ON STABLE ORBITAL INTEGRALS AND THE STEINBERG-HITCHIN BASE

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Abstract. This paper studies the geometric side of the stable trace formula. First, we express the elliptic regular terms over the Steinberg-Hitchin base, following the suggestion of Frenkel-Langlands-Ngô. Then we prove splitting and descent formulas for singular stable orbital integrals, and write down general stable germ expansions. Finally, we give a condition for the convergence of the limiting distributions on the geometric side involved in the Beyond Endoscopy limit.

Contents

1. Introduction

Let G be a reductive group over a number field F. The stable trace formula is an identity of distributions on the adelic group $G(\mathbf{A})$, and has led to the resolution of the endoscopic classification of automorphic representations of classical groups, following the work of many, most of all Arthur, Moeglin, and Waldspurger on the stabilisation of the invariant trace formula and its twisted form. The invariant trace formula I^G is an identity of distributions on G, referred to as the spectral and geometric expansions. It has a decomposition into stable distributions,

$$I^G(f) = \sum_{G'} \iota(G, G') \hat{S}^{\tilde{G}'}(f')$$

where the sum G' runs over isomorphism classes of elliptic endoscopic data, and the function f' is the Langlands-Shelstad transfer of the test function f from G to fixed central extension $\tilde{G}'$ of G'. The problem now known as Beyond Endoscopy, proposed by Langlands [Lan04], is to further refine the trace formula in order to access cases of functoriality beyond the endoscopic setting. Arthur [Art17] has

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formalised this as a two-step problem: first establishing an r-stable trace formula $S^r_{\mathrm{cusp}}(f)$, where r is a representation of the L-group ${}^LG$ of G, whose spectral side consists only of automorphic representations in the cuspidal part of the stable trace formula $S_{\mathrm{cusp}}(f) = S^1_{\mathrm{cusp}}(f)$ for which the associated automorphic L-function has a pole at $s = 1$; then secondly, a primitivisation of this trace formula, in analogy with the twisted endoscopic situation

$$S^r_{\mathrm{cusp}}(f) = \sum_{G'} \iota(r, G')\hat{P}^{\tilde{G}'}_{\mathrm{cusp}}(f'), \tag{1.1}$$

where G' is a suitable generalisation of endoscopic data and $\iota(r, G')$ a suitable generalisation of the endoscopic coefficient. The precise formulation of this is still tentative, as a host of unresolved difficulties arise along the way. Nonetheless, one hopes to obtain a decomposition in terms of primitive, stable distributions P^G_{cusp} which represent the spectral contribution to the stable trace formula of tempered, cuspidal automorphic representations that are primitive, in the sense that they are not functorial images from some smaller group. The function f' would then be a the stable transfer of f from G to $\tilde{G}'$, that is naturally described by the local Langlands correspondence. As a first step towards establishing an r-trace formula, one needs first to remove the contribution of noncuspidal representations, which promises to be both a difficult and valuable problem in its own right. In particular, being able to isolate the distribution $S^G_{\mathrm{cusp}}(f)$ essentially amounts to establishing the Ramanujan conjecture on average for G.

This paper lays the ground work in anticipation of the proposal of Frenkel, Langlands, and Ngô [FLN10, Lan13] to apply an additive Poisson summation formula to the regular elliptic terms in the stable trace formula. The purpose of such a formula is to cancel the contribution of the noncuspidal spectrum as desired. This was solved in a special case for GL_2 by Altuğ [Alt15a] using a variation of this method, continuing the initial analysis of Langlands in [Lan04]. The stable trace formula is an identity of stable distributions

$$\sum_{t\geq 0}\sum_{M\in\mathcal{L}} |W^M_0||W^G_0|^{-1}\int_{\Phi_t(M,V,\zeta)} b^M(\phi)S_M(\phi, f)d\phi$$

and

$$\sum_{M\in\mathcal{L}} |W^M_0||W^G_0|^{-1}\sum_{\delta\in\Delta(M,V,\zeta)} b^M(\delta)S_M(\delta, f),$$

where we refer to [Art02] for precise definitions. The contribution of the regular elliptic terms, which we shall write as $S_{\mathrm{ell}}(f)$ can be indexed by the Steinberg-Hitchin base $\mathcal{A}(G, V, \zeta)$ which is parametrised by semisimple stable conjugacy classes of G, and contains a linear vector space $\mathcal{B}(G, V, \zeta)$ over which the summation will take place.

We shall provide a construction in Section 3 of the Steinberg-Hitchin base in terms of stable distributions, in the generality of Arthur's stable trace formula, and index the regular elliptic terms using this base. We then prove splitting and descent formulas for singular stable orbital integrals along the lines of Arthur's work [Art99]. These singular distributions enter in when extending the distributions beyond the elliptic locus of the Steinberg-Hitchin base. In Section 4, we give a treatment of the stable germ expansions for orbital integrals, of the kind used in [Art02] but remain unpublished. Good control over the germ expansions is needed to study the asymptotics of the Fourier transform of orbital integrals. Finally, in Section 5 we discuss the convergence of the distributions arising in the geometric expansion of $S^r_{\mathrm{cusp}}(f)$. Using some of the results in Sections 3 and 4, we provide a condition for their existence in Theorem 5.1, the main result of this paper, which represents a first exploration in establishing a general r-trace formula. We mention that [Mok18] has obtained such a formula in a special case by working directly with the spectral side, which sidesteps the analytic problems on the geometric side which we discuss.

The astute reader will observe that we have not broached the core problem, which is the Poisson summation formula. This has only been proven in the case of GL_2 using an approximate functional equation [Alt15a], but appears to pose problems in higher rank [GKM$^+$18]. Besides this, the existence of such a formula has proved difficult and has simply been taken as a working hypothesis such as in [FLN10, Art18]. Nonetheless, our study of stable orbital integrals will have wider applications to the study of the stable trace formula, for example in the study of basic functions initiated in [Ngô14].

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2. Geometric side: regular elliptic terms

For convenience of the reader, in this section we shall review certain definitions and basic constructions from [Art02] required for the geometric side of the stable trace formula, and refer there for more details. We use this to give a simple expression for the regular elliptic contribution to the trace formula.

2.1. Definitions and distributions.

2.1. Definitions and distributions. Let G be a connected, quasisplit reductive group over a number field F. As usual, a Levi subgroup is taken to mean the F-rational Levi component of a parabolic subgroup of G. Given a Levi subgroup M, denote by $\mathcal{L}(M)$ be the finite set of Levi subgroups of G containing M, and let $\mathcal{L}^0(M)$ be the set of proper Levi subgroups of G containing M. Also, we shall identify the Weyl group of (G, A_M) with the quotient of the normaliser of M by M, thus

$$W^G(M) = \mathrm{Norm}_G(M)/M.$$

Let M_0 be a minimal Levi subgroup, which we shall assume to be fixed, and denote $\mathcal{L} = \mathcal{L}(M_0), \mathcal{L}^0 = \mathcal{L}^0(M_0)$, and $W_0^G = W^G(M_0)$.

Let $X^*(G)_F$ be the group of F-rational characters of G. Let A_G be the F-split component of the centre of G, and set $\mathfrak{a}_G = \mathrm{Hom}(X^*(G)_F, \mathbf{R})$. For any subgroup H of G, define

$$G(\mathbf{A})^H = \{x \in G(\mathbf{A}) : H_G(x) \in \mathrm{Im}(\mathfrak{a}_H \to \mathfrak{a}_G)\}$$

where H_G is the canonical map from $G(\mathbf{A})$ to $\mathfrak{a}_G$, defined by $e^{\langle H_G(x), \chi\rangle} = |\chi(x)|$, for $\chi \in X^*(G)_F$ and $x \in G(F)$. For any $c \in G$, we denote by $G_{c,+}$ the centraliser of c in G, and G_c the connected component of the identity in $G_{c,+}$. We say a semisimple element c is F-elliptic if $A_{G_c} = A_G$.

2.1.1. *z-extensions.* We call a torus induced if it is a finite product of $\mathrm{Res}_{E/F}(\mathbf{G}_m)$, where E is a finite extension of F, and central if it embeds into the center $Z(G)$ of G. Recall that a z-extension of G is a surjective homomorphism $\alpha : G' \to G$, where G' is a connected reductive group over F whose derived group G'_{der} is simply connected, and $\ker(\alpha)$ is an induced torus.

Define the central data (Z,ζ) of G to be a fixed central induced torus Z of G, and a character ζ of $Z(\mathbf{A})/Z(F)$. Note that if $H=1$, then $G(\mathbf{A})^1=\{x\in G(\mathbf{A}): H_G(x)=0\}$, and if $H\supset Z$, then $Z(\mathbf{A})\subset G(\mathbf{A})^H$. The group Z acts on functions f and distributions D on G by $(zf)(x)=f(z^{-1}x)$ and $(xD)(f)=D(z^{-1}f)$ for any $z\in Z$. Then we say that f is ζ^{-1}-equivariant if $zf=\zeta(z)^{-1}f$ for all $z\in Z$, and that D is ζ-equivariant if $zD=\zeta(z)^{-1}D$ for for all $z\in Z$.

Let V be a finite set of valuations of F that contains the places over which G, Z, and ζ are ramified, and write G_V^Z for the subgroup of elements of x in $G_V=G(F_V)$ with $F_V=\prod_{v\in V}F_v$, such that $H_G(x)$ lies in the image of $\mathfrak{a}_Z$ in $\mathfrak{a}_G$, and ζ_V for the restriction of ζ to the subgroup Z_V of $Z(\mathbf{A})$. We shall also denote $\bar{G}_V=(G/Z)(F_V)$, which is equal to G_V/Z_V since Z is an induced torus.

2.1.2. *Multiple groups.* In fact, we shall be working in the setting where G is a K-group in the sense of [Art02, §4]. That is, $G=\coprod_\alpha G_\alpha$ for $\alpha\in\pi_0(G)$, such that any connected redutive group is a component of a K-group that is unique up to weak isomorphism. It comes with a local product structure

$$G_V=\prod_{v\in V}\coprod_{\alpha_v\in\pi_0(G_v)}G_{v,\alpha_v}.$$

We similarly extend the definitions of Z_V and ζ_V. Namely, we let Z be an induced torus of F with central embeddings $Z\simeq Z_\alpha$ into the center $Z(G_\alpha)$ of G_α for each $\alpha\in\pi_0(G)$ over F, that are compatible with the isomorphisms $\psi_{\alpha,\beta}:G_\beta\to G_\alpha$. Then any character ζ of $Z(\mathbf{A})/Z(F)$ determines a character ζ_α on $Z_\alpha(\mathbf{A})/Z_\alpha(F)$ for each α. We have the quotient

$$\bar{G}=G/Z=\coprod_{\alpha\in\pi_0(G)}(G_\alpha/Z_\alpha),$$

which is again a K-group, and note that for any central extension of K-groups $\tilde{G}$ by a central torus $\tilde{Z}$, we also have $G=\tilde{G}/\tilde{Z}$. We also fix a compatible family of inner twists $\psi_\beta:G_\beta\to G^*$, where G^* is a quasisplit inner twist of G. We say G is quasisplit if it contains a component G_β that is quasisplit.

2.1.3. *Hecke algebras.* Let $\mathcal{C}(G_V^Z,\zeta_V)$ be the space of ζ_V^{-1}-equivariant Schwartz functions on G_V^Z, and the Hecke subalgebra

$$\mathcal{H}(G,V,\zeta)=\mathcal{H}(G_V^Z,\zeta_V)=\coprod_{\alpha_V}\mathcal{H}(G^Z_{V,\alpha_V},\zeta_{V,\alpha_V})$$

of compactly supported, bi-K_∞-invariant functions in $\mathcal{C}(G_V^Z,\zeta_V)$. Here $K_\infty=K_{V_\infty}$ is a chosen maximal compact subgroup of G_{V_∞}, and V_∞ denotes the archimedean valuations in V. We define a projection from the non-equivariant Hecke algebra $\mathcal{H}(G_V^Z)$ onto $\mathcal{H}(G_V^Z,\zeta_V)$ by

$$f\mapsto f^\zeta=\int_{Z(F)}\zeta(z)f_z dz,\quad f\in\mathcal{H}(G_V^Z),$$

where f_z is defined by $f_z(x)=f(zx)$ for any $x\in G_V^Z$. Given a function $\dot{f}^1$ in $\mathcal{H}(G)=\mathcal{H}(G(\mathbf{A})^1)$, we define a projection to $\dot{f}^\zeta$ in $\mathcal{H}(G,\zeta)=\mathcal{H}(G(\mathbf{A})^Z,\zeta)$ given by

$$\dot{f}^\zeta(x)=\int_{Z(\mathbf{A})^x}\dot{f}^1(zx)\zeta(x)dz,\quad x\in G(\mathbf{A})^Z,$$

where $Z(\mathbf{A})^x = \{z \in Z(\mathbf{A}) : H_G(zx) = 0\}$. On the other hand, there is a map $f \to \dot{f} = f \times u^V$ from $\mathcal{H}(G, V, \zeta)$ to $\mathcal{H}(G, \zeta)$, where u^V is the function on $G^V = \prod_{v \notin V} G_v$ with support equal to $K^V Z^V$ and such that

$$u^V(kz) = \zeta(z)^{-1}, \quad k \in K^V, z \in Z^V.$$

This allows us to relate linear forms on $\mathcal{H}(G)$ to those on $\mathcal{H}(G, V, \zeta)$. Let $\mathcal{H}_{\mathrm{adm}}(G, V, \zeta)$ be the subspace of functions in $\mathcal{H}(G, V, \zeta)$ whose support is an admissible subset of G_V in the sense of [Art02, §1].

2.1.4. *Invariant distributions.* Let $\mathcal{D}(G_V^Z, \zeta_V)$ be the vector space of distributions on G_V^Z that are (i) invariant under conjugation by G_V^Z, (ii) ζ_V-equivariant under translation by Z_V, and (iii) supported on the preimage in G_V^Z of a finite union of conjugacy classes in $\bar{G}_V^Z = G_V^Z / Z_V$. Let c belong to the subset $\Gamma_{\mathrm{ss}}(\bar{G}_V^Z)$ of semisimple conjugacy classes in $\bar{G}_V^Z$. Denote by $\mathcal{D}_c(G_V^Z, \zeta_V)$ the subspace of distributions in $\mathcal{D}(G_V^Z, \zeta_V)$ for which the conjugacy classes in (iii) all have semisimple parts equal to c. It is non-zero only if the class c consists of images of semisimple conjugacy classes in G_V^Z whose stabiliser in Z_V lies in the kernel of ζ_V. There is a decomposition

$$\mathcal{D}(G_V^Z, \zeta_V) = \bigoplus_c \mathcal{D}_c(G_V^Z, \zeta_V) \tag{2.1}$$

where the direct sum is taken over classes c in $\Gamma_{\mathrm{ss}}(\bar{G}_V^Z)$. Also, define the semisimple part of a distribution D in $\mathcal{D}(G_V^Z, \zeta_V)$ to be the union of those classes c for which the image of D in $D_c(G_V^Z, \zeta_V)$ is nonzero.

