).

We then have the analogous germ expansions due to Arthur \([\text{Art16}]\), where in place of an actual identity, we have an asymptotic formula

\[
f_G(\gamma) \sim \sum_{\rho \in R_c(G, \zeta)} \rho^\vee(\gamma)f_G(\rho), \quad f \in \mathcal{E}(G, \zeta).
\]

As before, the germs \( \rho^\vee(\gamma) \), for each \( \rho \in R_c(G, \zeta) \) can be treated as homogeneous functions of \( \gamma \). For weighted orbital integrals, the germ expansion is formulated in terms of the family germs \( g_M^G(\gamma, \rho) \) which belong to a certain space of formal germs \( \mathcal{H}_c(M, G) \) \([\text{Art16} \ \S 4]\). It consists of germs of smooth \( W(M)M(\mathbb{R}) \)-invariant functions on invariant neighbourhoods of \( c \) in \( M_{G_{\text{reg}}}(\mathbb{R}) \) with bounded growth near the boundary. To incorporate the central character datum \( (Z, \zeta) \), one introduces the corresponding space \( \mathcal{H}_c(M, G, \zeta) \) of formal germs of \( \zeta^{-1} \)-equivariant functions. To do so, we simply restrict the smooth \( W(M)M(\mathbb{R}) \)-invariant functions on invariant neighbourhoods of \( c \) in \( M_{G_{\text{reg}}}(\mathbb{R}) \) above to only \( \zeta^{-1} \)-equivariant ones. Then from \([\text{Art16} \ \text{Theorem 5.1}]\) we have

\[
J_M(\gamma, f) \sim \sum_{L \in \mathcal{Z}(M)} \sum_{\rho \in R_c(L)} g_M^L(\gamma, \rho)J_L(\rho, f)
\]

and the invariant analogue \([\text{Art16} \ \text{Corollary 10.1}]\)

\[
I_M(\gamma, f) \sim \sum_{L \in \mathcal{Z}(M)} \sum_{\rho \in R_c(L)} g_M^L(\gamma, \rho)I_L(\rho, f).
\]
We shall require the quantitative form of this asymptotic expansion. To state it, we first recall that the universal space of formal germs \( \hat{\mathcal{G}}_c(M,G,\zeta) \) is constructed \([Art16 \, \S 4]\) using spaces \( \mathcal{F}_{c,n}^\alpha(V,G,\zeta) \) of \( \zeta^{-1}\)-equivariant functions on \( V_{G,\text{reg}} = V \cap G_{\text{reg}}(\mathbb{R}) \). It is obtained as the direct limit of spaces of formal \( \alpha \)-germs

\[
\hat{\mathcal{G}}^\alpha_c(M,G,\zeta) = \lim_{\alpha} \hat{\mathcal{G}}_c^\alpha(M,G,\zeta)
\]

where \( \alpha \) are weight functions on invariant differential operators \( X \) on a maximal torus \( T \) of \( M \). The formal \( \alpha \)-germs are in turn obtained as a projective limit

\[
\hat{\mathcal{G}}^\alpha_c(M,G,\zeta) = \lim_{n} \mathcal{G}^\alpha_{c,n}(M,G,\zeta)
\]

for \( n \geq 0 \), where

\[
\mathcal{G}^\alpha_{c,n}(M,G,\zeta) = \mathcal{G}^\alpha_c(M,G,\zeta) / \mathcal{G}^\alpha_{c,n}(M,G,\zeta)
\]

is the space of \((\alpha,n)\)-jets for \((M,G)\) at \( c \), and

\[
\mathcal{G}^\alpha_{c,n}(M,G,\zeta) = \lim_{V} \mathcal{F}^\alpha_{c,n}(V,G,\zeta)
\]

is the space of \( \alpha \)-germs. Here the direct limit is taken over \( W(M)M(\mathbb{R}) \)-invariant neighborhoods \( V \) of \( c \) in \( M(\mathbb{R}) \). Then the invariant germ expansion can be interpreted as there exists a weight function \( \alpha \) such that for any \( n \geq 0 \),

\[
f \rightarrow I^\alpha_M(\gamma,f) - I^n_M(\gamma,f)
\]

is a continuous linear mapping from \( \mathcal{C}(G,\zeta) \) to \( \mathcal{F}^\alpha_{c,n}(V,G) \) where

