THE DENSITY OF ELLIPTIC DEDEKIND SUMS

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Abstract. Elliptic Dedekind sums were introduced by R. Sczech as generalizations of classical Dedekind sums to complex lattices. We show that for any lattice with real $j$-invariant, the values of suitably normalized elliptic Dedekind sums are dense in the real numbers. This extends an earlier result of Ito for Euclidean imaginary quadratic rings. Our proof is an adaptation of the recent work of Kohnen, which gives a new proof of the density of values of classical Dedekind sums.

1. Introduction

1.1. The density of classical Dedekind sums. The classical Dedekind sum $s(m,n)$ is defined for $m, n \in \mathbb{Z}$, $(m, n) = 1$, $n \neq 0$, by

$$s(m,n) := \frac{1}{4n} \sum_{k=1}^{n-1} \cot \left( \pi \frac{mk}{n} \right) \cot \left( \pi \frac{k}{n} \right).$$

These sums were first studied by Dedekind due to their appearance in the transformation law for the logarithm of the Dedekind $\eta$ function. Specifically, for integers $a, b, c,$ and $d$ with $c \neq 0$ and $ad - bc = 1$,

$$\log \eta \left( \frac{a\tau + b}{c\tau + d} \right) = \log \eta(\tau) + \frac{1}{2} \log \left( \frac{ct + d}{1 \text{ sign}(c)} \right) + \frac{\pi i}{12} \phi \left( \begin{array}{cc} a & b \\ c & d \end{array} \right),$$

where $\phi$ is a map from $SL_2(\mathbb{Z})$ to $\mathbb{Z}$ given by

$$\phi \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \begin{cases} \frac{k}{2} - 12(s(d, |c|)) & \text{if } c = 0 \\ \frac{k}{c} - 12(s(d, |c|)) & \text{if } c \neq 0 \end{cases},$$

and $\tau$ is any point in the upper half-plane. The logarithm is taken with respect to the principle branch. The $\phi$ map is a quasi-homomorphism satisfying

$$\phi(A_1) = \phi(A_2) + \phi(A_3) - 3 \text{ sign}(c_1c_2c_3),$$

where $A_1 = A_2A_3$ and $c_i$ denotes the bottom left entry of $A_i$. This near homomorphism law gives rise to many interesting results in the study of Dedekind sums and makes them an important tool in number theory.

It was conjectured by Grosswald and Rademacher that the values of $s(m,n)$ are dense in $\mathbb{R}$, and moreover the graph $(m/n, s(m,n))$ is dense in $\mathbb{R}^2$. The latter statement (which implies the former) was proved by Hickerson. Recently, Kohnen gave a new proof of the former using a result of Girstmair on the three-term relation of Dedekind sums and Dirichlet’s theorem on arithmetic progressions of primes, providing a direct approximation of any rational number.
1.2. The density of elliptic Dedekind sums. In this paper, we use Kohnen’s method to provide an equivalent result for normalized elliptic Dedekind sums for any imaginary quadratic field. Let \( L \) be a non-degenerate lattice in \( \mathbb{C} \). We define
\[
E_k(z) = \sum_{\substack{l \in L, \cr l+z \neq 0}} (l+z)^{-k} |l+z|^{-s}_s = 0,
\]
where the value of the sum at \( s = 0 \) is evaluated by means of analytic continuation. We remark on the similarity between these elliptic functions and the cotangent function used to define classical Dedekind sums.

Define \( \mathcal{O}_L := \{ m \in \mathbb{C} : mL \subset L \} \). This set is known as the ring of multipliers for \( L \) and is either equal to the integers or an order in an imaginary quadratic field (see exercise 12.3 of Neukirch [5] for reference). Then, following Sczech [7], the elliptic Dedekind sums for \( L \) are defined as
\[
D_L(h,k) = \frac{1}{k} \sum_{\mu \in L/kL} E_1\left(\frac{h\mu}{k}\right) E_1\left(\frac{\mu}{k}\right),
\]
for \( h, k \in \mathcal{O}_L, k \neq 0 \). It is important to note that \( D_L(h,k) \) is dependent on the choice of the lattice \( L \).