Given $f \in C_c^\infty(G_V^Z)$ and $\gamma_V \in G_V^Z$, the (normalised) orbital integral of f at γ_V is

$$f_G(\gamma_V) = |D(\gamma_V)|^{\frac{1}{2}} \int_{(G_{\gamma_V} \cap G_V^Z)\backslash G_V^Z} f(x^{-1}\gamma_V x) dx$$

where $D(\gamma_V)$ is the Weyl discriminant, and for some choice of invariant measure on $(G_{\gamma_V} \cap G_V^Z)\backslash G_V^Z$. We can also define the ζ_V-equivariant orbital integral

$$\int_{Z_V} \zeta(z) f_G(z\gamma_V) dz$$

in $\mathcal{D}(G_V^Z, \zeta_V)$. Let $\mathcal{D}_{\mathrm{orb}}(G_V^Z, \zeta_V)$ be the subspace of $\mathcal{D}(G_V^Z, \zeta_V)$ spanned by such distributions. Fix a basis $\Gamma(G_V^Z, \zeta_V)$ of $\mathcal{D}(G_V^Z, \zeta_V)$ satisfying natural compatibility conditions as in [Art02, §1]. For example, any $\gamma \in \Gamma(G_V^Z, \zeta_V)$ is assumed to have a decomposition $\gamma = \prod_{v \in V} \gamma_v$, where $\gamma_v \in \Gamma(G_v^Z, \zeta_v)$, relative to fixed bases $\Gamma(G_v^Z, \zeta_v)$ of $\mathcal{D}(G_v^Z, \zeta_v)$. Also, the semisimple part of any element in $\Gamma(G_V^Z, \zeta_V)$ is a single class c, that is, the intersection

$$\Gamma_c(G_V^Z, \zeta_V) = \Gamma(G_V^Z, \zeta_V) \cap \mathcal{D}_c(G_V^Z, \zeta_V), \qquad c \in \Gamma_{\mathrm{ss}}(\bar{G}_V^Z, \zeta_V)$$

is a basis for $\mathcal{D}_c(G_V^Z, \zeta_V)$. Similarly, we require $\Gamma_{\mathrm{orb}}(G_V^Z, \zeta_V)$ to be a basis of $\mathcal{D}_{\mathrm{orb}}(G_V^Z, \zeta_V)$, giving a sequence of inclusions

$$\Gamma_{\mathrm{orb}}(G_V^Z, \zeta_V) \supset \Gamma_{\mathrm{ss}}(G_V^Z, \zeta_V) \supset \Gamma_{\mathrm{reg}}(G_V^Z, \zeta_V) \supset \Gamma_{\mathrm{ell}}(G_V^Z, \zeta_V),$$

corresponding to subsets of clases in $\Gamma_{\mathrm{orb}}(\bar{G}_V^Z, \zeta_V)$ that are, respectively, semisimple, strongly regular, and strongly regular elliptic. Now let $\Gamma_{\mathrm{ell}}(G, V, \zeta)$ be the set of elements $\gamma \in \Gamma_{\mathrm{orb}}(G_V^Z, \zeta)$ such that there is a $\dot{\gamma} \in G(F)$ satisfying

(ii) the semisimple part of $\dot{\gamma}$ is F-elliptic in G,
(ii) the conjugacy class of $\dot{\gamma}$ in G_V maps to γ, and

(iii) for every $v \notin V$, the image of $\dot{\gamma}$ in G_v lies in a compact subgroup.

Given $\mu \in \Gamma_{\mathrm{ell}}(G,V,\zeta)$, define μ^G to be the distribution in $\Gamma(G_V^Z,\zeta)$ such that $f_G(\mu^G) = f_M(\mu)$ for any $f \in \mathcal{H}(G_V^Z,\zeta_V)$. Then define the subset $\Gamma(G,V,\zeta) \subset \Gamma(G_V^Z,\zeta_V)$ consisting of induced distributions μ^G for each $\mu \in \Gamma_{\mathrm{ell}}(M,V,\zeta)$ and $M \in \mathcal{L}$.

2.1.5. *Stable distributions.* We call a distribution on G_V^Z is stable if it lies in the closed linear span of the strongly regular, stable orbital integrals

$$f^G(\delta_V) = \sum_{\gamma_V \sim \delta_V} f_G(\gamma_V). \tag{2.2}$$

Here δ_V is a strongly regular, stable conjugacy class in G_V^Z and the summation is taken over the finite set of distinct conjugacy classes γ_V in δ_V in the set $\Gamma_{\mathrm{reg}}(\bar{G}_V^Z) = \Gamma_{\mathrm{reg}}(\bar{G}_V^Z,\zeta_V)$ that map to δ. The basis $\Gamma(G_V^Z,\zeta_V)$ is chosen so that

$$f^G(\delta_V) = (\delta_V/\delta)f^G(\delta), \quad f_G(\gamma_V) = (\gamma_V/\gamma)f_G(\gamma).$$

Here (γ_V/γ) is the ratio of the invariant measure on γ_V and the signed measure on γ_V that comes with γ, and $(\delta_V/\delta) = (\gamma_V/\gamma)$ where γ_V is the conjugacy class in δ_V that maps to the given class γ. In particular, the distribution δ is stable, and we have identified $\Delta_{\mathrm{reg}}(G_V^Z)$ with a subset $\Delta_{\mathrm{reg}}(G_V^Z,\zeta_V)$ of $\mathcal{SD}(G_V^Z,\zeta_V)$, which we define to be the subspace of stable distributions in $\mathcal{D}(G_V^Z,\zeta_V)$.

We fix a suitable basis

$$\Delta(G_V^Z,\zeta_V) = \Delta^{\mathcal{E}}(G_V^Z,\zeta_V) \cap \mathcal{SD}(G_V^Z,\zeta_V)$$

for $\mathcal{SD}(G_V^Z,\zeta_V)$, where the endoscopic basis $\Delta^{\mathcal{E}}(G_V^Z,\zeta_V)$ is also a basis for $\mathcal{D}(G_V^Z,\zeta_V)$ which is constructed as a quotient of G-relevant pairs in (G',δ') where $G'_V \in \mathcal{E}(G,V)$ and $\delta' \in \Delta((G'_V)^{Z'},\zeta'_V)$ [Art02, §4]. By restricting to the subset of stable distributions, we can form the basis $\Delta_{\mathrm{orb}}(G_V^Z,\zeta_V)$ of $\mathcal{SD}_{\mathrm{orb}}(G_V^Z,\zeta_V)$, giving a chain of inclusions

$$\Delta_{\mathrm{orb}}(G_V^Z,\zeta_V) \supset \Delta_{\mathrm{ss}}(G_V^Z,\zeta_V) \supset \Delta_{\mathrm{reg}}(G_V^Z,\zeta_V) \supset \Delta_{\mathrm{ell}}(G_V^Z,\zeta_V),$$

parallel to that of $\Gamma(G_V^Z,\zeta_V)$ above.

As a K-group, we have a decomposition

$$\Delta(G_V,\zeta_V) = \coprod_{\alpha\in\pi_0(G)} \Delta(G_{V,\alpha_V},\zeta_{V,\alpha_V}) \tag{2.3}$$

of the vector space

$$\mathcal{SD}(G_V,\zeta_V) = \bigoplus_{\alpha\in\pi_0(G)} \mathcal{SD}(G_{V,\alpha_V},\zeta_{V,\alpha_V})$$

of ζ_V-equivariant stably invariant distributions on G_V.

Then as before, we define the elliptic subset $\Delta^{\mathcal{E}}_{\mathrm{ell}}(G,V,\zeta)$ to be the set of $\delta \in \Delta^{\mathcal{E}}(G_V^Z,\zeta_V)$ such that either the transfer factor $\Delta_G(\gamma,\delta) \neq 0$ for some $\gamma \in \Gamma_{\mathrm{ell}}(G,V,\zeta)$, or δ is in the image in $\Delta^{\mathcal{E}}(G_V^Z,\zeta_V)$ of some element $\delta' \in \Delta_{\mathrm{ell}}(\tilde{G}',V,\tilde{\zeta})$ for some proper elliptic endoscopic group $G' \in \mathcal{E}^0_{\mathrm{ell}}(G,V)$. Then define the intersection $\Delta_{\mathrm{ell}}(G,V,\zeta) = \Delta^{\mathcal{E}}_{\mathrm{ell}}(G,V,\zeta) \cap \Delta(G_V^Z,\zeta_V)$ and using this construct the larger subset $\Delta(G,V,\zeta)$ parallel to $\Gamma(G,V,\zeta)$.

Finally, we define the invariant Hecke space

$$\mathcal{I}(G_V^Z,\zeta_V) = \mathcal{I}(G,V,\zeta) = \{f_G : f \in \mathcal{H}(G_V^Z,\zeta_V)\},$$

and the stably invariant Hecke space

$$\mathcal{SI}(G_V^Z,\zeta_V) = \mathcal{SI}(G,V,\zeta) = \{f^G : f \in \mathcal{H}(G_V^Z,\zeta_V)\}$$

which we will need later. We view the objects in these spaces as functions on $\Gamma_{\mathrm{reg}}(G_V^Z,\zeta_V)$ and $\Delta_{\mathrm{reg}}(G_V^Z,\zeta_V)$ respectively.

2.1.6. *Endoscopic data.* Finally, we recall the elliptic endoscopic data $G' \in \mathcal{E}_{\mathrm{ell}}(G)$ for G over F, which represents the tuple $(G',\mathcal{G}',s',\xi')$. Here G' is a quasisplit group over F, $\mathcal{G}'$ is a split extension of the Weil group W_F by the dual group $\hat{G}'$ of G',

$$1 \to \hat{G}' \to \mathcal{G}' \to W_F \to 1,$$

s' is a semisimple element in $\hat{G}$, and $\xi' : \mathcal{G}' \to {}^L G$ is an L-embedding. It is required that $\xi'(\hat{G})$ is equal to the connected centraliser of s' in $\hat{G}$, and that $\xi'(u')s' = s'\xi'(u')a(u')$ where $u' \in \mathcal{G}'$ and a is a 1-cocycle from W_F to $Z(\hat{G})$ that is locally trivial in the sense that its image in $H^1(W_{F_v},Z(\hat{G}))$ is trivial for every v. We say the endoscopic datum is elliptic in the sense that the image of ξ' in ${}^L G$ in not contained in ${}^L M$ for any proper Levi subgroup M of G over F. The endoscopic datum also comes with the auxiliary data $(\tilde{G}',\tilde{\xi}')$ where $\tilde{G}'$ is a central extension of G' by an induced torus $\tilde{C}'$ over F,

$$1 \to \tilde{C}' \to \tilde{G}' \to G' \to 1,$$

and $\tilde{\xi}' : \mathcal{G}' \to {}^L\tilde{G}'$ is an L-embedding. We shall also denote G^* to be the quasisplit inner form of G.

2.2. **The geometric side.** We now state the geometric side of the stable trace formula for G quasisplit over F. It is given by the stable linear form

$$S(f) = \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\delta\in\Delta(M,V,\zeta)} b^M(\delta)S_M(\delta,f), \tag{2.4}$$

where $S_M(\delta,f)$ and $b^M(\delta)$ are defined in (2.6) and (2.7) respectively below. The theory of endoscopy reduces the study of automorphic forms on reductive groups to that on quasisplit groups. Thus our primary interest will be in the stable trace formula for quasisplit groups. Implicitly, we shall also require a finite set $S \supset V$ such that the support of f is S-admissible in the sense of [Art02, §1]. The strongly regular elliptic stable classes are contained in the summands corresponding to $M = G$. The stable distribution is denoted $S_{\mathrm{orb}}(f)$, the so-called orbital part of the $S(f)$, thus

$$S^G_{\mathrm{orb}}(f) = \sum_{\delta\in\Delta(G,V,\zeta)} b^G(\delta)S_G(\delta,f)$$

where $S_G(\delta,f)$ is equal to the stable orbital integral

$$f^{\mathcal{E}}_G(\delta) = \sum_{\gamma\in\Gamma(G,\zeta)} \Delta(\delta,\gamma)f_G(\gamma),$$

where $\Delta(\delta,\gamma)$ is the usual transfer factor. Suppose G' is an endoscopic datum for G (relative to an implicitly chosen dual $\hat{G}$ of G). The transfer factor depends also on a choice of auxiliary datum $(\tilde{G}',\tilde{\zeta}')$ for G'. It is a smooth function Δ of $\delta \in \Delta_{G\text{-reg}}(\tilde{G}')$ and $\gamma \in \Gamma_{\mathrm{reg}}(G)$, determined uniquely up to a multiplicative constant of absolute value 1.

2.2.1. *The stable distributions.* We shall take M' to represent an elliptic endoscopic datum $(M', \mathcal{M}', s'_M, \xi'_M)$ for M over F as in [Art02, §4]. We assume implicitly that M' has been equipped with auxiliary data $(\tilde{M}', \tilde{\xi}')$ required for transfer, so that $\tilde{M}' \to M'$ is a central extension by an induced torus over F, while $\tilde{\xi}' : \mathcal{M}' \to {}^L\tilde{M}'$ is an L-embedding. Define $\mathcal{E}_{M'}(G)$ to be the set of endoscopic data $(G', \mathcal{G}', s', \xi')$ for G where s' lies in $s'_M(Z(\hat{M})^\Gamma)$, and $\mathcal{G}' = \mathcal{M}'\hat{G}'$. We also define $\mathcal{E}^0_{M'}(G) = \mathcal{E}_{M'}(G) - \{G^*\}$, for G quasisplit.

$$\iota_{M'}(G, G') = |Z(\hat{M}')^\Gamma / Z(\hat{M})^\Gamma||Z(\hat{G}')^\Gamma / Z(\hat{G})^\Gamma|^{-1}, \quad G' \in \mathcal{E}_{M'}(G), \tag{2.5}$$

which vanish unless G' is elliptic.

We then define a family of stable distributions $S^G_M(M', \delta', f)$ for any $\delta' \in \Delta^{\mathcal{E}}(\tilde{M}', \tilde{\zeta}')$ inductively by the formula

$$I^{\mathcal{E}}_M(\delta', f) = \sum_{G' \in \mathcal{E}^0_{M'}(G)} \iota_{M'}(G, G') \hat{S}^{\tilde{G}'}_{\tilde{M}'}(\delta', f) + \varepsilon(G) S^G_M(M', \delta', f') \tag{2.6}$$

where $\varepsilon(G) = 1$ if G is quasisplit and 0 otherwise. We also require that $I^{\mathcal{E}}_M(\delta', f) = I_M(\delta, f)$ if G is quasisplit and δ' maps to $\delta \in \Delta^{\mathcal{E}}(M, \zeta)$. If G is quasisplit and $M' = M^*$, we set

$$S^G_M(\delta, f) = S^G_M(M^*, \delta^*, f).$$

By the local theorems of [Art02] we know that these linear forms are stable, so we have the linear form $\hat{S}^{G^*}_{M^*}(\delta^*, f^*) = S^G_M(\delta, f)$ on $\mathcal{SI}(G^*, \zeta^*)$, which occurs in the inductive definition above. In any case, we shall be mainly concerned with the simplest case, whereby $M = G$ and G quasisplit, so that $S^G_G(\delta, f) = f^G(\delta)$.

2.2.2. *The stable geometric coefficient.* The global coefficient $b^G(\delta)$ is given by

$$b^G(\delta) = \sum_{M \in \mathcal{L}} |W^M_0||W^G_0|^{-1} \sum_{\ell \in \mathcal{L}^V_{\mathrm{ell}}(M/Z, S)} b^M_{\mathrm{ell}}(\delta_M \times \ell) s^G_M(\ell). \tag{2.7}$$

It is stable in the sense that is supported on the basis $\Delta(G, V, \zeta)$, and is independent of the set S. The 'elliptic' coefficient b^M_{ell} is a modification of the stable coefficient $b^{\tilde{G}'}_{\mathrm{ell}}(\dot{\delta}'_S)$ so as to be indexed by admissible elements in $\Delta(G^Z_S, \zeta_S)$ rather than the (G, S)-equivalence classes in $(G(F))_{G,S}$. Let $\mathcal{L}(\bar{G}^V_S)$ be the set of conjugacy classes in $\bar{G}^V_S$ that are bounded, are contained in a compact subgroup for each $v \in S - V$. Given $\ell \in \mathcal{L}(G^V_S)$ we obtain a distribution $\delta^V_S(\ell)$, and given $\delta \in \Delta(G^Z_V, \zeta_V)$, we write $\delta \times \ell = \delta \times \delta^V_S(\ell)$. Finally $s^G_M(\ell)$ is an unramified stable orbital integral which comes by the weighted fundamental lemma.