\[
(11.5) \quad I^n_M(\gamma,f) = \sum_{L \in \mathcal{L}(M)} \sum_{\rho \in R_c(L)} g^L_{M,\gamma}(\rho) J_L(\rho,f),
\]

belongs to \( \mathcal{F}^\alpha_{c}(V,G,\zeta) = \mathcal{F}^\alpha_{c-1}(V,G,\zeta) \) \([Art16 \, \text{Corollary 10.1}]\). Here \( g^L_{M,\gamma}(\rho) \) is the projection of \( g^L_{M}(\rho) \) onto the quotient of \( \mathcal{G}^\alpha_{c}(M,L,\zeta) \) by the kernel of its projection onto \( \mathcal{G}^\alpha_{c,n}(M,G) \), and \( g^L_{M,\gamma}(\rho) \) is a representative of \( g^L_{M}(\rho) \) in \( \mathcal{F}^\alpha_{c}(V,L,\zeta) \). We assume that \( g^L_{M,\gamma}(\rho) = 0 \) if \( g^L_{M}(\rho) = 0 \). It follows then that the inner sum of \( I^n_M(\gamma,f) \) can be taken over a finite set. It is uniquely determined up to a finite sum

\[
\sum_i \phi_i(f) J_i(f)
\]

for tempered distributions \( J_i(f) \) and functions \( \phi_i(\gamma) \) in \( \mathcal{F}^\alpha_{c,n}(V,G) \).

### 11.2. Stable germ expansions.

The stable germ expansion of orbital integrals was used in \([Art03 \, \S 5]\) for the nonarchimedean case, and invariant germ expansions were developed in \([Art16]\) for the archimedean case. The proofs of the stable germ expansions remain unpublished; a twisted stable version was proved in \([MW16a \, \S 2]\) and \([MW16a \, \S 6]\) for the nonarchimedean and archimedean cases respectively, but for a simpler subset of distributions in \( \mathcal{D}(G,\zeta) \) in the archimedean case. This space essentially consists of orbital distributions \( \mathcal{D}_{\text{orb}}(G,\zeta) \) on \( G \) and transfers of orbital distributions from elliptic endoscopic groups of \( G \). We shall provide a version of this construction here.

We now turn to stable germ expansions. Fix \( d^* \in \Delta_{\text{ss}}(\hat{M}^*) \). The direct sum

\[
\mathcal{D}_{d^*}(G,\zeta) = \bigoplus_c \mathcal{D}_c(G,\zeta)
\]
taken over classes \( c \in \Gamma_{ss}(\tilde{M}) \) whose image in \( \Delta_{ss}(\tilde{M}^*) \) equals \( d^* \), has an orthonormal basis given by the union
\[
R_{d^*}(G, \zeta) = \bigoplus_c R_c(G, \zeta).
\]

Any distribution \( \delta \in \mathcal{D}_d(G, \zeta) \) has semisimple component equal to \( d \in \Delta_{ss}(\tilde{G}) \), and unipotent part equal to \( \beta \in \mathcal{D}_{\text{unip}}(G_d, \zeta) = \mathcal{D}_1(G_d, \zeta) \). The choice of \( \beta \) is unique up to the action of the finite group \( \bar{G}_{d,+}(F)/\bar{G}_d(F) \). Once again, this Jordan decomposition \( \delta = d\beta \) depends on a suitably chosen section \( d \to \bar{d} \) from \( \Delta_{ss}(\tilde{G}) \) to \( G(F) \).

We shall assume that such a section has been fixed. The elements \( \rho \in R_{\bar{d}^*}(G, \zeta) \) parametrise a family of germs of functions \( g_M^G(\gamma, \rho) \) of points \( \gamma \in \Gamma_{G,\text{reg}}(M, \zeta) \) that are close to \( d^* \), in the sense that the image in \( M'(F_v) \) of the support of \( \gamma \) is close to \( d^* \). If \( \rho \) belongs to a subset \( R_{\bar{d}}(G, \zeta) \) of \( R_{\bar{d}^*}(G, \zeta) \), we set \( g_M^G(\gamma, \rho) \) equal to zero for \( \gamma \) outside some fixed, small neighbourhood of \( c \in \Gamma_{G,\text{reg}}(M, \zeta) \). It follows that for any \( \gamma \), the germs \( g_M^G(\gamma, \rho) \) can be nonzero for elements \( \rho \) in at most one subset \( R_c(G, \zeta) \) of \( R_{\bar{d}^*}(G, \zeta) \).