As with the classical case, we define a map \( \Phi \) from \( SL_2(\mathcal{O}_L) \) to \( \mathbb{C} \) given by
\[
\Phi\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) := \begin{cases} E_2(0)I\left(\frac{a+d}{c}\right) - D_L(a,c) & c \neq 0 \\ E_2(0)I\left(\frac{b}{d}\right) & c = 0 \end{cases},
\]
where \( I(z) = z - 3 \). It was shown by Sczech [7] that \( \Phi \) is a homomorphism in the additive group of complex numbers and that it is trivial for \( \mathcal{O}_L = \mathbb{Z}, \mathbb{Z}[i], \mathbb{Z}[\rho] \), where \( \rho = (-1 + \sqrt{-3})/2 \).

For our purposes it is advantageous to follow Ito [3] and define normalized elliptic Dedekind sums as
\[
\tilde{D}_L(a,c) = (i \sqrt{|d_L|} E_2(0))^{-1} D_L(a,c).
\]

Our departure from Ito is in our choice of \( d_L \). In particular, any ring of multipliers \( \mathcal{O}_L \) is an order of some imaginary quadratic field \( K = \mathbb{Q}(\sqrt{-n}) \) with a ring of integers having discriminant \( d_K \). Recall that the ring of integers is the maximal order in \( K \). Hence, we may write \( \mathcal{O}_L = \mathbb{Z}[f^{d_K+\sqrt{d_K}}] \) (for this result see Neukirch [5]). We therefore take \( d_L = f^2 d_K \) with a positive integer \( f \), called the conductor of \( \mathcal{O}_L \), instead of only using the discriminant \( d_K \) of \( K \).

Ito [3] proved that the values of the normalized elliptic Dedekind sums are dense in \( \mathbb{R} \) when \( \mathcal{O}_L \) is the ring of integers of \( \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-5}), \) or \( \mathbb{Q}(\sqrt{-7}) \). In fact, for these cases, Ito showed that the set \( (a/c, \tilde{D}_L(a,c)) \) is dense in \( \mathbb{C} \times \mathbb{R} \), using the continued fraction method of Hickerson [2]. As continued fraction expansions are less well-developed for non-Euclidean rings, especially for \( \mathcal{O}_L \) with class number greater than 1, Ito’s method does not immediately generalize. In any case, Ito’s result implies that \( \tilde{D}(a,c) \) is dense in \( \mathbb{R} \) for the cases listed above.

Using the method introduced by Kohnen [4] for the case of classical Dedekind sums, we provide an alternate proof that the normalized elliptic Dedekind sums are dense in the real line, and we show that the density result holds more generally, for every lattice \( L \) with a real \( j \)-invariant \( j(L) \).

**Theorem 1.** Let \( L \) be a lattice with \( j \)-invariant \( j(L) \in \mathbb{R} \) and multiplier ring \( \mathcal{O}_L = \mathbb{Z}[f^{d_K+\sqrt{d_K}}] \) as above, and \( \mathcal{O}_L \) different from \( \mathbb{Z}, \mathbb{Z}[i], \) and \( \mathbb{Z}[\frac{1-\sqrt{-3}}{2}] \). Under
these conditions, the values of the normalized elliptic Dedekind sum \( \tilde{D}_L(a, c) \) with respect to \( L \) are dense in \( \mathbb{R} \).

Note that if the lattice \( L \) is similar to \( \mathcal{O}_L \), then \( j(L) \) is real. In particular, this includes all imaginary quadratic ring of integers \( \mathcal{O}_K \).

We shall first prove in Section 2 a lemma that provides us with an elliptic Dedekind sum identity analogous to the three term relation of Girstmair. After proving the lemma, we will be prepared to show that \( \tilde{D}_L(a, c) \) is dense in the real numbers in Section 3. We will use a subset of the rationals that is dense in the reals with the idea of approximating values in this subset. The theorem then follows as an application of Dirichlet’s theorem and quadratic reciprocity.