Having Global Theorem 1 of [Art02] in hand, we can define the stable coefficients $b^{\tilde{G}'}_{\mathrm{ell}}(\dot{\delta}'_S)$ inductively by the endoscopic coefficient

$$a^{G,\mathcal{E}}_{\mathrm{ell}}(\dot{\gamma}_S) = \sum_{G'} \sum_{\dot{\delta}'_S} \iota(G, G') b^{\tilde{G}'}_{\mathrm{ell}}(\dot{\delta}'_S) \Delta_G(\dot{\delta}'_S, \dot{\gamma}_S)$$

with G' and $\dot{\delta}'_S$ are summed over $\mathcal{E}_{\mathrm{ell}}(G, S)$ and $\Delta_{\mathrm{ell}}(\tilde{G}', S, \tilde{\zeta}')$ respectively [Art01, (1.1*)]. The coefficients $b^{\tilde{G}'}_{\mathrm{ell}}(\dot{\delta}'_S)$ are defined inductively by $a^{G,\mathcal{E}}_{\mathrm{ell}}(\dot{\gamma}_S) = a^G_{\mathrm{ell}}(\dot{\gamma}_S)$ and $b^{G^*}_{\mathrm{ell}}(\dot{\delta}^*_S) = b^G_{\mathrm{ell}}(\dot{\delta}_S)$. Here

$$a^G_{\mathrm{ell}}(\dot{\gamma}) = \sum_{\{\dot{\gamma}\}} |Z(F, \dot{\gamma})|^{-1} a^G(S, \dot{\gamma})(\dot{\gamma}/\dot{\gamma}_S),$$

where the sum is taken over $Z_{S,\mathfrak{o}}$-orbits in $(G(F))_{G,S}$ that map to $\dot\gamma_S$ and such that the $G(\mathbf{A}^S)$-conjugacy class of $\dot\gamma$ in $G(\mathbf{A}^S)$ meets K^S. Here $a^G_{\mathrm{ell}}(\dot\gamma)$ is supported on the set $\Gamma_{\mathrm{ell}}(G,V,\zeta)$, which consists of $\gamma\in\Gamma_{\mathrm{orb}}(G^Z_V,\zeta_V)$ such that (i) the semisimple part of $\dot\gamma$ is F-elliptic in G, (ii) the conjugacy class of $\dot\gamma$ in G_V maps to γ, and (iii) for each $v\notin V$, the image of $\dot\gamma$ in G_v lies in a compact subgroup. Given the standard Jordan decomposition $\dot\gamma=c\dot\alpha$ of a (G,S)-equivalence class in $G(F)$,

$$a^G(S,\dot\gamma)=i^G(S,c)|\mathrm{Stab}(c,\dot\alpha)|^{-1}a^{G_c}(S,\dot\alpha).$$

Here $i^G(S,c)$ is 1 if c is F-elliptic in G and the $G(\mathbf{A}^S)$-conjugacy class of c meets K^S, and otherwise it is 0. $\mathrm{Stab}(c,\dot\alpha)$ is the stabiliser of $\dot\alpha$ in the finite group $G_{c,+}(F)/G_c(F)$, acting on the set of unipotent conjugacy classes in $G_c(F_S)$. If $\dot\gamma$ is regular elliptic then $c=1$, then $a^{G_c}(S,1)=\mathrm{vol}(G_c(F)\backslash G_c(\mathbf{A})^1)$, whereas in general, one does not have an explicit formula for these coefficients.

2.2.3. *Regular elliptic terms.* We first want to separate the contribution of semisimple elements to the geometric side of the trace formula. This can be seen in terms of the Jordan decomposition of the stable distributions supported on $\delta=d\beta$ where $d\in\Delta_{\mathrm{ss}}(\bar M)$ and $\beta\in\Delta_{\mathrm{unip}}(\bar M)$ for each $M\in\mathcal{L}$. We shall describe this in detail later below (see also 4.2). The contribution of strongly regular elliptic part of the trace formula corresponds to the terms in (2.4) with $M=G$ and $\delta\in\Delta(G,V,\zeta)$ strongly regular elliptic. We may express this as

$$\begin{aligned} S_{\mathrm{ell}}(f)&=\sum_{\Delta_{G\text{-reg,ell}}(G,V,\zeta)} b^G(\delta)S_G(\delta,f)\\ &=\sum_{\Delta_{G\text{-reg,ell}}(G,V,\zeta)} \mathrm{vol}(G_\delta(F)\backslash G_\delta(\mathbf{A})^1)\iota(G_\delta,G)f^G(\delta), \end{aligned}\tag{2.8}$$

as given in [Art05, (27.11)] when $G'=G^*$, except that we have now replaced the sum over strongly G-regular elliptic stable conjugacy classes δ, with the subset of stable distributions $\Delta_{G\text{-reg,ell}}(G,V,\zeta)$ of $\mathcal{SD}(G^Z_V,\zeta_V)$ as described above. The coefficient $\iota(G_\delta,G)$ is equal to the ratio of absolute Tamagawa numbers $\tau(G)\tau(G_\delta)^{-1}$ by [Kot84, 8.3.1], where the definition of the Tamagawa number can be taken to be

$$\tau(G)=|\pi_0(Z(\hat G)^{\Gamma_F})||\ker^1(F,Z(\hat G))|^{-1}.$$

We briefly recall the Tamagawa measure. Let G be a connected reductive group over F. For each place v, choose a Haar measure on F_v such that for almost every finite v the ring of integers $\mathcal{O}_V$ has unit measure. The product of these measures gives a measure on $\mathbf{A}$, and further assume that $\mathrm{vol}(F\backslash\mathbf{A})=1$. Now fix a nonzero differential form of top degree on $\mathfrak{g}$ over F. For each place v, this differential form and the measure on F_v induces a measure dX_v on $\mathfrak{g}(F_v)$. The set of Haar measures on $\mathfrak{g}(F_v)$ is identified with that on $G(F_v)$ where two measures correspond if and only if the Jacobian of the exponential map is 1 at $0\in\mathfrak{g}(F_v)$ with the given measures. This gives a measure dg_v on $G(F_v)$. Fix a finite set of places V containing the archimedean places of F such that G is unramified outside of V. Denote by σ_G the representation of Γ_F acting on $X^*(G)\otimes_{\mathbf{Z}}\mathbf{C}$, and let $L_v(\sigma_G,s)$ be the associated local Artin L-function. Let r be the order of the pole of the partial product $L^V(\sigma_G,s)$ outside of V at $s=1$, and set

$$\ell^V_G=\lim_{s\to1}(s-1)^rL^V(\sigma_G,s).$$

For every $v \notin V$, fix a hyperspecial compact subgroup $K_v \subset G(F_v)$ and a canonical measure dg_v^{can} on $G(F_v)$ such that the measure on K_v is 1. Then the Tamagawa measure dg_V^{Tam} on $G(F_V)$ is given by

$$(\ell_G^V)^{-1} \prod_{v \notin V} L_v(\sigma_G, 1) dg_v \prod_{v \in V} dg_v,$$

so that $dg_V^{\mathrm{Tam}} \otimes (\otimes_{v \notin V} dg_v^{\mathrm{can}})$ is equal to the Tamagawa measure on $G(\mathbf{A})$.

Let $\mathfrak{A}_G$ be the neutral component of $A_{G_{\mathbf{Q}}}(\mathbf{R})$, so that $A_G^0 \subset A_G(F_\infty)$. We may identify it with $\mathrm{Hom}(X^*(G)^{\Gamma_F}, \mathbf{R})$ where $X^*(G)$ is the group of algebraic characters of G, and Γ_F the absolute Galois group of F. Define the lattice $\mathfrak{A}_{G,\mathbf{Z}} = \mathrm{Hom}(X^*(G)^{\Gamma_F}, \mathbf{Z})$, and denote by $\mathrm{covol}(\mathfrak{A}_{G,\mathbf{Z}})$ its covolume in $\mathfrak{A}_G$. If the measure chosen on $G(\mathbf{A})$ and $\mathfrak{A}_G$ is the Tamagawa measure, then $\mathrm{covol}(\mathfrak{A}_{G,\mathbf{Z}}) = 1$ and $\mathrm{vol}(\mathfrak{A}_G G(F)\backslash G(\mathbf{A})) = \tau(G)$. But the Tamagawa measure on $\mathfrak{A}_G$ is inconvenient, as noted in [MW16b, VII.4.1], thus it will be better to fix the Tamagawa measure on $G(\mathbf{A})$ but not on $\mathfrak{A}_G$, which then gives the relation

$$\mathrm{vol}(\mathfrak{A}_G G(F)\backslash G(\mathbf{A})) = \mathrm{covol}(\mathfrak{A}_{G,\mathbf{Z}})^{-1}\tau(G).$$

Since δ is strongly regular elliptic, it follows that $\mathrm{covol}(\mathfrak{A}_{G,\mathbf{Z}}) = \mathrm{covol}(\mathfrak{A}_{G_\delta,\mathbf{Z}})$ [FLN10, p.224], and following their notation we shall denote this by m_G. There is an isomorphism of the two quotients $G_\delta(F)\backslash G_\delta(\mathbf{A})^1$ and $\mathfrak{A}_{G_\delta} G_\delta(F)\backslash G_\delta(\mathbf{A})$ induced by the inclusion map from $G_\delta(\mathbf{A})^1$ into $G_\delta(\mathbf{A})$, so the measure on the two spaces are equivalent. It follows then that

$$\mathrm{vol}(G_\delta(F)\backslash G_\delta(\mathbf{A})^1)\tau(G)\tau(G_\delta)^{-1} = m_G \tau(G).$$

We have now arrived at the desired expression for the regular elliptic contribution.

Lemma 2.1. *Let G be a quasisplit reductive K-group over F, and V a finite set of places as above. Fix the Tamagawa measure on G_δ for each regular elliptic δ. The regular elliptic part of the stable trace formula for G can be expressed as*

$$S_{\mathrm{reg,ell}}(f) = m_G \tau(G) \sum_{\delta \in \Delta_{G\text{-reg,ell}}(G,V,\zeta)} f^G(\delta). \tag{2.9}$$

Proof. This expression also follows from the definition of the geometric coefficients. Given δ a strongly regular elliptic conjugacy class in $G'(F)$, we have $b_{\mathrm{ell}}^G(\delta) = \tau(G)$ [Art03, p.847], which follows from the unipotent descent formula for the elliptic geometric coefficient

$$b_{\mathrm{ell}}^G(\dot{\delta}_V) = \sum_d j^{G^*}(V,d) b_{\mathrm{ell}}^{G_d^*}(\dot{\beta})$$

of [Art01, Theorem 1.1(b)], where $j^{G^*}(V,d) = i^{G^*}(V,d)\tau(G^*)\tau(G_d^*)^{-1}$, where G_d^* represents a quasisplit connected centraliser of an appropriate representative of the class d. The sum d is taken over the set of elements in $\Delta_{\mathrm{ss}}(G^*)$ whose image in $\Delta_{\mathrm{ss}}(G_V)$ equals d_V, and $\dot{\beta}$ is the image of $\dot{\beta}_V$ in $\Delta_{\mathrm{unip}}(G_{d,V}^*)$. Here we are using the Jordan decomposition for elements in $\Delta(G_V, \zeta_V)$, whereby the semisimple part of any such element $\dot{\delta}_V$ is given by a semisimple stable conjugacy class $d_V \in \Delta_{\mathrm{ss}}(\bar{G}_V^*)$, with representative chosen such that G_d^* is quasisplit, while the unipotent part is defined as an element $\dot{\beta}_V$ in the subset of $\Delta_{\mathrm{unip}}(G_{d,V}^*)$ of $\Delta(G_V, \zeta_V)$ with semisimple part equal to 1. The Jordan decomposition is then denoted by the formal product

$$\dot{\delta}_V = d_V \dot{\beta}_V.$$

These definitions naturally extend to the set $\Delta(G, V, \zeta)$.

Also, it follows from [Kot84, Theorem 8.3.1] that

$$\iota(G, G')b_{\mathrm{ell}}^{G'}(\delta') = \tau(G)|\mathrm{Out}_G(G')|^{-1}.$$

In particular, the right-hand side is independent of δ'. Recall that $\mathrm{Out}_G(G') = \mathrm{Aut}_G(G')/\hat{G}$ where $\mathrm{Aut}_G(G')$ is the group of automorphisms $\alpha : G' \to G'$ of G' as an endoscopic datum of G. □

Remark 2.2. A similar expression for the regular elliptic terms can be found in [MW16b, p.792] for the twisted case, which has also been previously computed by Labesse, Kottwitz, and Shelstad. In the present setting, the expression here should be compared with the expression given in [FLN10, (3.25)], which differs from ours due to the choice of measure. A word of caution is due here. It seems as if we have made the summand into a product of local distributions, but in fact there remans a global factor that remains in the definition of the measure dg_V, which obstructs the use of a Poisson summation formula. That is, for the elliptic torus G_δ, using a formula due to T. Ono,

$$\mathrm{vol}(G_\delta(F)\backslash G_\delta(\mathbf{A})^1) = L(1, \sigma_{G_\delta})\tau(G_\delta)$$

where σ_{G_δ} is the Artin representation obtained from the action of $\mathrm{Gal}(\bar{F}/F)$ on the roots $X^*(G_\delta) \otimes \bar{F}$. Here the measure is chosen so as to be compatible with the Tamagawa measure.

3. Over the Steinberg-Hitchin base

3.1. **The base.** Our first step is to replace the sum over $\Delta_{\mathrm{reg,ell}}(G, V, \zeta)$ with the Steinberg-Hitchin base, which was introduced in [FLN10]. We shall summarise its construction here. Let $C = Z(G)^0$ be the connected component of the identity in the center of G, and set $A = C \cap G_{\mathrm{der}}$. Let T be a maximal split torus in G, and W the Weyl group of G relative to T. The Chevalley isomorphism $\bar{F}[G]^G \simeq \bar{F}[T]^W$ gives a morphism

$$G \to T/W$$

that we view as the characteristic polynomial map. If G is semisimple, that is, $G = G_{\mathrm{der}}$, and split over F, then there is a correspondence between the semisimple stable conjugacy classes of G with T/W. If moreover G is simply connected, then there is an isomorphism $T/W \simeq \mathbb{A}^r$, where $\mathbb{A}^r$ denotes r-dimensional affine space and $r = \mathrm{rank}(G_{\mathrm{der}})$. Its coordinates are parametrised by the irreducible, finite dimensional representations of G attached to fundamental dominant weights (relative to some splitting of G).