The space \( \mathcal{D}(G, \zeta) \) naturally has a subspace of stable distributions
\[
S\mathcal{D}(G, \zeta) = \bigoplus_{d \in \Delta_{ss}(\tilde{G})} S\mathcal{D}_d(G, \zeta) = \bigoplus_{d^* \in \Delta_{ss}(\tilde{G}^*)} S\mathcal{D}_{d^*}(G, \zeta).
\]

with a corresponding orthonormal basis \( \Sigma_d(G, \zeta) \).

11.2.1. A subspace of distributions. Let \( G \) be a quasisplit reductive group over \( \mathbb{R} \). We shall construct the subspace \( \mathcal{D}_{\text{tr-orb}}(G, \zeta) \) of \( \mathcal{D}(G, \zeta) \) inductively on \( \dim(G_{sc}) \), as in [MW16a, V.2.1]. Assuming that \( \mathcal{D}_{\text{tr-orb}}(G, \zeta) \) is given, we may define the subset
\[
S\mathcal{D}_{\text{tr-orb}}(G, \zeta) = \mathcal{D}_{\text{tr-orb}}(G, \zeta) \cap S\mathcal{D}(G, \zeta).
\]

Then we define \( \mathcal{D}_{\text{tr-orb}}(G, \zeta) \) to be the subspace of \( \mathcal{D}(G, \zeta) \) generated by \( \mathcal{D}_{\text{orb}}(G, \zeta) \) and the images of the transfer of \( S\mathcal{D}_{\text{tr-orb}}(G', \zeta) \) for each elliptic endoscopic datum \( G' \in \delta(G) \), with \( G' \neq G \). These spaces depend only on \( G' \), and are independent of the auxiliary endoscopic datum. We may also extend the definition of \( \mathcal{D}_{\text{tr-orb}}(G, \zeta) \) to \( z \)-extensions \( \tilde{G} \) of \( G \) by the natural image of \( \mathcal{D}_{\text{tr-orb}}(G, \zeta) \) in \( D(\tilde{G}) \), where \( \tilde{\zeta} \) is chosen to be compatible with \( \zeta \). Parallel to the decompositions of \( \mathcal{D}(G, \zeta) \) and \( S\mathcal{D}(G, \zeta) \) in (11.1) and (11.6), we obtain
\[
\mathcal{D}_{\text{tr-orb}}(G, \zeta) = \bigoplus_{c \in \Gamma_{ss}(\tilde{G})} \mathcal{D}_{\text{tr-orb}}^c(G, \zeta)
\]
and
\[
S\mathcal{D}_{\text{tr-orb}}(G, \zeta) = \bigoplus_{d \in \Delta_{ss}(\tilde{G})} S\mathcal{D}_{d}^{\text{tr-orb}}(G, \zeta)
\]
where \( \mathcal{D}_{\text{tr-orb}}^c(G, \zeta) \) is given by \( \mathcal{D}_{\text{tr-orb}}(G, \zeta) \cap \mathcal{D}_c(G, \zeta) \) and similarly for \( S\mathcal{D}_d^{\text{tr-orb}}(G, \zeta) \). Then the induction homomorphism \( \mu \mapsto \mu^G \) yields maps from \( \mathcal{D}_{\text{tr-orb}}(M, \zeta) \) to \( \mathcal{D}_{\text{tr-orb}}(G, \zeta) \), and \( S\mathcal{D}_{\text{tr-orb}}(M, \zeta) \) to \( S\mathcal{D}_{\text{tr-orb}}(G, \zeta) \). These definitions extend as usual to \( K \)-groups through the direct sum
\[
\mathcal{D}_{\text{tr-orb}}(G, \zeta) = \bigoplus_{\alpha \in \pi_0(G)} \mathcal{D}_{\text{tr-orb}}(G_{\alpha}, \zeta_{\alpha})
\]
and
\[
S\mathcal{D}_{\text{tr-orb}}(G, \zeta) = \bigoplus_{\alpha \in \pi_0(G)} S\mathcal{D}_{\text{tr-orb}}(G_{\alpha}, \zeta_{\alpha}).
\]
We remark here that a stable germ expansion and transfer of distributions for real groups is provided in the literature by [MW16a, V] using these simpler subspaces and a more direct argument, which includes all the distributions necessary in the stabilisation of the trace formula. The treatment we shall give here follows the spirit of Arthur’s stable germ expansions, which coincide with [MW16a, II] in the nonarchimedean case of stable Shalika germs.