2. An application of the three-term relation

We first prove a lemma that establishes a simple formula for certain values of elliptic Dedekind sums. This lemma makes use of the homomorphism property of \( \Phi \), and follows the method of Girstmair [1] and Kohnen [4].

**Lemma 1.** Let \( A_i := \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in SL_2(\mathcal{O}_L) \) for \( i \in \{1, 2, 3\} \). If \( A_1 = A_2 \cdot A_3 \) with the conditions \( c_1 = c_2 = c \neq 0 \) and \( a_1a_2 \equiv 1 \pmod{c} \) satisfied, then

\[
D_L(a_3, c_3) = E_2(0)I \left( \frac{2}{c_3} \frac{c_1}{c_3} \right).
\]

**Proof.** Let \( A_i \) be as above. Since \( A_i \in SL_2(\mathcal{O}_L) \), the following equations must hold

\[
\begin{align*}
(1) & \quad a_2d_2 - b_2c = 1, \\
(2) & \quad a_1d_1 - b_1c = 1.
\end{align*}
\]

In Section 1.2 of [7], Sczech states that \( \Phi \), as defined above, is a homomorphism from the multiplicative group \( SL_2(\mathcal{O}_L) \) to the additive group of complex numbers.

Therefore,

\[
0 = \Phi(Id) = \Phi(A_1^{-1}A_2A_3) = -\Phi(A_1) + \Phi(A_2) + \Phi(A_3).
\]

After expanding and reordering terms as needed, we arrive at the three term relation

\[
D_L(a_1, c) = D_L(a_2, c) + D_L(a_3, c_3) + E_2(0)I \left( \frac{a_1 + d_1}{c} \frac{a_2 + d_2}{c} - \frac{a_3 + d_3}{c} \right).
\]

Fix \( c_1 = c_2 \) and \( a_1a_2 \equiv 1 \pmod{c} \). We claim that \( D_L(a_1, c_1) = D_L(a_2, c_2) \). This follows from

\[
D_L(a_1, c) = \frac{1}{c} \sum_{k \in L/cL} E_1 \left( \frac{a_1k}{c} \right) E_1 \left( \frac{k}{c} \right) .
\]

With the change of variables \( k' = a_1k \), this becomes

\[
\frac{1}{c} \sum_{k' \in L/cL} E_1 \left( \frac{k'}{c} \right) E_1 \left( \frac{a_2k'}{c} \right) = D_L(a_2, c).
\]

Since \( D_L(a_1, c) = D_L(a_2, c) \), we can conclude that

\[
D_L(a_3, c_3) = E_2(0)I \left( \frac{a_2 + d_2}{c} + \frac{a_3 + d_3}{c} - \frac{a_1 + d_1}{c} \right).
\]
Now note that by assumption \( A_3 = A_7^{-1} A_1 \), so we may write

\[
A_3 = \begin{pmatrix}
    a_1 d_2 - b_2 c & d_2 b_1 - d_1 b_2 \\
    a_2 c - a_1 c & d_1 a_2 - b_1 c
\end{pmatrix}.
\]

Hence,

\[
a_2 + a_2 + a_3 + a_4 - a_1 + a_1 = \frac{(a_2 - a_1)(a_2 + a_2)}{(a_2 - a_1)c} + \frac{a_3 d_2 - b_2 c + d_1 a_2 - b_1 c}{(a_2 - a_1)c} - \frac{(a_2 - a_1)(a_1 + a_1)}{(a_2 - a_1)c},
\]

which by (1) and (2) reduces to

\[
\frac{2 + a_2^2 - 2a_1 a_2 + a_1^2}{a_2 - a_1 c} = \frac{2}{c_3} + \frac{c_3}{c^2}.
\]

Therefore, by substitution we arrive at

\[
D_L(a_3, c_3) = E_2(0) I \left( \frac{2}{c_3} + \frac{c_3}{c^2} \right)
\]

as desired.

\[\square\]