Now let G be a general reductive group and quasisplit over F, whose derived group G_{der} is simply connected. There is an exact sequence

$$1 \to A \to G_{\mathrm{der}} \times C \to G \to 1,$$

which descends to the maximal torus

$$1 \to A \to T_{\mathrm{der}} \times C \to T \to 1.$$

Then the Steinberg-Hitchin base associated to $G(F)$ is an algebraic variety whose F-points are given by

$$\mathcal{A}(G) = \coprod_\eta \mathcal{A}_\eta(G) = \coprod_\eta (\mathcal{B}_\eta(G) \times C_\eta(F))/A(F), \tag{3.1}$$

where for a fixed cocycle $\eta \in H^1(F, A)$, the products $\mathcal{B}_\eta(G) \times C_\eta(F)$ are twisted forms of $\mathcal{B}(G) \times C(F)$. Here $\mathcal{B}(G) = T_{\mathrm{der}}/W$ is a linear vector space of dimension equal to $\mathrm{rank}(G_{\mathrm{der}})$ over F, and $C_\eta(F)$ is a $C(F)$-torsor. The group $A(F)$ acts on $B_\eta(G)$ and $C_\eta(F)$ as a subgroup of $G_{\mathrm{der}}(F)$ and $C(F)$ respectively. The action of the cocycle η is described explicitly in [FLN10, §3.3], and the union over η is taken over the subset $\mathfrak{h}$ consisting of elements $\eta \in H^1(F, A)$ given by the images of cocycles $\sigma(g)^{-1}g$ with $g \in G_{\mathrm{der}}(F), \sigma \in \mathrm{Gal}(\bar{F}^{\mathrm{sep}}/F)$. In particular, if G is split over F, then only the trivial cocyle occurs and $\mathfrak{h}$ is trivial.

We have thus a fibration

$$c : G(F) \to \mathcal{A}(G),$$

sending any element $g \in G(F)$ to the semisimple part of its stable conjugacy class, given by $a \in \mathcal{A}(G)$. We have a sequence of inclusions

$$\mathcal{A}_{\mathrm{ell}}(G) \subset \mathcal{A}_{\mathrm{reg}}(G) \subset \mathcal{A}(G),$$

where $\mathcal{A}_{\mathrm{ell}}(G)$ is the image of the strongly regular elliptic elements in $G(F)$, and $\mathcal{A}_{\mathrm{reg}}(G)$ the image of strongly regular semisimple elements, which forms a dense open subset of $\mathcal{A}(G)$. Its complement is defined by the zero locus of $D(\gamma)$, the generalised Weyl discriminant

$$D(\gamma) = \det(I - \mathrm{Ad}(\gamma^{-1}))|_{\mathfrak{g}/\mathfrak{t}} = \prod_\alpha (1 - \alpha(\gamma)), \tag{3.2}$$

The product is taken over all roots α of T, and satisfying the product formula $\prod_v |D(\gamma_v)| = 1$. It is indepdendent of choice of γ_v in the stable conjugacy class of δ_v. Also define $\Delta(\gamma) = \gamma^{-\rho} \prod_{\alpha>0}(\alpha(\gamma) - 1)$, for some integer ρ, whereby $|D(\gamma)| = |\Delta(\gamma)|^2$. The functions depend only on the stable conjugacy class δ of γ, so one may also write $\Delta(\delta)$ and $D(\delta)$.

More generally, suppose that G is a reductive K-group, whose derived group is simply connected. Then we can associate to it the base

$$\mathcal{A}(G) = \coprod_{\alpha \in \pi_0(G)} \mathcal{A}(G_\alpha) = \coprod_{\alpha \in \pi_0(G)} \coprod_{\eta \in \mathfrak{h}_\alpha} (\mathcal{B}_{\alpha,\eta}(G) \times C_{\alpha,\eta}(F))/A_\alpha(F),$$

where each term is the Steinberg-Hitchin base (3.1) associated to the connected group G_α, and η belongs to the subset $\mathfrak{h}_\alpha \subset H^1(F, A_\alpha), \alpha \in \pi_0(G)$. We shall also write for the direct sum of vector spaces

$$\mathcal{B}_\eta(G) = \bigoplus_{\alpha \in \pi_0(G)} \mathcal{B}_\eta(G_\alpha).$$

Let now M be a standard Levi subgroup of G, then its embedding into G induces a map of characteristic polynomials

$$\pi_M : \mathcal{A}(M) \to \mathcal{A}(G)$$

that is finite of degree $|W_0^G||W_0^M|^{-1}$ and étale over $\mathcal{A}_{\mathrm{reg}}(G)$. There is a decomposition

$$\mathcal{A}_{\mathrm{reg}}(G) = \coprod_{M \in \mathcal{L}} \pi_M(\mathcal{A}_{\mathrm{ell}}(M)) \tag{3.3}$$

where $\mathcal{L}$ is the set of standard Levi subgroups of G.

The base $\mathcal{A}(G)$ parametrises semisimple stable conjugacy classes in $G(F)$. Similarly, the fibre over a_v in $\mathcal{A}(G_v)$ for any v forms a stable conjugacy class in $G(F_v)$.

Given the product group G_V we shall then associate the base $\mathcal{A}(G_V)$, and the related constructions are completely parallel to those given above.

3.1.1. *Distributions.* Recall that the geometric terms in the stable trace formula are indexed by distributions in $\Delta(G, V, \zeta)$, rather than stable conjugacy classes. As our goal is to index the elliptic part of the stable trace formula using the Steinberg-Hitchin base instead, we therefore have to replace the base $\mathcal{A}(G_V)$ with an appropriate set $\mathcal{A}(G, V, \zeta)$ of distributions on G_V.

By construction, the basis of semisimple stable conjugacy classes $\Delta_{\mathrm{ss}}(\bar{G}_v)$ is in bijection with $\mathcal{A}(G_v)$, and similarly $\Delta_{\mathrm{ss}}(\bar{G}^Z_V)$ with $\mathcal{A}(G^Z_V)$. Following the constructions in 2.1.5, we shall define the space $\mathcal{A}(G^Z_V, \zeta_V)$ to be the subset of the basis $\Delta(G_V, \zeta_V)$, consisting of ζ_V-equivariant stable distributions supported on the preimage of the equivalence classes in $\Delta_{\mathrm{ss}}(\bar{G}_V, \zeta_V)$. There is then a decomposition

$$\mathcal{A}(G^Z_V, \zeta_V) = \coprod_{\alpha \in \pi_0(G)} \prod_{v \in V} \mathcal{A}(G^Z_{\alpha,v}, \zeta_{\alpha,v})$$

parallel to (2.3). Define also $\mathcal{A}_{\mathrm{ell}}(G^Z_V, \zeta_V)$ to be the distributions corresponding to $\Delta_{\mathrm{ell}}(G, V, \zeta)$. Then given a $\mu \in \mathcal{A}_{\mathrm{ell}}(G, V, \zeta)$, define μ^G to be the distribution in $\mathcal{A}(G^Z_V, \zeta)$ such that $f^G(\mu^G) = f^M(\mu)$ for any $f \in \mathcal{H}(G^Z_V, \zeta_V)$. Then define the subset

$$\mathcal{A}(G, V, \zeta) \subset \mathcal{A}(G^Z_V, \zeta_V)$$

consisting of all such induced distributions μ^G, for each $\mu \in \mathcal{A}_{\mathrm{ell}}(M, V, \zeta)$ and $M \in \mathcal{L}$. Equivalently, we can also write

$$\mathcal{A}(G, V, Z) = \mathcal{A}(G^Z_V) \cap \Delta(G, V, \zeta),$$

which follows from the definitions. We shall again refer to the distributions $\mathcal{A}(G, V, \zeta)$ as the Steinberg-Hitchin base associated to G_V, as it indeed parametrises the semisimple distributions in the stable trace formula.

The map $c : G(F_V) \to \mathcal{A}(G_V)$ induces a pushforward of stable distributions from $G(F_V)$ to $\mathcal{A}(G_V)$. We may view the elements of $\mathcal{A}(G, V, \zeta)$ as distributions on $\mathcal{A}(G_v)$. While the stable orbital integrals can be viewed as distributions in $\mathcal{A}(G.V, \zeta)$, the measure implicitly used in defining the orbital integral is different from the natural one on the fibre over $a \in \mathcal{A}$. So if $c(\delta) = a$, we shall write $f^G(a) = f^G(\delta)$ both locally and globally, with integration taken with respect to the canonical measure, which we discuss below.

Parallel to the linear subspace $\mathcal{B}(G_v)$ of $\mathcal{A}(G_v)$, we shall also construct a linear subspace $\mathcal{B}(G, V, \zeta)$ from $\mathcal{A}(G, V, \zeta)$ as follows: we can view the space

$$\mathcal{B}(G) = \coprod_{\alpha} \coprod_{\eta} \mathcal{B}_\eta(G_\alpha)$$

as parametrizing semisimple elements in $G = (G_{\mathrm{der}} \times C)/A$ with trivial c component. This gives a subset of the basis $\Delta_{\mathrm{ss}}(G^Z_V)$, and in turn parametrises a subset of ζ_V-equivariant distributions in $\mathcal{A}(G^Z_V, \zeta_V)$, which we shall call $\mathcal{B}(G^Z_V, \zeta_V)$. It is again a vector space over F_V. Finally, we also define

$$\mathcal{B}(G, V, \zeta) = \mathcal{B}(G^Z_V) \cap \mathcal{A}(G, V, \zeta),$$

which is the space that we will be primarily interested in.

3.2. **A family of distributions.** By the Jordan decomposition, the geometric terms in the trace formula can be seen as fibering over the base $\mathcal{A}(G,V,\zeta)$ by sending any distribution $\delta \in \Delta(G,V,\zeta)$ to its semisimple part, reminiscent of the semisimple descent formulas. But for the Poisson summation we are only interested in the semisimple distributions $\delta \in \Delta_{\mathrm{ss}}(G,V,\zeta)$, and more specifically, the elliptic ones. For such δ we have a family of distributions over the Steinberg-Hitchin base, obtained from the family of stable linear forms at semisimple elements, defined by

$$S_M^G(a,f) = S_M^G(d,f). \tag{3.4}$$

Here $d \in \Delta_{\mathrm{ss}}(M,V,\zeta)$ maps to $a \in \mathcal{A}(M,V,\zeta)$, and if $M = G$ we simply write $S_M^G(a,f) = f^G(a)$. We first show that the semisimple distributions $S_M^G(\delta,f)$ can indeed be parametrised by the Steinberg-Hitchin base for a general G, in particular, including the case where G_{der} is not simply connected.

Lemma 3.1. *The linear forms $f^G(\delta)$ and the global coefficient $b^G(\delta)$ for $\delta \in \Delta_{\mathrm{ss}}(G,V,\zeta)$ are parametrised by the Steinberg-Hitchin base $\mathcal{A}(G,V,\zeta)$.*

Proof. We first reduce to the case where G_{der} is simply connected. If G_{der} is not simply-connected, we know that there is a z-extension, i.e., an extension $\tilde{G}$ of G by a central induced torus $\tilde{S}$ over F such that $\tilde{G}_{\mathrm{der}}$ is simply connected. We write $\tilde{Z}$ for the preimage of Z in $\tilde{G}$, and $\tilde{\zeta}$ for the pullback of ζ to $\tilde{Z}(\mathbf{A})/\tilde{Z}(F)$. We have that $G_V \simeq \tilde{G}_V/\tilde{S}_V$, so that distributions on G_V can be viewed as $\tilde{Z}_V$-invariant distributions on $\tilde{G}_V$, and there is a canonical isomorphism $f \to \tilde{f}$ from $\mathcal{H}(G,V,\zeta)$ to $\mathcal{H}(\tilde{G},V,\tilde{\zeta})$. We can assume that the bases $\Gamma(\tilde{G}_V,\tilde{\zeta}_V)$ and $\Delta(\tilde{G}_V,\tilde{\zeta}_V)$ are the images of the corresponding bases $\Gamma(G_V,\zeta_V)$ and $\Delta(G_V,\zeta_V)$ under the canonical maps $\gamma_V \mapsto \tilde{\gamma}_V$, and $\delta_V \mapsto \tilde{\delta}_V$ of $\tilde{\zeta}$-equivariant distributions.

From the proof of [Art01, §2], we obtain the identities

$$b_{\mathrm{ell}}^G(\dot{\delta}_V) = b_{\mathrm{ell}}^G(\tilde{\dot{\delta}}_V)$$

for $\dot{\delta}_V \in \Delta_{\mathrm{ell}}(G,V,\zeta)$, which follows from the identity

$$S_{\mathrm{ell}}^G(\dot{f}_V) = S_{\mathrm{ell}}^{\tilde{G}}(\tilde{\dot{f}}_V)$$

where $\dot{f}_V \in \mathcal{H}_{\mathrm{adm}}(G,V,\zeta)$ as in [Art02, §2]. Or, stated locally,

$$S_M^G(M',\delta',f) = S_{\tilde{M}}^{\tilde{G}}(\tilde{M}',\tilde{\delta}',\tilde{f}), \quad M' \in \mathcal{E}_{\mathrm{ell}}(M), \delta' \in \Delta_{G\text{-reg,ell}}(\tilde{M}',\tilde{\zeta}').$$

Also, we have from [Art03, (7.2)] the relation

$$S_M^G(M',\delta'_\zeta,f^\zeta) = (\delta'/\delta'_\zeta)^{-1}\int_{Z(F)} S_M^G(M',\delta',f_z)\zeta(z)dz$$

for $M' \in \mathcal{E}_{\mathrm{ell}}(M), \delta'_\zeta \in \Delta_{G\text{-reg,ell}}(\tilde{M}',\tilde{\zeta}')$, and an element $\delta' \in \Delta_{G\text{-reg,ell}}(\tilde{M}',\tilde{\eta}')$ that maps to δ'_ζ. Then the right-hand side corresponds to the case where Z is trivial. It follows then that the statement for arbitrary (Z,ζ) is implied by the case where Z is trivial.

It remains to show that the restriction of the coefficient $b^M(\delta)$ to semisimple elements δ in the fiber over $\mathcal{A}(M_V^Z,\zeta_V)$ vanishes outside the set $\mathcal{A}(M,V,\zeta)$. From Global Theorem 1' of [Art02] we know that $b^M(\delta)$ is stable in the sense that it vanishes outside the subset $\Delta(M,V,\zeta)$ of the basis $\Delta(M_V^Z,\zeta_V)$ of $\mathcal{SD}(M_V^Z,\zeta_V)$. Thus if we view the restriction of $b^M(\delta)$ to $\Delta_{\mathrm{ss}}(M_V^Z) = \mathcal{A}(M_V^Z)$, the previous condition then implies that $b^M(\delta)$ is supported on the subset $\mathcal{A}(M,V,\zeta)$. □

3.2.1. *Splitting and descent.* The distributions $S_M^G(\delta, f)$ are first constructed as stable linear forms on a connected reductive group, and then extended to a product of several copies of G. It is these compound forms that occur in the stable trace formula. They satisfy splitting formulas, and if δ belongs to a proper Levi subgroup, descent formulas that reduce the study of the compound distributions to the special case of the simple ones, in which δ is elliptic in the corresponding Levi subgroup.