11.2.2. Stable and endoscopic germs. Let $M'$ be an elliptic endoscopic datum for $M$, and $d' \in \Delta_{ss}(M')$ a stable semisimple conjugacy class in $M'_*(F_c)$ whose image in $\Delta_{ss}(M^*)$ equals $d'$. Given $M'$ and $d'$, assume inductively that for each element $G' \in \delta_{M'}^0(G)$ with auxiliary datum $(\tilde{G}', \tilde{\xi}')$, we have defined a family

$$h_{\tilde{M}_1}^{\tilde{G}'}(\delta', \sigma'), \quad \sigma' \in \Sigma_{d'}(\tilde{G}', \tilde{\xi}')$$

of germs of functions of points $\delta' \in \Delta_{G, reg}(\tilde{M}', \tilde{G}', \tilde{\xi}')$ near $d'$, parametrised by a basis $\Sigma_{d'}(\tilde{G}', \tilde{\xi}')$ that is constructed analogously to $R_{d'}(G, \zeta)$. We view these as conjugate linear forms in $\sigma' \in S\mathfrak{D}_{d'}(\tilde{G}', \tilde{\xi}')$, which extend continuously to the completion $\hat{\mathfrak{D}}_{d'}(\tilde{G}', \tilde{\xi}')$. The germs satisfy the symmetry condition $h_{\tilde{M}_1}^{\tilde{G}'}(\theta' \delta', \theta' \sigma') = h_{\tilde{M}_1}^{\tilde{G}'}(\delta', \sigma')$ for any $\theta' \in \text{Out}(\tilde{G}', \tilde{M}', \tilde{\xi}')$, the group of outer automorphisms of $\tilde{G}'$ fixing the central character datum $(\tilde{Z}', \tilde{\xi}')$ and leave $\tilde{M}'$ invariant. We also require the compatibility condition $h_{\tilde{M}_1}^{\tilde{G}'}(\delta'_1, \omega' \sigma') = h_{\tilde{M}_1}^{\tilde{G}'}(\delta'_1, \sigma')$ for a second auxiliary datum $(\tilde{G}'_1, \tilde{\xi}'_1)$ and for any character $\omega'$ of $\tilde{G}'$ and $\delta'_1 \in \Delta_{G, reg}(\tilde{M}'_1, \tilde{\xi}'_1)$ near $d'$.

The stable and endoscopic germs are defined inductively by

$$g_{\tilde{M}}^{G, \mathfrak{d}'}(\delta', \rho) = \sum_{G' \in \delta_{\tilde{M}_1}^0(G)} t_{M'}(G, G')h_{\tilde{M}_1}^{\tilde{G}'}(\delta', \rho') + \varepsilon(G)h_{\tilde{M}}^{G}(M', \delta', \rho)$$

with the condition that $g_{\tilde{M}}^{G, \mathfrak{d}'}(\delta', \rho) = g_{\tilde{M}_1}^{G'}(\delta, \rho)$ if $G$ is quasisplit and $\delta'$ maps to $\delta \in \mathfrak{D}_{G, \mathfrak{d}'}(M, \zeta)$.

Here

$$g_{\tilde{M}}^{G}(\delta, \rho) = \sum_{\gamma \in \Gamma_{G, \mathfrak{d}'}(M, \zeta)} \Delta_M(\delta, \gamma)g_{\tilde{M}}^{G}(\gamma, \rho), \quad \rho \in R_{d'}(G, \zeta)$$

for any point $\delta \in \Delta_{G, \mathfrak{d}'}(M, \zeta)$ that is close to $d^*$ in the sense that the image in $M^*_*(F)$ of its support is close to $d^*$. Here $\Delta_{G, \mathfrak{d}'}(M, \zeta)$ is the endoscopic analogue of the $\Gamma_{G, \mathfrak{d}'}(M, \zeta)$, and is a subset of the general basis $\Delta_M(M, \zeta)$ of $\mathfrak{D}(M, \zeta)$. Taking $G$ to be quasisplit, we set

$$h_{\tilde{M}}^{G}(M', \delta', \sigma) = \sum_{\rho \in R_{d'}(G, \zeta)} h_{\tilde{M}}^{G}(M', \delta', \rho)\Delta(\rho, \sigma)$$

for $\sigma \in \Delta_M^0(G, \zeta)$. We note that the fact that these stable germs are indeed stable distributions follows from the stability of the linear forms $S_{M}^G(\delta, f)$.