3. Proof of Theorem 1

We first explain the choice of the normalization of \( D_L(h, k) \) that we employ. It is worth noting that all values taken by normalized elliptic Dedekind sums over imaginary quadratic fields are real, provided \( j(L) \in \mathbb{R} \), and in fact are real algebraic. Ito [3] states that normalized elliptic Dedekind sums over imaginary quadratic fields take values in \( \mathbb{Q}(j) \), where \( j \) is equal to the \( j \)-invariant of the lattice \( L \). Since we assumed \( j \) to be real in the statement of Theorem 1, \( \mathbb{Q}(j) \) is a subset of the real numbers. In the important case where \( L \) is similar to its own multiplier ring \( O_L \) in an imaginary quadratic field, we can easily see that \( j(L) \) is real. Recall that \( j \) is defined by

\[
j(\tau) = 1728 \frac{g_2(\tau)^3}{\Delta(\tau)},
\]

where

\[
\Delta(\tau) = g_2(\tau)^3 - 27 g_3(\tau)^2
\]

and \( g_2, g_3 \) are Eisenstein series. The \( q = e^{2 \pi i \tau} \) expansions of these series have purely real coefficients. Since

\[
\tau = f \frac{d_K + \sqrt{d_K}}{2},
\]

we see that \( q = e^{2 \pi i \tau} \) is real if and only if \( f d_K \) is an integer. The last condition is true by assumptions on \( f \) and \( d_K \). Therefore, \( j(\tau) \) and the normalized elliptic Dedekind sums are also real.

One can show more generally that \( j(L) \) is real if and only if the similarity class of \( L \) is invariant under complex conjugation. Since complex conjugation acts on ideal classes by taking an ideal class into its inverse, the condition \( j(L) \in \mathbb{R} \) means that the order of the ideal class represented by the lattice \( L \) is at most 2 in the class group of \( O_L \).

With this, we are prepared to prove density. In the following proof we recall the convention \( d := d_L = f^2 d_K \) introduced above.
Proof of Theorem 1. Since the rationals are dense in the reals, it suffices to show that the range of $\tilde{D}_L$ is dense in $\{ \frac{a}{b} \in \mathbb{Q} : (b, 2d) = 1 \}$, which is a dense subset of the rationals. Let $x = \frac{a}{b}$ be a rational number written in lowest terms with $(b, 2d) = 1$. Let $\tilde{a}$ be the inverse of $a \pmod{b}$. The assumptions on $a$ and $b$ allow us to find an infinite sequence of primes $p$ satisfying the requirement that $d^2e^2 + 4d$ be a square mod $p$, where $e := (ap - 1)/b$.

Indeed, since $(b, 2d) = 1$, $b$ is odd and shares no common factors with $d$. So $(b, 4(4b^2d + d^2)) = (b, 4b^2d + d^2) = (b, d) = 1$. By Dirichlet’s theorem on primes in arithmetic progressions and the Chinese Remainder Theorem, there exist infinitely many primes $p$ such that

$$p \equiv 1 \pmod{4(4b^2d + d^2)}, \quad p \equiv \tilde{a} \pmod{b},$$

where $\tilde{a}$ is the inverse of $a \pmod{b}$. The first congruence condition guarantees that $p \equiv 1 \pmod{4}$ and $p \equiv 1 \pmod{4b^2d + d^2}$. As above, define $e := (ap - 1)/b$. This is an integer by the second congruence condition. Then

$$\left(\frac{d^2e^2 + 4d}{p}\right) = \left(\frac{d^2 + 4b^2d}{p}\right) = \left(\frac{p}{d^2 + 4b^2d}\right) = \left(\frac{1}{d^2 + 4b^2d}\right) = 1.$$