It is follows from 3.4 that the descent and splitting properties of the original distributions $S_M^G(\delta, f)$ carry over to the family of distributions $S_M^G(a, f)$, and along the way, we shall also establish formulas for the endoscopic linear forms. To state the result, we first require some definitions. Let $V = V_1 \sqcup V_2$ be a finite set of places of F, and fix a function $f_V \in \mathcal{C}(G_V, \zeta_V)$ where

$$f_V = f_{V_1} \times f_{V_2}, \quad f_{V_i} \in \mathcal{C}(G_{V_i}, \zeta_{V_i}).$$

Given Levi subgroups $L_1, L_2 \in \mathcal{L}(M)$, we define

$$\mathrm{e}_M^G(L_1, L_2) = d_M^G(L_1, L_2)|Z(\hat{L}_1)^\Gamma \cap Z(\hat{L}_2)^\Gamma / Z(\hat{G})^\Gamma|^{-1}.$$

If $d_M^G(L_1, L_2)$ is nonzero, then $\mathfrak{a}_{L_1}^* \cap \mathfrak{a}_{L_2} = \mathfrak{a}_G^*$, so the identity component of $Z(\hat{L}_1)^\Gamma \cap Z(\hat{L}_2)^\Gamma$ is equal to that of $Z(\hat{G})^\Gamma$, therefore $\mathrm{e}_M^G(L_1, L_2)$ is also nonzero. The same also holds true for $\mathrm{e}_{M_1}^G(M, L_2)$.

Moreover, let M_1 be a proper Levi subgroup of M, with a fixed dual subgroup $\hat{M}_1 \subset M$. Any Levi subgroup $L_1 \in \mathcal{L}(M_1)$ comes with a dual Levi subgroup $\hat{L}_1 \subset \hat{G}$ containing $\hat{M}_1$, by which we define the stable coefficient

$$\mathrm{e}_{M_1}^G(M, L_1) = d_M^G(M, L_1)|Z(\hat{M})^\Gamma \cap Z(\hat{L}_1)^\Gamma / Z(\hat{G})^\Gamma|^{-1} \tag{3.5}$$

where the coefficient $d_M^G(M, L_1)$ is defined as in [Art99].

The formulas are a direct consequence of the splitting and descent formulas for $S_M^G(\delta, f)$. For δ strongly G-regular, they are given by Theorem 6.1 and Theorem 7.1 of [Art99], and provide a decomposition of the compound distributions over $\Delta((\tilde{M}'_V)^{\tilde{Z}'}, \tilde{\zeta}'_V)$ into simple linear forms over $\Delta_{G\text{-reg}}(\tilde{M}', \tilde{\zeta}')$. For the general case, the formal structure of the arguments are similar to that for strongly G-regular elements. We provide the argument here for the singular distributions $S_M^G(\delta, f)$.The general descent formulas are described in unpublished work of Arthur, and we shall provide some of the arguments here.

Proposition 3.2. *(a) Suppose G is arbitrary. Then we have the descent formula*

$$I_M^{G,\mathcal{E}}(\delta', f) = \sum_{L_1 \in \mathcal{L}(M_1)} d_{M_1}^G(M, L_1) \hat{I}_{M_1}^{L_1,\mathcal{E}}(\delta_1, f^{L_1}).$$

(b) Suppose that G is quasisplit, and that $M' = M^*$. Then

$$S_M^G(\delta, f) = \sum_{L_1 \in \mathcal{L}(M_1)} \mathrm{e}_{M_1}^G(M, L_1) \hat{S}_{M_1}^{L_1}(\delta_1, f^{L_1}). \tag{3.6}$$

for $\delta = \delta', \delta_1 = \delta'_1$. If $M' \neq M^*$, then the distribution $S_M^G(M', \delta', f)$ vanishes.

Proof. For any $\delta' \in \Delta((\tilde{M}'_V)^{\tilde{Z}'}, \tilde{\zeta}'_V), M' \in \mathcal{E}_{\mathrm{ell}}(M, V)$, and $f \in \mathcal{H}(G, V, \zeta)$, we view the stable linear forms $S_M^G(M', \delta', f)$ as an element of $\mathcal{A}(G, V, \zeta)$. We shall first show the descent formula (3.6). We begin with the difference

$$I_M^{\mathcal{E}}(\delta', f) - \varepsilon(G) S_M^G(M', \delta', f') = \sum_{G' \in \mathcal{E}_{M'}^0(G)} \iota_{M'}(G, G') \hat{S}_{\tilde{M}'}^{\tilde{G}'}(\delta', f). \tag{3.7}$$

Applying (3.6) inductively to the terms in the right-hand sum, we write

$$\hat{S}^{\tilde{G}'}_{\tilde{M}'}(\delta', f') = \sum_{L'_1 \in \mathcal{L}^{G'}(M'_1)} \mathrm{e}^{G'}_{M'_1}(M', L'_1)\hat{S}^{\tilde{L}'_1}_{\tilde{M}'_1}(\delta'_1, f'_1).$$

where $f'_1 = (f')^{L'_1}$. We can assume that the endoscopic datum G' for G is elliptic. There is a canonical Levi subgroup $G' \in \mathcal{L}(M_1)$ for which L'_1 is an elliptic endoscopic datum. On the other hand, if L_1 is given, then L'_1 is uniquely determined by G' and there is a mapping $G' \to L'_1$ from $\mathcal{E}_{M'}(G)$ to $\mathcal{E}_{M'_1}(L_1)$, sending $s' \in s'_M Z(\hat{M})^\Gamma / Z(\hat{G})^\Gamma$ to the point $s'_{M_1} Z(\hat{M}_1)^\Gamma / Z(\hat{L}_1)^\Gamma$. We may therefore write (3.7) as the sum

$$\sum_{L_1 \in \mathcal{L}(M_1)} \sum_{G' \in \mathcal{E}^0_{M'}(G)} \iota_{M'}(G, L'_1)\mathrm{e}^{G'}_{M'_1}(M', L'_1)\hat{S}^{\tilde{L}'_1}_{\tilde{M}'_1}(\delta'_1, f_1).$$

If we let $\varepsilon_{M'}(G) = 1$ if G is quasisiplit and $M' = M^*$, and 0 otherwise, then we can replace the inner sum above with the complete set $\mathcal{E}_{M'}(G)$ if we also subtract the contribution

$$\varepsilon_{M'}(G) \sum_{L_1 \in \mathcal{L}(M_1)} \mathrm{e}^G_{M_1}(M, L_1)\hat{S}^{L_1}_{M_1}(\delta_1, f^{L_1}). \tag{3.8}$$

Then substituting (3.5), and using the fact that $d^{G'}_{M'_1}(M', L'_1) = d^G_{M_1}(M, L_1)$, we conclude that (3.7) is equal to the difference of the sum

$$\sum_{L_1 \in \mathcal{L}(M_1)} d^G_{M_1}(M, L_1) \sum_{G' \in \mathcal{E}_{M'}(G)} \frac{|Z(\hat{M}')^\Gamma / Z(\hat{M})^\Gamma|}{|Z(\hat{M}')^\Gamma \cap Z(\hat{L}'_1)^\Gamma / Z(\hat{G})^\Gamma)|} \hat{S}^{\tilde{L}'_1}_{\tilde{M}'_1}(\delta'_1, f'_1) \tag{3.9}$$

and the expression (3.8).

The sum over $G' \in \mathcal{E}_{M'}(G)$ depends only on the image of G' in $\mathcal{E}_{M'_1}(G)$, so we may replace the sum if we multiply by the order of the preimage of $L'_1 \in \mathcal{E}_{M'_1}(G)$ in $\mathcal{E}_{M'}(G)$. Also, by [Art99, Lemma 1.1], we have $Z(\hat{M}_1)^\Gamma = Z(\hat{M})^\Gamma Z(\hat{L}_1)^\Gamma$. It then follows that $\mathcal{E}_{M'}(G)$ maps surjectively onto $\mathcal{E}_{M'_1}(L_1)$, and the finite group $Z(\hat{M})^\Gamma \cap Z(\hat{L}_1)^\Gamma / Z(\hat{G})^\Gamma$ acts simply transitively on the fibres. Therefore the order of this preimage is equal to the order of the finite group. For the resulting product of coefficients, observe that there is a surjective map

$$(Z(\hat{M}')^\Gamma / Z(\hat{M})^\Gamma) \times (Z(\hat{L}'_1)^\Gamma / Z(\hat{L}_1)^\Gamma) \to Z(\hat{M}'_1)^\Gamma / Z(\hat{M}_1)^\Gamma$$

whose kernel is isomorphic to the quotient

$$(Z(\hat{M}')^\Gamma \cap Z(\hat{L}'_1)^\Gamma) / (Z(\hat{M})^\Gamma \cap Z(\hat{L}_1)^\Gamma).$$

It follows then that

$$\begin{aligned} &|Z(\hat{M}')^\Gamma / Z(\hat{M})^\Gamma||Z(\hat{M}')^\Gamma \cap Z(\hat{L}'_1)^\Gamma / Z(\hat{G})^\Gamma|^{-1}|Z(\hat{M})^\Gamma \cap Z(\hat{L}_1)^\Gamma / Z(\hat{G})^\Gamma| \\ &= |Z(\hat{L}'_1)^\Gamma / Z(\hat{L}_1)^\Gamma|^{-1}|Z(\hat{M}'_1)^\Gamma / Z(\hat{M}_1)^\Gamma| = \iota_{M'_1}(L_1, M_1). \end{aligned}$$

We can therefore write (3.9) as

$$\sum_{L_1 \in \mathcal{L}(M_1)} d^G_{M_1}(M, L_1) \sum_{L'_1 \in \mathcal{E}_{M'_1}(G)} \iota_{M'_1}(L_1, M_1)\hat{S}^{\tilde{L}'_1}_{\tilde{M}'_1}(\delta'_1, f'_1),$$

which is equal to

$$\sum_{L_1\in\mathcal{L}(M_1)} d_{M_1}^G(M,L_1)\hat{I}_{M_1}^{L_1,\mathcal{E}}(\delta_1',f_{L_1})$$

by the inductive definition (2.6). In the case $\varepsilon(G)=1$, it follows then that if we let δ and δ_1 be the images of δ' and δ_1' in $\mathcal{D}^{\mathcal{E}}(M,\zeta)$ and $\mathcal{D}^{\mathcal{E}}(M_1,\zeta)$ respectively, we obtain

$$I_M(\delta,f)=\sum_{L_1\in\mathcal{L}(M_1)} d_{M_1}^G(M,L_1)\hat{I}_{M_1}^{L_1}(\delta_1,f_{L_1}),$$

and the descent formula (3.6) follows from the identity between $S_M^G(M',\delta',f')$ in (3.7) and (3.8). $\square$

We now turn to the splitting formula.

Proposition 3.3. *(a) Suppose that G is arbitrary, and that $\gamma_V=(\gamma_{V_1},\gamma_{V_2})\in\Gamma_{G_V}(M_V)$. Then*

$$I_M^{G,\mathcal{E}}(\gamma_V,f_V)=\sum_{L_1,L_2\in\mathcal{L}(M)} d_M^G(L_1,L_2)\hat{I}_M^{L_1,\mathcal{E}}(\gamma_{V_1},f_{V_1}^{L_1})\hat{I}_M^{L_2,\mathcal{E}}(\gamma_{V_2},f_{V_2}^{L_2}).$$

(b) Suppose that G is quasisplit, and that $\delta_V=(\delta_{V_1},\delta_{V_2})\in\Delta_{G_V}(M_V)$. Then

$$S_M^G(\delta_V,f_V)=\sum_{L_1,L_2\in\mathcal{L}(M)} \mathrm{e}_M^G(L_1,L_2)\hat{S}_M^{L_1}(\delta_V,f_{V_1}^{L_1})\hat{S}_M^{L_2}(\delta_V,f_{V_2}^{L_2}). \tag{3.10}$$

Proof. We shall proceed as in the proof of the descent formula. Beginning again with the difference

$$I_M^{\mathcal{E}}(\delta_V',f_V)-\varepsilon(G)S_M^G(M_V',\delta_V',f_V')=\sum_{G_V'\in\mathcal{E}_{M'}^0(G_V)} \iota_{M'}(G,G')\hat{S}_{\tilde{M}'}^{\tilde{G}'}(\delta_V',f_V), \tag{3.11}$$

we apply (3.10) inductively to the right-hand sum to get

$$\hat{S}_{\tilde{M}'}^{\tilde{G}'}(\delta',f)=\sum_{L_1',L_2'\in\mathcal{L}^{G'}(M')} \mathrm{e}_{M'}^{G'}(L_1',L_2')\hat{S}_{\tilde{M}'}^{\tilde{L}_1'}(\delta_V',f_{V_1}^{L_1'})\hat{S}_{\tilde{M}'}^{\tilde{L}_2}(\delta_V',f_{V_2}^{L_2'}).$$

where $f_{V_i}^{G_i'}=(f')_{V_i}^{G_i'}, i=1,2$. We may assume as before that G' is elliptic. As Levi subgroups of G', there are a canonical Levi subgroups $L_i\in\mathcal{L}(M), i=1,2$ for which L_i' is an elliptic endoscopic datum. To switch the sum over G_V' with the sums over L_1',L_2', we observe that there is a natural map of endoscopic data

$$\mathcal{E}_{M_V'}(G_V)\to\mathcal{E}_{M_{V_1}'}(L_{1,V_1})\times\mathcal{E}_{M_{V_2}'}(L_{2,V_2}) \tag{3.12}$$

given by $G_V'\to(L_1',L_2')$. If G_V' corresponds to $s_V'\in s_{M_V}'Z(\hat{M})^\Gamma/Z(\hat{G})^\Gamma$, then (L_1',L_2') corresponds to the element

$$(s_{L_1}',s_{L_2}'),\qquad s_{L_i}'\in s_{M_{V_i}}'Z(\hat{M})^\Gamma/Z(\hat{L}_i)^\Gamma, i=1,2,$$

by projecting s_V' onto each factor $s_{M_{V_i}}'Z(\hat{M})^\Gamma/Z(\hat{L}_i)^\Gamma$. It follows then that we may write the sum in (3.11) as

$$\sum_{L_1,L_2\in\mathcal{L}(M)}\ \sum_{G_V'\in\mathcal{E}_{M'}^0(G_V)} \iota_{M'}(G,G')\mathrm{e}_{M'}^{G'}(L_1',L_2')\hat{S}_{\tilde{M}'}^{\tilde{L}_1'}(\delta_V',f_{V_1}^{L_1'})\hat{S}_{\tilde{M}'}^{\tilde{L}_2}(\delta_V',f_{V_2}^{L_2'}). \tag{3.13}$$

Moreover, we may take the inner sum to be over $\mathcal{E}_{M'}(G)$ instead of $\mathcal{E}^0_{M'}(G)$ if we subtract the contribution

$$\varepsilon_{M'}(G) \sum_{L_1,L_2\in\mathcal{L}(M)} \mathrm{e}^G_M(L_1,L_2)\hat{S}^{L_1}_M(\delta_V, f^{L_1}_{V_1})\hat{S}^{L_2}_M(\delta_V, f^{L_2}_{V_2})$$

of $G'=G^*$, in the case G is quasisplit.