We record the following descent and splitting formulas for endoscopic and stable germs, which are derived in a similar manner to Propositions 10.1 and 10.2 respectively. See also Lemme 2.12 and 2.13 of [MW16a, II].

**Lemma 11.1** (Descent). (a) Let $G$ be arbitrary. Then

$$g_{\tilde{M}}^{G, \mathfrak{d}'}(\delta', \rho) = \sum_{G_1 \in \mathfrak{S}(M_1)} d_{G_1}^{G_1}(M, G_1)g_{\tilde{M}_1}^{G_1, \mathfrak{d}'}(\delta'_1, \rho_{G_1}), \quad \rho \in R_{d'}(G, \zeta).$$
(b) Let $G$ be quasisplit. If $M' = M^*$, then

$$h^G_M(\delta, \sigma) = \sum_{G_1 \in \mathcal{G}(M')} e^{G_1}_M(M, G_1) h^{G_1}_M(\delta_1, \rho_{G_1}), \quad \sigma \in \Delta^G_{d^*}(G, \zeta),$$

and if $M' \neq M^*$ then $g^G_M(M', \delta', \sigma) = 0$ for all $\sigma \in \Delta^G_{d^*}(G, \zeta)$.

**Lemma 11.2** (Splitting). (a) Let $G$ be arbitrary. Then

$$g^G_M(\delta', \rho) = \sum_{G_1 \in \mathcal{G}(M')} d^{G_1}_M(M, G_1) g^{G_1}_M(\delta_1', \rho_{G_1}), \quad \rho \in R_{d^*}(G, \zeta).$$

(b) Let $G$ be quasisplit. If $M' = M^*$, then

$$h^G_M(\delta, \sigma) = \sum_{G_1 \in \mathcal{G}(M')} e^{G_1}_M(M, G_1) h^{G_1}_M(\delta_1, \rho_{G_1}), \quad \sigma \in \Delta^G_{d^*}(G, \zeta),$$

and if $M' \neq M^*$ then $g^G_M(M', \delta', \sigma) = 0$ for all $\sigma \in \Delta^G_{d^*}(G, \zeta)$.

In particular, the splitting formula allows us again to reduce to the case of simple forms.

Let $G$ be quasisplit over $F$ and $\gamma \in \mathcal{G}(G, \zeta)$ to be a strongly $G$-regular element close to $d'$, in the sense that the image of its support in $M'$ is close to $d'$. In the special case that $M' = M^*$, we shall take $d = d'$ and $\delta = \delta'$. We now state the stable and endoscopic germ expansions, we include the proof as it will lead to a similar quantitative statement as in the invariant case.

**Proposition 11.3.** If $G$ is arbitrary, and $\gamma \in \Gamma_{G, \text{reg}}(M, \zeta)$ is close to $d'$, there is a germ expansion

$$I^G_M(\gamma, f) \sim \sum_{L \in \mathcal{Z}(M)} \sum_{\rho \in R_d}(L, \zeta) g^L_M(\gamma, \rho) I^\rho_L(\rho, f).$$

If $G$ is quasisplit, and $\delta' \in \Delta_{G, \text{reg}}(M', \zeta')$ near $d'$, there is a stable germ expansion

$$S^G_M(M', \delta', f') \sim \sum_{L \in \mathcal{Z}(M)} \sum_{L' \in \mathcal{Z}(L)} \sum_{\sigma \in \Sigma_{d'}(L', \zeta')} \iota_{M'}(L, L') h^L_{M'}(\delta', \sigma') S^G_{L'}(\sigma', f').$$

If $F$ nonarchimedean, the relation $\sim$ is an equality in both cases.