Therefore, $d^2e^2 + 4d$ is a square (mod $p$). So there exists an $\ell$ with $(\ell, p) = 1$ such that

$$d^2e^2 + 4d \equiv (2\ell - de)^2 \pmod{p},$$

which simplifies to

$$(1 + \ell e)d \equiv \ell^2 \pmod{p}.$$

Let $k$ be the inverse of $\ell \pmod{p}$. Then $k(k + e)d \equiv 1 \pmod{p}$. We set

$$a_1 := k\sqrt{d} \quad \text{and} \quad a_2 := (k + e)\sqrt{d}.$$

Then $(a_i, p) = 1$, and

$$a_1 a_2 = k(k + e)d \equiv 1 \pmod{p}.$$

Since $(p, k) = (p, d) = 1$, we have $(p, kd) = 1$. Therefore, there exist integers $x_1, y_1$ such that $px_1 + kdy_1 = 1$, and thus

$$px_1 + a_1(y_1\sqrt{d}) = 1.$$

Similarly, there exist integers $x_2, y_2$ such that $px_2 + a_2y_2\sqrt{d} = 1$. Let

$$A_1 = \begin{pmatrix} a_1 & -x_1 \\ p & y_1\sqrt{d} \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_2 & -x_2 \\ p & y_2\sqrt{d} \end{pmatrix},$$

and

$$A_3 = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} = A_2^{-1}A_1.$$
By construction, each of these matrices are in $SL_2(\mathcal{O}_L)$. So we may apply Lemma 1 to find
\[ D_L(a_3, c_3) = E_2(0) \left( \frac{2}{pe\sqrt{d}} + \frac{e\sqrt{d}}{p} \right) \]
\[ = E_2(0)2^{\sqrt{d}} \left( \frac{2}{ep} + \frac{e}{p} \right). \]
Thus,
\[ \tilde{D}_L(a_3, c_3) = \frac{2}{ep} + \frac{e}{p}, \]
and by substituting $e = (ap - 1)/b$, we have
\[ \frac{2b}{p(ap - 1)} + \frac{ap - 1}{bp}. \]
Taking the limit as $p \to \infty$ and using the fact that
\[ E_2(0) I \left( x\sqrt{d} \right) = 2E_2(0)x\sqrt{d}, \]
we have
\[ \lim_{p \to \infty} \tilde{D}_L(1 - kd, pe\sqrt{d}) = 2x. \]
Therefore, since $2x$ is a rational number, we can conclude that the Dedekind sums are dense in the rational numbers. This concludes the proof.

Before concluding, it is of interest to mention that while our result extends Ito’s in a certain sense, it remains open whether the graph $(a/c, \tilde{D}_L(a/c))$ is dense in $\mathbb{C} \times \mathbb{R}$ for general $L$. Indeed, in [4], Kohnen also poses the question of whether his method can be used with classical Dedekind sums to prove the density statement for the graph.

Remark 1. We are grateful to the anonymous referee for providing the following insightful comments.

1. There is an alternative proof for Theorem 1 using properties of the group homomorphism $\Phi$ established in [7]. It was shown in that paper that the set of values of $\Phi$ forms a subgroup of the additive group of complex numbers which has rank greater or equal to $h = h(\mathcal{O}_L)$ when viewed as a $\mathbb{Z}$-module. So when the normalized values are all real, then the values can not be all rational in the case $h > 1$. Some of these values must be irrational if $h > 1$. From this it follows easily that some of the normalized Dedekind sums must be irrational as well since the other term contributing to $\Phi(a, b, c, d)$ is rational (after normalization). From this it follows that the values of the normalized Dedekind sums are dense on the real line in the case of a lattice $L$ with a real $j$-invariant $j(L)$ and $h(\mathcal{O}_L) > 1$. A variation of this argument also shows the density of the normalized values of the Dedekind sums in the excluded case $h = 1$. The key observation here is that the values $I((a + d)/c)/\sqrt{d}$ are dense in the field of rational numbers.

2. The density result also has a geometric interpretation. If $\mathcal{O}_L$ is fixed, then one can chose a lattice $L$ in each similarity class of lattices with that fixed multiplier ring $\mathcal{O}_L$. This gives a set of $h$ homomorphisms $\Phi_L$. The set of these $h$ homomorphisms take values in $\mathbb{C}^h$. In fact, the set of values in
\( \mathbb{C}^h \) forms a discrete lattice. The situation is analogous to the case of the ring of algebraic integers in a real quadratic field. These integers are dense on the real line once a specific real embedding of the real quadratic field was chosen. However, if both embeddings are considered simultaneously, the integers form a lattice in \( \mathbb{R}^2 \).

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