We can assume that the coefficient $\mathrm{e}^{G'}_{M'}(L_1',L_2')$ is nonzero, otherwise the summand indexed by L_1,L_2 would vanish. Therefore $d^{G'}_{M'}(L_1',L_2') = d^G_M(L_1,L_2)$ is nonzero, and it follows that the connected component$(Z(\hat{M}')^\Gamma)^0$ is equal to the product of $(Z(\hat{L}_1)^\Gamma)^0$ with $(Z(\hat{L}_2)^\Gamma)^0$. On the other hand, $Z(\hat{M}')^\Gamma = (Z(\hat{M}')^\Gamma)^0 Z(\hat{G}')^\Gamma$, so by [Art99, Lemma 1.1], we have again $Z(\hat{M})^\Gamma = Z(\hat{L}_1)^\Gamma Z(\hat{L}_1)^\Gamma$. It follows that the map (3.12) is surjective. Moreover, the finite group

$$Z(\hat{L}_1)^\Gamma \cap Z(\hat{L}_2)^\Gamma/Z(\hat{G})^\Gamma \tag{3.14}$$

acts simply transitively on the fibres. The sum over G'_V in (3.13) depends only on (L_1',L_2'), therefore we can replace the sum over $\mathcal{E}_{M'_V}(G_V)$ with a sum over $\mathcal{E}_{M'_{V_1}}(L_{1,V_1})\times\mathcal{E}_{M'_{V_2}}(L_{2,V_2})$ if we multiply the summand by the order of the group (3.14). To compute the resulting product of coefficients, we observe that

$$(Z(\hat{L}_1')^\Gamma/Z(\hat{L}_1)^\Gamma)\times(Z(\hat{L}_2')^\Gamma/Z(\hat{L}_2)^\Gamma)\to Z(\hat{M}')^\Gamma/Z(\hat{M})^\Gamma$$

is a surjective map with kernel isomorphic to the quotient of $Z(\hat{L}_1')^\Gamma\cap Z(\hat{L}_2')^\Gamma$ by $Z(\hat{L}_1)^\Gamma\cap Z(\hat{L}_2)^\Gamma$. It follows that the product is given by

$$\begin{aligned}
&|Z(\hat{M}')^\Gamma/Z(\hat{M})^\Gamma||Z(\hat{L}_1')^\Gamma\cap Z(\hat{L}_2')^\Gamma/Z(\hat{G})^\Gamma|^{-1}|Z(\hat{L}_1)^\Gamma\cap Z(\hat{L}_2)^\Gamma/Z(\hat{G})^\Gamma|\\
&=|Z(\hat{M}')^\Gamma/Z(\hat{M})^\Gamma||Z(\hat{L}_1')^\Gamma/Z(\hat{L}_1)^\Gamma|^{-1}|Z(\hat{M}')^\Gamma/Z(\hat{M})^\Gamma||Z(\hat{L}_2')^\Gamma/Z(\hat{L}_2)^\Gamma|^{-1}\\
&=\iota_{M'}(L_1,L_1')\iota_{M'}(L_2,L_2').
\end{aligned}$$

We can therefore write (3.13) as the sum over $L_1,L_2\in\mathcal{L}(M)$ of

$$d^G_M(L_1,L_2)\prod_{i=1,2}\Big(\sum_{L_1'}\iota_{M'}(L_i,L_i')\hat{S}^{\tilde{L}_i}_{\tilde{M}'}(\delta_V, f^{L_i'}_{V_i})\Big)$$

where the sum over L_i' runs over $\mathcal{E}_{M'_{V_i}}(L_{i,V_i})$, which is equal to

$$d^G_M(L_1,L_2)\hat{I}^{L_1,\mathcal{E}}_M(\delta'_{V_1}, f_{V_1,L_1})\hat{I}^{L_2,\mathcal{E}}_M(\delta'_{V_2}, f_{V_2,L_2})$$

by definition.

□

We now have the following reformulation of the elliptic contribution as an immediate consequence of our constructions.

Lemma 3.4. *The expression* (2.9) *can be written as*

$$m_G\tau(G)|A(F_V)|^{-1}\sum_{\eta\in\mathfrak{h}}\sum_{c\in C_\eta(G_V)}\sum_{b\in\mathcal{B}_{\eta,\mathrm{ell}}(G,V,\zeta)} f^G(\delta_{b,c,\eta}) \tag{3.15}$$

where $a=(b,c,\eta)$ belongs to $\mathcal{A}_{\mathrm{ell}}(G,V,\zeta)$, and parametrises the element $\delta_{b,c,\eta}$ in $\Delta(G,V,\zeta)$.

Proof. Using Lemma 3.1, we may rewrite the summation in (2.9) as

$$\tau(G)\sum_{a\in\mathcal{A}_{\mathrm{ell}}(G,V,\zeta)} f^G(a).$$

The element a is determined by the triple (b,c,η) for $\eta\in H^1(F,A)$ and (b,c) in $(\mathcal{B}_{\eta,\mathrm{ell}}(G,V,\zeta)\times C_\eta(G_V))/A(F_V)$. To isolate the linear part $\mathcal{B}(G,V,\zeta)$ of the base $\mathcal{A}(G,V,\zeta)$, we separate the sum over a into sums over η, b, and c, resulting in the triple sum as desired. Note that since $f\in\mathcal{H}(G,V,\zeta)$ is compactly supported, it follows that the summands are nonzero for only finitely many η. □

3.3. **Measures.** We next describe two natural measures that that may be chosen on the base. First, there is the geometric measure discussed in [FLN10] which naturally arises from a fixed differential form of top degree, a product measure which normalises this geometric measure. In the first case, choose invariant differential forms of top degree ω_G on G over F, and $\omega_{\mathcal{A}}$ on $\mathcal{A}$ inducing a form ω_a on the fibre $c^{-1}(a)$ for $a\in\mathcal{A}_{\mathrm{reg}}(G)$, the regular semisimple locus, such that $\omega_G=\omega_a\wedge\omega_{\mathcal{A}}$ for any such a. Indeed, there is an exact sequence of tangent spaces

$$0\to T_\gamma(c^{-1}(a))\to T_\gamma G\to T_a(\mathcal{A}(G))\to 0,$$

where c maps γ to a. It follows that

$$\wedge^d\mathfrak{g}=\wedge^n T_a(\mathcal{A}(G))\otimes\wedge^{d-n}T_\gamma(c^{-1}(a))$$

where $d=\dim(G)$ and $n=\dim(\mathcal{A}(G))$. We call this the geometric measure. Notice that integrating along the fibre gives

$$\int_{G(F_v)} f(g)dg=\int_{\mathcal{A}(G_v)}\int_{c^{-1}(a_v)} f(g)d|\omega_{a_v}|d|\omega_{\mathcal{A}_v}| \tag{3.16}$$

for any compactly supported function f on $G(F_v)$, from which we see the character of the trivial representation.

Secondly, there is a canonical measure that is compatible with the Tamagawa measure. It can be obtained from the measure chosen on the tori of G, which follows from Arthur's work on the trace formula, for example [Art96], which we now discuss. The Weyl group W_0 acts on the Levi subgroups in $\mathcal{L}$ by conjugation. We write $\mathcal{L}/W_0$ for the set of orbits. The strongly regular semisimple conjugacy classes of G can then be written as a disjoint union of W_0-orbits of elliptic G-regular conjugacy classes in Levi subgroups of G. Thus,

$$\Gamma_{G\text{-rs}}(G_v,\zeta_v)=\coprod_{M\in\mathcal{L}/W_0}(\Gamma_{G\text{-reg,ell}}(G_v,\zeta_v)/W(M)),$$

and similarly for stable classes we have

$$\Delta_{G\text{-rs}}(G_v,\zeta_v)=\coprod_{M\in\mathcal{L}/W_0}(\Delta_{G\text{-reg,ell}}(G_v,\zeta_v)/W(M)).$$

Let $\mathcal{T}_G^{\mathrm{st}}$ be a set of representatives of stable conjugacy classes of elliptic maximal tori in G over F, where by an elliptic torus T we mean that $T(F_v)/A_G(F_v)$ is compact. Also let $W_F(G,T)$ be the subgroup of the elements in the absolute Weyl group of (G,T) defined over F. We denote by n_T^G its cardinality. In the work of

Arthur, for example on the local stable trace formula, there is a typical measure chosen on the set $\Delta_{G\text{-reg,ell}}(G, V, \zeta)$ determined by the local measures on the tori,

$$\int_{\Delta_{G\text{-reg,ell}}(G_v,\zeta_v)} \beta(\sigma)d\sigma = \sum_{T\in\mathcal{T}_G^{\mathrm{st}}} \frac{1}{n_T^G} \int_{T(F)} \beta(t)dt \tag{3.17}$$

for any $\beta \in C_c(\Delta_{G\text{-reg}}(G_v, \zeta_v))$. We then obtain a measure over the larger set of strongly regular semisimple stable distributions $\Delta_{G\text{-rs}}(G, V, \zeta)$,

$$\int_{\Delta_{G\text{-rs}}(G,V,\zeta)} \beta(\sigma)d\sigma = \sum_{M\in\mathcal{L}/W_0} \frac{1}{n_M^G} \int_{\Delta_{G\text{-reg,ell}}(M,V,\zeta)} \beta(\sigma_M)d\sigma_M, \tag{3.18}$$

where $n_M^G = |W(M)|$. We may also choose analogous measures on the analogous set $\Gamma_{G\text{-reg}}(G, V, \zeta)$, replacing stable conjugacy with ordinary conjugacy. They will be related to the measures on $\Delta_{G\text{-rs}}(G, V, \zeta)$ by the formula

$$\int_{\Delta_{G\text{-rs}}(G,V,\zeta)} \sum_{\gamma\sim\sigma} \alpha(\gamma) = \int_{\Gamma_{G\text{-rs}}(G,V,\zeta)} \alpha(\gamma)d\gamma$$

for any $\alpha \in C_c(\Gamma_{G\text{-reg}}(G, V, \zeta))$. Note that the set $\Delta_{G\text{-rs}}(G, V, \zeta)$ is in bijection with the set $\Delta_{G\text{-rs}}(G/Z)$. We therefore have a measure on $\mathcal{A}_{\mathrm{reg}}(G, V, \zeta)$ by the relation

$$\int_{\mathcal{A}_{\mathrm{reg}}(G,V,\zeta)} \beta(a)da = \int_{\Delta_{G\text{-rs}}(G,V,\zeta)} \beta(\sigma)d\sigma, \tag{3.19}$$

and hence also $\mathcal{B}_{\mathrm{reg}}(G, V, \zeta)$. In other words, the measure on the base shall be determined by the invariant measure chosen on the maximal tori T in G.

Up to a set of measure zero, $\mathcal{A}(G_v)$ is a disjoint union of images of stable conjugacy classes of tori $T(F_v)$ in $G(F_v)$, and the restriction of c to T is $|W_T|$-to-one. So if φ is any function on $\mathcal{A}(G_v)$, we can write

$$\int_{\mathcal{A}(G_v,\zeta_v)} \varphi(a)da = \sum_{\eta} \sum_{T\in\mathcal{T}_G^{\mathrm{st}}} \frac{1}{n_T^G} \int_{T_v/Z_v} \varphi(t)dt$$

Moreover, organising the tori into the Levi subgroups M obtained as centralisers we can further express this as

$$\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{T\in\mathcal{T}_M^{\mathrm{st}}} \frac{1}{n_T^M} \int_{T_v} \varphi(t)dt. \tag{3.20}$$

This reflects the geometric decomposition (3.3) earlier.

Remark 3.5. Note that in [FLN10] the normalisation $L_v(1, \sigma_G)d|\omega_{G,v}|$ is used, and taking the product over all places v gives the so-called product measure on G. This normalisation can be seen as an intermediate normalisation between the canonical and geometric measures, and leads to the formula appearing before [FLN10, (3.23)], which we have mentioned in Remark 2.2.

4. Stable germ expansions

We now turn to the stable germ expansions for orbital integrals developed. Particular attention will be paid to the archimedean setting, since in that case the distributions are more complicated and the expansions are asymptotic rather than exact.

4.1. **Invariant germ expansions.** We first recall the ordinary germ expansions. Fix a conjugacy class $c \in \Gamma_{\mathrm{ss}}(\bar{G})$, and let $\mathcal{U}_c(G)$ the union of conjugacy classes $\Gamma_c(G)$ in $G(F)$ whose semisimple part maps to the conjugacy class of c. Then define $\mathcal{D}_c(G,\zeta)$ to be the space of ζ-equivariant invariant distributions on $G(F)$ supported on $\mathcal{U}_c(G)$. Parallel to (2.1), we refer to the space

$$\mathcal{D}(G,\zeta) = \bigoplus_{c\in\Gamma_{\mathrm{ss}}(\bar{G})} \mathcal{D}_c(G,\zeta) \tag{4.1}$$

as the space of singular invariant distributions on G, with a suitably chosen basis

$$\Gamma(G,\zeta) = \coprod_{c\in\Gamma_{\mathrm{ss}}(\bar{G})} \Gamma_c(G,\zeta).$$

The semisimple component of a distribution $\gamma \in \mathcal{D}_c(G,\zeta)$ is the underlying conjugacy class c, whereas the unipotent component α is a distribution in the space $\mathcal{D}_{\mathrm{unip}}(G_c,\zeta) = \mathcal{D}_1(G_c,\zeta)$ and is determined only up to the action of the finite group $\bar{G}_{c,+}(F)/\bar{G}_c(F)$. Note that because of the central datum (Z,ζ), the Jordan decomposition $\gamma = c\alpha$ is not canonical, and depends the choice of a suitable section $c \mapsto \tilde{c}$ from $\Gamma_{\mathrm{ss}}(\bar{G})$ to $G(F)$. We shall assume that such a section has been fixed.

4.1.1. *Nonarchimedean case.* If F is nonarchimedean, $\mathcal{D}_c(G,\zeta)$ is finite dimensional and has a basis of singular invariant orbital integrals

$$f \mapsto f_G(\rho), \qquad \rho \in \Gamma_c(G,\zeta)$$

taken over classes in $\Gamma_c(G,\zeta) = \Gamma_c(G)$. We have the ordinary Shalika germ expansion for the invariant orbital integral, decomposing it into a finite linear combination of functions parametrised by conjugacy classes

$$f_G(\gamma) = \sum_{\rho\in\Gamma_c(G,\zeta)} \rho^\vee(\gamma) f_G(\rho), \qquad f \in C_c^\infty(G(F)), \tag{4.2}$$

where γ is a strongly regular point close to c, in a sense that depends on f. The terms $\rho^\vee(\gamma) = g_G^G(\gamma,\rho)$ for each $\rho \in \Gamma_c(G,\zeta)$ are known as Shalika germs, but are in fact homogeneous functions defined on a fixed neighbourhood of c.

For weighted orbital integrals

$$J_M(\gamma,f) = |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x)dx,$$

there is a finite expansion

$$J_M(\gamma,f) = \sum_{L\in\mathcal{L}(M)} \sum_{\rho\in\Gamma_c(L,\zeta)} g_M^L(\gamma,\rho) J_L(\rho,f)$$

which follows from [Art88b]. The terms $g_M^L(\gamma,\rho)$ are defined as germs of functions of γ in $M(F)\cap G_{\mathrm{reg}}(F)$ near to c, and the coefficients $J_L(\rho,f)$ are singular weighted orbital integrals.