**Proof.** We assume inductively that

$$S^G_{M'}(\delta', f') \sim \sum_{L' \in \mathcal{Z}(M')} \sum_{\sigma' \in \Sigma_{d'}(L', \zeta')} h_{M'}^L(\delta', \sigma') S^G_{L'}(\sigma', f')$$

for $\delta' \in \Delta_{G', \text{reg}}(M', \zeta)$ near $d'$ and $G' \in \mathcal{G}(G)$. Note that by Local Theorem 1 of [Art03] we know that the linear form $S^G_{M'}(\delta', f')$ is stable. Let $G$ be arbitrary and $\delta'$ as above. Then by definition we have

$$I^G_M(\delta', f) - e(G) S^G_{M'}(M', \delta', f) = \sum_{G' \in \mathcal{G}_{\text{ell}}(G)} \iota_{M'}(G, G') S^G_{M'}(\delta', f').$$

Also, since

$$\sum_{\sigma \in \Sigma_{d'}(G, \zeta)} h^G_M(M', \delta', \sigma) f^G_G(\sigma) = \sum_{\rho \in R_d(G, \zeta)} g^G_M(M', \delta', \sigma) f_G(\rho),$$
we can express

\[(11.11) \quad \sum_{\rho \in R_{\ast}(G, \zeta)} g^G_\delta^{\epsilon, \rho}(\delta', \rho) f_G(\rho) = \sum_{\sigma \in \Sigma^\epsilon_\rho(G, \zeta)} \varepsilon(G) \delta^G_{M}(M', \delta', \sigma)) f^\rho_G(\rho)\]

as

\[
\begin{align*}
&= \sum_{\rho \in R_{\ast}(G, \zeta)} \sum_{G' \in \mathcal{E}_{\rho}^0(G)} \iota_{M'}(G, G') h_{M'}^G(\delta', \rho) \\
&= \sum_{G' \in \mathcal{E}_{\rho}^0(G)} \sum_{\sigma' \in \Sigma^\epsilon_{\rho}(G', \zeta')} h_{M'}^G(\delta', \sigma').
\end{align*}
\]

Putting (11.10) and (11.11) together, we can write

\[(11.12) \quad I_M^G(\delta', f) = \sum_{\rho \in R_{\ast}(G, \zeta)} g^G_\delta^{\epsilon, \rho}(\delta', \rho) f_G(\rho)\]

as the sum of

\[(11.13) \quad \sum_{G' \in \mathcal{E}_{\rho}^0(G)} \iota_{M'}(G, G')(\hat{S}_{M'}^G(\delta', f') - \sum_{\sigma' \in \Sigma^\epsilon_{\rho}(G', \zeta')} h_{M'}^G(\delta', \sigma'))\]

and

\[
\varepsilon(G)(S_{M'}^G(M', \delta', f) - \sum_{\sigma \in \Sigma^\epsilon_\rho(G, \zeta)} h_{M'}^G(M', \delta', \sigma) f^\rho_G(\rho)).
\]

Using the induction assumption, we apply the expansion (11.9) to (11.13), which leads to

\[(11.14) \quad \sum_{G' \in \mathcal{E}_{\rho}^0(G)} \iota_{M'}(G, G') \sum_{L' \in \mathcal{L}^0'(M')} \sum_{\sigma' \in \Sigma^\epsilon_{\rho}(G', \zeta')} h_{M'}^G(\delta', \sigma') \hat{S}_{L'}^G(\sigma', f')\]

for \(\delta \in \Delta_{G, \text{reg}}(M')\) close to \(d'\), and \(\mathcal{L}^0'(M')\) denotes the complement of \(\{G'\}\) in \(\mathcal{L}^0'(M')\). Note that any Levi subgroup \(L' \in \mathcal{L}^0'(M')\) of \(G'\) determines a Levi subgroup \(L \in \mathcal{L}(M)\) of \(G\) with \(A_{L'} = A_L\), so we can rewrite the sum over \(G'\) and \(L'\) as a sum over \(L \in \mathcal{L}^0(M), L' \in \mathcal{E}_{\rho}^0(L), \) and \(G' \in \mathcal{E}_{\rho}^0(G)\). Then using the identity \(\iota_{M'}(G, G') = \iota_{M'}(L, L') \iota_{L'}(G, G')\), it follows that we can write (11.14) as

\[
\sum_{L \in \mathcal{L}^0(M)} \sum_{L' \in \mathcal{E}_{\rho}^0(L)} \iota_{M'}(L, L') \delta_{M'}^G(\delta', \sigma') \sum_{G' \in \mathcal{E}_{\rho}^0(G)} \iota_{L'}(G, G') \hat{S}_{L'}^G(\sigma', f').
\]

By definition (10.1) the inner sum is equal to

\[(11.15) \quad I_L^G(\delta', f') = \varepsilon(G) S_{L'}^G(L', \delta', f').\]