More generally, the functions g_M^L will belong the space $\hat{\mathcal{G}}_c(M,G,\zeta)$, consisting of germs of smooth $W(M)M(F_v)$-invariant, ζ^{-1}-equivariant functions on invariant neighbourhoods of c in $M_{G\text{-reg}}(F_v)$. The analogous expansion then also holds for invariant distributions [Art88a, (2.5)]

$$I_M(\gamma,f) = \sum_{L\in\mathcal{L}(M)} \sum_{\rho\in\Gamma_c(L,\zeta)} g_M^L(\gamma,\rho) I_L(\rho,f).$$

4.1.2. *Archimedean case.* If $F = \mathbf{R}$, then $\mathcal{D}_c(G,\zeta)$ is infinite dimensional, containing normal derivatives of orbital integrals, as well as more general distributions associated to harmonic differential operators, and has a basis $R_c(G)$ described in [Art16, §1]. Denote by $R_c(G,\zeta)$ by the subset of ζ-equivariant distributions. We describe the elements of $\mathcal{D}_c(G,\zeta)$ in further detail. Let $\mathcal{T}_c(G)$ be a fixed set of representatives of $G_{c,+}(\mathbf{R})$-orbits of maximal tori in G_c over $\mathbf{R}$, or equivalently, a fixed set of representatives of the $G(\mathbf{R})$-orbits of maximal tori in G over $\mathbf{R}$ that contain c. Denote by $S_c(G)$ the set of triplets $\sigma = (T,\Omega,X)$ where $T \in \mathcal{T}_c(G)$, $\Omega \in \pi_{0,c}(T_{\text{reg}}(\mathbf{R}))$ the set of connected components of $T_{\text{reg}}(\mathbf{R})$ whose closure contains c, and X is an invariant differential operator on $T(\mathbf{R})$. Given and f in the Schwartz space $\mathcal{C}(G)$ of $G(\mathbf{R})$, by Harish-Chandra we know that the orbital integral

$$f_G(\gamma), \qquad f \in \mathcal{C}(G), \gamma \in \Omega$$

extends to a continuous linear map from $\mathcal{C}(G)$ to the space of smooth functions on the closure of Ω. It follows from this that the limit

$$f_G(\sigma) = \lim_{\gamma \to c} (X f_G)(\gamma)$$

exists and is continuous in f. If f is compactly supported and vanishes on a neighbourhood of $\mathcal{U}_c(G)$, then $f_G(\sigma) = 0$, therefore the linear form $f \mapsto f_G(\sigma)$ belongs to $\mathcal{D}_c(G,\zeta)$, and in fact spans $\mathcal{D}_c(G,\zeta)$.

We then have the analogous germ expansions due to Arthur [Art16], where in place of an actual identity, we have an asymptotic formula

$$f_G(\gamma) \sim \sum_{\rho \in R_c(G,\zeta)} \rho^\vee(\gamma) f_G(\rho), \qquad f \in \mathcal{C}(G,\zeta). \tag{4.3}$$

As before, the germs $\rho^\vee(\gamma)$, for each $\rho \in R_c(G,\zeta)$ can be treated as homogeneous functions of γ. For weighted orbital integrals, the germ expansion is formulated in terms of the family germs $g_M^G(\gamma,\rho)$ which belong to a certain space of formal germs $\hat{\mathcal{G}}_c(M,G)$ [Art16, §4]. It consists of germs of smooth $W(M)M(\mathbf{R})$-invariant functions on invariant neighbourhoods of c in $M_{G\text{-reg}}(\mathbf{R})$ with bounded growth near the boundary. To incorporate the central character datum (Z,ζ), one introduces the corresponding space $\hat{\mathcal{G}}_c(M,G,\zeta)$ of formal germs of ζ^{-1}-equivariant functions. To do so, we simply restrict the smooth $W(M)M(\mathbf{R})$-invariant functions on invariant neighbourhoods of c in $M_{G\text{-reg}}(\mathbf{R})$ above to only ζ^{-1}-equivariant ones. Then from [Art16, Theorem 5.1] we have

$$J_M(\gamma,f) \sim \sum_{L \in \mathcal{L}(M)} \sum_{\rho \in R_c(L)} g_M^L(\gamma,\rho) J_L(\rho,f)$$

and the invariant analogue [Art16, Corollary 10.1]

$$I_M(\gamma,f) \sim \sum_{L \in \mathcal{L}(M)} \sum_{\rho \in R_c(L)} g_M^L(\gamma,\rho) I_L(\rho,f). \tag{4.4}$$

We shall require the quantitative form of this asymptotic expansion. To state it, we first recall that the universal space of formal germs $\hat{\mathcal{G}}_c(M,G,\zeta)$ is constructed [Art16, §4] using spaces $\mathcal{F}^\alpha_{c,n}(V,G,\zeta)$ of ζ^{-1}-equivariant functions on $V_{G\text{-reg}} = V \cap G_{\text{reg}}(\mathbf{R})$. It is obtained as the direct limit of spaces of formal α-germs

$$\hat{\mathcal{G}}_c(M,G,\zeta) = \varinjlim_{\alpha} \hat{\mathcal{G}}^\alpha_c(M,G,\zeta)$$

where α are weight functions on invariant differential operators X on a maximal torus T of M. The formal α-germs are in turn obtained as a projective limit

$$\hat{\mathcal{G}}^{\alpha}_{c}(M,G,\zeta) = \varprojlim_{n} \mathcal{G}^{\alpha,n}_{c}(M,G,\zeta)$$

for $n \geq 0$, where

$$\mathcal{G}^{\alpha,n}_{c}(M,G,\zeta) = \mathcal{G}^{\alpha}_{c}(M,G,\zeta)/\mathcal{G}^{\alpha}_{c,n}(M,G,\zeta)$$

is the space of (α,n)-jets for (M,G) at c, and

$$\mathcal{G}^{\alpha}_{c,n}(M,G,\zeta) = \varinjlim_{V} \mathcal{F}^{\alpha}_{c,n}(V,G,\zeta)$$

is the space of α-germs. Here the direct limit is taken over $W(M)M(\mathbf{R})$-invariant neighborhoods V of c in $M(\mathbf{R})$. Then the invariant germ expansion can be interpreted as there exists a weight function α such that for any $n \geq 0$,

$$f \to I_M(\gamma,f) - I^n_M(\gamma,f)$$

is a continuous linear mapping from $\mathcal{C}(G,\zeta)$ to $\mathcal{F}^{\alpha}_{c,n}(V,G)$ where

$$I^n_M(\gamma,f) = \sum_{L\in\mathcal{L}(M)} \sum_{\rho\in R_c(L)} g^{L,n}_M(\gamma,\rho) I_L(\rho,f), \tag{4.5}$$

belongs to $\mathcal{F}^{\alpha}_{c}(V,G,\zeta) = \mathcal{F}^{\alpha}_{c,-1}(V,G,\zeta)$ [Art16, Corollary 10.1]. Here $g^{L,n}_M(\rho)$ is the projection of $g^L_M(\rho)$ onto the quotient of $\hat{\mathcal{G}}^{\alpha}_{c}(M,L,\zeta)$ by the kernel of its projection onto $\mathcal{G}^{\alpha,n}_{c}(M,G)$, and $g^{L,n}_M(\gamma,\rho)$ is a representative of $g^{L,n}_M(\rho)$ in $\mathcal{F}^{\alpha}_{c}(V,L,\zeta)$. We assume that $g^{L,n}_M(\gamma,\rho) = 0$ if $g^{L,n}_M(\rho) = 0$. It follows then that the inner sum of $I^n_M(\gamma,f)$ can be taken over a finite set. It is uniquely determined up to a finite sum

$$\sum_i \phi_i(f) J_i(f)$$

for tempered distributions $J_i(f)$ and functions $\phi_i(\gamma)$ in $\mathcal{F}^{\alpha}_{c,n}(V,G)$.

4.2. **Stable germ expansions.** We now turn to stable germ expansions. Fix $d^* \in \Delta_{\mathrm{ss}}(\bar{M}^*)$. The direct sum

$$\mathcal{D}_{d^*}(G,\zeta) = \bigoplus_{c} \mathcal{D}_c(G,\zeta)$$

taken over classes $c \in \Gamma_{\mathrm{ss}}(\bar{M})$ whose image in $\Delta_{\mathrm{ss}}(\bar{M}^*)$ equals d^*, has an orthonormal basis given by the union

$$R_{d^*}(G,\zeta) = \coprod_c R_c(G,\zeta).$$

Any distribution $\delta \in \mathcal{D}_d(G,\zeta)$ has semisimple component equal to $d \in \Delta_{\mathrm{ss}}(\bar{G})$, and unipotent part equal to $\beta \in \mathcal{D}_{\mathrm{unip}}(G_d,\zeta) = \mathcal{D}_1(G_d,\zeta)$. The choice of β is unique up to the action of the finite group $\bar{G}_{d,+}(F)/\bar{G}_d(F)$. Once again, this Jordan decomposition $\delta = d\beta$ depends on a suitably chosen section $d \to \tilde{d}$ from $\Delta_{\mathrm{ss}}(\bar{G})$ to $G(F)$. We shall assume that such a section has been fixed. The elements $\rho \in R_{d^*}(G,\zeta)$ parametrise a family of germs of functions $g^G_M(\gamma,\rho)$ of points $\gamma \in \Gamma_{G\text{-reg}}(M,\zeta)$ that are close to d^*, in the sense that the image in $\bar{M}'(F_v)$ of the support of γ is close to d^*. If ρ belongs to a subset $R_c(G,\zeta)$ of $R_{d^*}(G,\zeta)$, we set $g^G_M(\gamma,\rho)$ equal to zero for γ outside some fixed, small neighbourhood of $c \in \Gamma_{G\text{-reg}}(M,\zeta)$. It follows that for any γ, the germs $g_M^G(\gamma,\rho)$ can be nonzero for elements ρ in at most one subset $R_c(G,\zeta)$ of $R_{d^*}(G,\zeta)$.

The space $\mathcal{D}(G,\zeta)$ naturally has a subspace of stable distributions

$$\mathcal{SD}(G,\zeta)=\bigoplus_{d\in\Delta_{\mathrm{ss}}(\bar{G})}\mathcal{SD}_d(G,\zeta)=\bigoplus_{d^*\in\Delta_{\mathrm{ss}}(\bar{G}^*)}\mathcal{SD}_{d^*}(G,\zeta). \tag{4.6}$$

with a corresponding orthonormal basis $\Sigma_d(G,\zeta)$.

4.2.1. *A subspace of distributions.* Let G be a quasisplit reductive group over $\mathbf{R}$. We shall construct the subspace $\mathcal{D}_{\text{tr-orb}}(G,\zeta)$of $\mathcal{D}(G,\zeta)$ inductively on $\dim(G_{\mathrm{sc}})$, as in [MW16a, V.2.1]. Assuming that $\mathcal{D}_{\text{tr-orb}}(G,\zeta)$ is given, we may define the subset

$$\mathcal{SD}_{\text{tr-orb}}(G,\zeta)=\mathcal{D}_{\text{tr-orb}}(G,\zeta)\cap\mathcal{SD}(G,\zeta).$$

Then we define $\mathcal{D}_{\text{tr-orb}}(G,\zeta)$ to be the subspace of $\mathcal{D}(G,\zeta)$ generated by $\mathcal{D}_{\mathrm{orb}}(G,\zeta)$ and the images of the transfer of $\mathcal{SD}_{\text{tr-orb}}(G',\zeta)$ for each elliptic endoscopic datum $G'\in\mathcal{E}(G)$, with $G'\neq G$. These spaces depend only on G', and are independent of the auxiliary endoscopic datum. We may also extend the definition of $\mathcal{D}_{\text{tr-orb}}(G,\zeta)$ to z-extensions $\tilde{G}$ of G by the natural image of $\mathcal{D}_{\text{tr-orb}}(G,\zeta)$ in $D(\tilde{G},\tilde{\zeta})$, where $\tilde{\zeta}$ is chosen to be compatible with ζ. Parallel to the decompositions of $\mathcal{D}(G,\zeta)$ and $\mathcal{SD}(G,\zeta)$ in (4.1) and (4.6), we obtain

$$\mathcal{D}_{\text{tr-orb}}(G,\zeta)=\bigoplus_{c\in\Gamma_{\mathrm{ss}}(\bar{G})}\mathcal{D}_c^{\text{tr-orb}}(G,\zeta)$$

and

$$\mathcal{SD}_{\text{tr-orb}}(G,\zeta)=\bigoplus_{d\in\Delta_{\mathrm{ss}}(\bar{G})}\mathcal{SD}_d^{\text{tr-orb}}(G,\zeta)$$

where $\mathcal{D}_c^{\text{tr-orb}}(G,\zeta)$ is given by $\mathcal{D}_{\text{tr-orb}}(G,\zeta)\cap\mathcal{D}_c(G,\zeta)$ and similarly for $\mathcal{SD}_d^{\text{tr-orb}}(G,\zeta)$. Then the induction homomorphism $\mu\mapsto\mu^G$ yields maps from $\mathcal{D}_{\text{tr-orb}}(M,\zeta)$ to $\mathcal{D}_{\text{tr-orb}}(G,\zeta)$, and $\mathcal{SD}_{\text{tr-orb}}(M,\zeta)$ to $\mathcal{SD}_{\text{tr-orb}}(G,\zeta)$. These definitions extend as usual to K-groups through the direct sum

$$\mathcal{D}_{\text{tr-orb}}(G,\zeta)=\bigoplus_{\alpha\in\pi_0(G)}\mathcal{D}_{\text{tr-orb}}(G_\alpha,\zeta_\alpha)$$

and

$$\mathcal{SD}_{\text{tr-orb}}(G,\zeta)=\bigoplus_{\alpha\in\pi_0(G)}\mathcal{SD}_{\text{tr-orb}}(G_\alpha,\zeta_\alpha).$$

We remark here that a stable germ expansion and transfer of distributions for real groups is provided in the literature by [MW16a, V] using these simpler subspaces and a more direct argument, which includes all the distributions necessary in the stabilisation of the trace formula. The treatment we shall give here follows the spirit of Arthur's stable germ expansions, which coincide with [MW16a, II] in the nonarchimedean case of stable Shalika germs.

4.2.2. *Stable and endoscopic germs.* Let M' be an elliptic endoscopic datum for M, and $d'\in\Delta_{\mathrm{ss}}(\bar{M}')$ a stable semisimple conjugacy class in $\bar{M}'(F_v)$ whose image in $\Delta_{\mathrm{ss}}(\bar{M}^*)$ equals d^*. Given M' and d', assume inductively that for each element $G'\in\mathcal{E}^0_{M'}(G)$ with auxiliary datum $(\tilde{G}',\tilde{\xi}')$, we have defined a family

$$h_{\tilde{M}'}^{\tilde{G}'}(\delta',\sigma'),\qquad \sigma\in\Sigma_{d'}(\tilde{G}',\tilde{\zeta}')$$

of germs of functions of points $\delta' \in \Delta_{G\text{-reg}}(\tilde{M}', \tilde{G}', \tilde{\zeta}')$ near d', parametrised by a basis $\Sigma_{d'}(\tilde{G}', \tilde{\zeta}')$ that is constructed analogously to $R_{d^*}(G, \zeta)$. We view these as conjugate linear forms in $\sigma' \in \mathcal{SD}_{d'}(\tilde{G}', \tilde{\zeta}')$, which extend continuously to the completion $\hat{\mathcal{SD}}_{d'}(\tilde{G}', \tilde{\zeta}')$. The germs satisfy the symmetry condition $h^{\tilde{G}'}_{\tilde{M}'}(\theta'\delta', \theta'\sigma') = h^{\tilde{G}'}_{\tilde{M}'}(\delta', \sigma')$ for any $\theta' \in \mathrm{Out}(\tilde{G}', \tilde{M}', \tilde{\zeta}')$, the group of outer automorphisms of $\tilde{G}'$ fixing the central character datum $(\tilde{Z}', \tilde{\zeta}')$ and leave $\tilde{M}'$ invariant. We also require the compatibility condition $h^{\tilde{G}'_1}_{\tilde{M}'_1}(\delta'_1, \hat{\omega}'\sigma') = h^{\tilde{G}'}_{\tilde{M}'}(\delta'_1\hat{\omega}', \sigma')$ for a second auxiliary datum $(\tilde{G}'_1, \tilde{\xi}'_1)$ and for any character ω' of $\tilde{G}'$ and $\delta'_1 \in \Delta_{G\text{-reg}}(\tilde{M}'_1, \tilde{\zeta}'_1)$ near d'.