The first term is given by

\[
I_L^G(\sigma', f') = \sum_{\rho \in R_{\ast}(L, \zeta)} \Delta_L(\sigma', \rho) I_L^G(\rho, f),
\]

and using the relations

\[
\sum_{\sigma' \in \Sigma^\epsilon_{\rho}(L, \zeta')} h_{M'}^G(\delta', \sigma') \Delta_L(\sigma', \rho) = h_{M'}^G(\delta', \rho)
\]
and
\[ \sum_{L' \in \mathcal{E}_{d'}(L)} t_{M'}(L, L') h^{d'}_{M'}(\delta', \rho) = g^{d, \rho}_{M}(\delta', \rho), \]
it follows that the contribution of the first term in (11.15) is equal to
\[ (11.16) \]
\[ \sum_{L \in \mathcal{E}^{\rho}(M)} \sum_{\rho \in R_{d'}(L, \zeta)} g^{d, \rho}_{M}(\delta', \rho) \mathcal{I}_{L}(\rho, f). \]
Subtracting this from the difference (11.12) we see that (11.16) is equal to the product of \( \varepsilon(G) \) and \( (11.8) \).
Thus if \( \varepsilon(G) = 0 \), (11.16) vanishes and we deduce the endoscopic expansion (11.7). If \( \varepsilon(G) = 1 \), then (11.7) is equivalent to the usual invariant germ expansion (11.4). One sees then that (11.16) again vanishes, and (11.8) follows.

Finally, we derive the quantitative form of the archimedean stable germ expansion, which is the key result of this section. We first define the spaces
\[ \mathcal{G}_{d'}(M, G, \zeta) = \prod_{c} \hat{\mathcal{G}}_{c}(M, G, \zeta) \]
and
\[ \mathcal{F}_{c,n}(V, G) = \prod_{c} \mathcal{F}_{c,n}(V, G) \]
where the union is again taken over classes \( c \in \Gamma_{ss}(M) \) whose image in \( \Delta_{ss}(\hat{M}) \) equals \( d^{*} \). We observe that the germs \( g^{M, \rho}_{\sigma}(\delta', \rho) \) and \( h^{\sigma}_{M}(\delta', \sigma) \) belong to the space \( \mathcal{G}_{d'}(M, G, \zeta) \). Then for any \( n \geq 0 \), we set
\[ (11.17) \]
\[ S^{n}_{M}(\delta, f) = \sum_{L \in \mathcal{E}(M)} \sum_{\sigma \in \Sigma_{d}(L)} h^{L,n}_{M}(\delta, \sigma) S_{L}(\sigma, f) \]
where \( h^{L,n}_{M}(\delta, \rho) \) are fixed representatives, defined again as the projection of \( h^{L}_{M}(\sigma) \) onto \( \hat{\mathcal{G}}^{n}(M, L, \zeta) \), or equivalently, as the stable partial germ obtained from \( g^{L,n}_{M}(\gamma, \rho) \) defined parallel to \( h^{L}_{M}(\delta, \sigma) \). We note again that the sum can be taken over finite set, as with (11.5).

**Corollary 11.4.** There exists a weight function \( \alpha \) such that \( \alpha(1) = 0 \), and for any \( n \) the mapping
\[ f \rightarrow S_{M}(\delta, f) - S^{n}_{M}(\delta, f), \quad f \in \mathcal{E}(G) \]
is a continuous linear mapping from \( \mathcal{E}(G) \) to \( \mathcal{F}_{d', n}(V, G) \). In particular, \( S_{M}(f) \) has a formal germ expansion given by the sum
\[ \sum_{L \in \mathcal{E}(M)} h^{L}_{M}(S_{L, d}(f)) = \sum_{L \in \mathcal{E}(M)} \sum_{\sigma \in \Sigma_{d}(L)} h^{L}_{M}(\sigma) S_{L}(\sigma, f). \]

**Proof.** The second assertion follows immediately from the first, similar to Theorem 5.1 of [Art10]. To establish the first assertion, we express \( S_{M}(\delta, f) - S^{n}_{M}(\delta, f) \) as the difference between
\[ I_{M}(\delta', f) - \Gamma^{n}_{M}(\delta', f) \]
and
\[ \sum_{G \in \mathcal{E}_{M}(G)} t_{M'}(G, G') \left( \hat{S}^{G'}_{M}(\delta', f) - \hat{S}^{G'}_{M'}(\delta', f) \right), \]
then the desired result follows inductively from [Art10, Corollary 10.1].
References