The stable and endoscopic germs are defined inductively by

$$g^{G,\mathcal{E}}_M(\delta', \rho) = \sum_{G' \in \mathcal{E}^0_{M'}(G)} \iota_{M'}(G, G') h^{\tilde{G}'}_{\tilde{M}'}(\delta', \rho') + \varepsilon(G) h^G_M(M', \delta', \rho)$$

with the condition that $g^{G,\mathcal{E}}_M(\delta', \rho) = g^G_M(\delta, \rho)$ if G is quasisplit and δ' maps to $\delta \in \mathcal{D}^{\mathcal{E}}_{G\text{-reg}}(M, \zeta)$. Here

$$g^G_M(\delta, \rho) = \sum_{\gamma \in \Gamma_{G\text{-reg}}(M,\zeta)} \Delta_M(\delta, \gamma) g^G_M(\gamma, \rho), \qquad \rho \in R_{d^*}(G, \zeta)$$

for any point $\delta \in \Delta^{\mathcal{E}}_{G\text{-reg}}(M, \zeta)$ that is close to d^* in the sense that the image in $\bar{M}^*(F)$ of its support is close to d^*. Here $\Delta^{\mathcal{E}}_{G\text{-reg}}(M, \zeta)$ is the endoscopic analogue of the $\Gamma_{G\text{-reg}}(M, \zeta)$, and is a subset of the general basis $\Delta^{\mathcal{E}}(M, \zeta)$ of $\mathcal{D}^{\mathcal{E}}(M, \zeta)$. Taking G to be quasisplit, we set

$$h^G_M(M', \delta', \sigma) = \sum_{\rho \in R_{d^*}(G,\zeta)} h^G_M(M', \delta', \rho)\Delta(\rho, \sigma)$$

for $\sigma \in \Delta^{\mathcal{E}}_{d^*}(G, \zeta)$. We note that the fact that these stable germs are indeed stable distributions follows from the stability of the linear forms $S^G_M(\delta, f)$.

We record the following descent and splitting formulas for endoscopic and stable germs, which are derived in a similar manner to Propositions 3.2 and 3.3, respectively. See also Lemme 2.12 and 2.13 of [MW16a, II].

Lemma 4.1 (Descent). *(a) Let G be arbitrary. Then*

$$g^{G,\mathcal{E}}_M(\delta', \rho) = \sum_{G_1 \in \mathcal{L}(M_1)} d^G_{M_1}(M, G_1) g^{G_1,\mathcal{E}}_{M_1}(\delta'_1, \rho_{G_1}), \qquad \rho \in R_{d^*}(G, \zeta).$$

(b) Let G be quasisplit. If $M' = M^*$, then

$$h^G_M(\delta, \sigma) = \sum_{G_1 \in \mathcal{L}(M_1)} \mathrm{e}^G_{M_1}(M, G_1) h^{G_1}_{M_1}(\delta_1, \rho_{G_1}), \qquad \sigma \in \Delta^{\mathcal{E}}_{d^*}(G, \zeta),$$

and if $M' \neq M^*$ then $g^G_M(M', \delta', \sigma) = 0$ for all $\sigma \in \Delta^{\mathcal{E}}_{d^*}(G, \zeta)$.

Lemma 4.2 (Splitting). *(a) Let G be arbitrary. Then*

$$g^{G,\mathcal{E}}_M(\delta', \rho) = \sum_{G_1 \in \mathcal{L}(M_1)} d^G_{M_1}(M, G_1) g^{G_1,\mathcal{E}}_{M_1}(\delta'_1, \rho_{G_1}), \qquad \rho \in R_{d^*}(G, \zeta).$$

(b) Let G be quasisplit. If $M' = M^*$, then

$$h^G_M(\delta, \sigma) = \sum_{G_1 \in \mathcal{L}(M_1)} \mathrm{e}^G_{M_1}(M, G_1) h^{G_1}_{M_1}(\delta_1, \rho_{G_1}), \qquad \sigma \in \Delta^{\mathcal{E}}_{d^*}(G, \zeta),$$

and if $M' \neq M^*$ then $g^G_M(M', \delta', \sigma) = 0$ for all $\sigma \in \Delta^{\mathcal{E}}_{d^*}(G,\zeta)$.

In particular, the splitting formula allows us again to reduce to the case of simple forms.

Let G be quasisplit over F and $f \in \mathcal{H}(G,\zeta)^0$. Take $\delta' \in \Delta_{G\text{-reg}}(\tilde{M}', \tilde{\zeta}')$ to be a strongly G-regular element close to d', in the sense that the image of its support in $\bar{M}'$ is close to d'. In the special case that $M' = M^*$, we shall take $d = d'$ and $\delta = \delta'$. We now state the stable and endoscopic germ expansions, we include the proof as it will lead to a similar quantitative statement as in the invariant case.

Proposition 4.3. *If G is arbitrary, and $\gamma \in \Gamma_{G\text{-reg}}(M,\zeta)$ is close to d^*, there is a germ expansion*

$$I^{\mathcal{E}}_M(\gamma, f) \sim \sum_{L \in \mathcal{L}(M)} \sum_{\rho \in R_{d*}(L,\zeta)} g^{L,\mathcal{E}}_M(\gamma,\rho) I^{\mathcal{E}}_L(\rho, f). \tag{4.7}$$

If G is quasisplit, and $\delta' \in \Delta_{G\text{-reg}}(\tilde{M}', \tilde{\zeta}')$ near d', there is a stable germ expansion
(4.8)
$$S^G_M(M', \delta', f') \sim \sum_{L \in \mathcal{L}(M)} \sum_{L' \in \mathcal{E}_{M'}(L)} \sum_{\sigma' \in \Sigma_{d'}(\tilde{L}', \tilde{\zeta}')} \iota_{M'}(L, L') h^{\tilde{L}'}_{\tilde{M}'}(\delta', \sigma') S^G_L(L', \sigma', f).$$

If F nonarchimedean, the relation $\sim$ is an equality in both cases.

Proof. We assume inductively that

$$S^{\tilde{G}'}_{\tilde{M}'}(\delta', f') \sim \sum_{L' \in \mathcal{L}(M')} \sum_{\sigma' \in \Sigma_{d'}(\tilde{L}', \tilde{\zeta}')} h^{\tilde{L}'}_{\tilde{M}'}(\delta', \sigma') S^{\tilde{G}'}_{\tilde{L}'}(\sigma', f') \tag{4.9}$$

for $\delta' \in \Delta_{G'\text{-reg}}(M', \zeta)$ near d' and $G' \in \mathcal{E}^0_{\text{ell}}(G)$. Note that by Local Theorem 1 of [Art03] we know that the linear form $S^{\tilde{G}'}_{\tilde{M}'}(\delta', f')$ is stable. Let G be arbitrary and δ' as above. Then by definition we have

$$I^{\mathcal{E}}_M(\delta', f) - \varepsilon(G) S^G_M(M', \delta', f) = \sum_{G' \in \mathcal{E}^0_{M'}(G)} \iota_{M'}(G, G') \hat{S}^{\tilde{G}'}_{\tilde{M}'}(\delta', f'). \tag{4.10}$$

Also, since

$$\sum_{\sigma \in \Sigma^{\mathcal{E}}_{d^*}(G,\zeta)} h^G_M(M', \delta', \sigma) f^{\mathcal{E}}_G(\sigma) = \sum_{\rho \in R_{d*}(G,\zeta)} h^G_M(M', \delta', \sigma) f_G(\rho),$$

we can express

$$\sum_{\rho \in R_{d*}(G,\zeta)} g^{G,\mathcal{E}}_M(\delta', \rho) f_G(\rho) - \sum_{\sigma \in \Sigma^{\mathcal{E}}_{d^*}(G,\zeta)} \varepsilon(G) h^G_M(M', \delta', \sigma)) f^{\mathcal{E}}_G(\rho) \tag{4.11}$$

as

$$\begin{aligned}
&\sum_{\rho \in R_{d*}(G,\zeta)} (g^{G,\mathcal{E}}_M(\delta', \rho) - \varepsilon(G) h^G_M(M', \delta', \sigma)) f_G(\rho) \\
&= \sum_{\rho \in R_{d*}(G,\zeta)} \sum_{G' \in \mathcal{E}^0_{M'}(G)} \iota_{M'}(G, G') h^{\tilde{G}'}_{\tilde{M}'}(\delta', \rho') \\
&= \sum_{G' \in \mathcal{E}^0_{M'}(G)} \iota_{M'}(G, G') \sum_{\sigma' \in \Sigma_{d'}(\tilde{G}', \tilde{\zeta}')} h^{\tilde{G}'}_{\tilde{M}'}(\delta', \sigma').
\end{aligned}$$

Putting (4.10) and (4.11) together, we can write

$$I_M^{\mathcal{E}}(\delta', f) - \sum_{\rho \in R_{d*}(G,\zeta)} g_M^{G,\mathcal{E}}(\delta', \rho) f_G(\rho) \tag{4.12}$$

as the sum of

$$\sum_{G' \in \mathcal{E}^0_{M'}(G)} \iota_{M'}(G, G') \Big(\hat{S}_{\tilde{M}'}^{\tilde{G}'}(\delta', f') - \sum_{\sigma' \in \Sigma_{d'}(\tilde{G}', \tilde{\zeta}')} h_{\tilde{M}'}^{\tilde{G}'}(\delta', \sigma') \Big) \tag{4.13}$$

and

$$\varepsilon(G)(S_M^G(M', \delta', f) - \sum_{\sigma \in \Sigma^{\mathcal{E}}_{d*}(G,\zeta)} h_M^G(M', \delta', \sigma) f_G^{\mathcal{E}}(\rho)).$$

Using the induction assumption, we apply the expansion (4.9) to (4.13), which leads to

$$\sum_{G' \in \mathcal{E}^0_{M'}(G)} \iota_{M'}(G, G') \sum_{L' \in \mathcal{L}^{G',0}(M')} \sum_{\sigma' \in \Sigma_{d'}(\tilde{G}', \tilde{\zeta}')} h_{\tilde{M}'}^{\tilde{L}'}(\delta', \sigma') \hat{S}_{\tilde{L}'}^{\tilde{G}'}(\sigma', f') \tag{4.14}$$

for $\delta \in \Delta_{G\text{-reg}}(M')$ close to d', and $\mathcal{L}^{G',0}(M')$ denotes the complement of $\{G'\}$ in $\mathcal{L}^{G'}(M')$. Note that any Levi subgroup $L' \in \mathcal{L}^{G'}(M')$ of G' determines a Levi subgroup $L \in \mathcal{L}(M)$ of G with $A_{L'} = A_L$, so we can rewrite the sum over G' and L' as a sum over $L \in \mathcal{L}^0(M), L' \in \mathcal{E}_{M'}(L)$, and $G' \in \mathcal{E}^0_{L'}(G)$. Then using the identity $\iota_{M'}(G, G') = \iota_{M'}(L, L')\iota_{L'}(G, G')$, it follows that we can write (4.14) as

$$\sum_{L \in \mathcal{L}^0(M)} \sum_{L' \in \mathcal{E}_{M'}(L)} \sum_{\sigma' \in \Sigma_{d'}(\tilde{L}', \tilde{\zeta}')} \iota_{M'}(L, L') h_{\tilde{M}'}^{\tilde{L}'}(\delta', \sigma') \sum_{G' \in \mathcal{E}^0_{L'}(G)} \iota_{L'}(G, G') \hat{S}_{\tilde{L}'}^{\tilde{G}'}(\sigma', f').$$

By definition (2.6) the inner sum is equal to

$$I_L^{\mathcal{E}}(\sigma', f') - \varepsilon(G) S_L^G(L', \sigma', f'). \tag{4.15}$$

The first term is given by

$$I_L^{\mathcal{E}}(\sigma', f') = \sum_{\rho \in R_{d*}(L,\zeta)} \Delta_L(\sigma', \rho) I_L^{\mathcal{E}}(\rho, f),$$

and using the relations

$$\sum_{\sigma' \in \Sigma_{d'}(\tilde{L}', \tilde{\zeta}')} h_{\tilde{M}'}^{\tilde{L}'}(\delta', \sigma') \Delta_L(\sigma', \rho) = h_{\tilde{M}'}^{\tilde{L}'}(\delta', \rho)$$

and

$$\sum_{L' \in \mathcal{E}_{M'}(L)} \iota_{M'}(L, L') h_{\tilde{M}'}^{\tilde{L}'}(\delta', \rho) = g_M^{L,\mathcal{E}}(\delta', \rho),$$

it follows that the contribution of the first term in (4.15) is equal to

$$\sum_{L \in \mathcal{L}^0(M)} \sum_{\rho \in R_{d*}(L,\zeta)} g_M^{L,\mathcal{E}}(\delta', \rho) I_L^{\mathcal{E}}(\rho, f). \tag{4.16}$$

Subtracting this from the difference (4.12) we see that (4.16) is equal to the product of $\varepsilon(G)$ and (4.8).

Thus if $\varepsilon(G) = 0$, (4.16) vanishes and we deduce the endoscopic expansion (4.7). If $\varepsilon(G) = 1$, then (4.7) is equivalent to the usual invariant germ expansion (4.4). One sees then that (4.16) again vanishes, and (4.8) follows.

□

Finally, we derive the quantitative form of the archimedean stable germ expansion, which is the key result of this section. We first define the spaces

$$\hat{\mathcal{G}}_{d^*}(M,G,\zeta)=\coprod_c \hat{\mathcal{G}}_c(M,G,\zeta)$$

and

$$\mathcal{F}^{\alpha}_{d^*,n}(V,G)=\coprod_c \mathcal{F}^{\alpha}_{c,n}(V,G)$$

where the union is again taken over classes $c\in\Gamma_{\mathrm{ss}}(\bar{M})$ whose image in $\Delta_{ss}(\bar{M}^*)$ equals d^*. We observe that the germs $g_M^{G,\mathcal{E}}(\delta',\rho)$ and $h_M^G(M',\delta',\sigma)$ belong to the space $\hat{\mathcal{G}}_{d^*}(M,G,\zeta)$. Then for any $n\geq 0$, we set

$$S_M^n(\delta,f)=\sum_{L\in\mathcal{L}(M)}\sum_{\sigma\in\Sigma_d(L)} h_M^{L,n}(\delta,\sigma)S_L(\sigma,f) \tag{4.17}$$

where $h_M^{L,n}(\delta,\rho)$ are fixed representatives, defined again as the projection of $h_M^L(\sigma)$ onto $\hat{\mathcal{G}}_d^{\alpha,n}(M,L,\zeta)$, or equivalently, as the stable partial germ obtained from $g_M^{L,n}(\gamma,\rho)$ defined parallel to $h_M^L(\delta,\sigma)$. We note again that the sum can be taken over finite set, as with (4.5).

Corollary 4.4. *There exists a weight function α such that $\alpha(1)=0$, and for any n the mapping*

$$f\to S_M(\delta,f)-S_M^n(\delta,f),\qquad f\in\mathcal{C}(G)$$

is a continuous linear mapping from $\mathcal{C}(G)$ to $\mathcal{F}^{\alpha}_{d^*,n}(V,G)$. In particular, $S_M(f)$ has a formal germ expansion given by the sum

$$\sum_{L\in\mathcal{L}(M)} h_M^L(S_{L,d}(f))=\sum_{L\in\mathcal{L}(M)}\sum_{\sigma\in\Sigma_d(L)} h_M^L(\sigma)S_L(\sigma,f).$$

Proof. The second assertion follows immediately from the first, similar to Theorem 5.1* of [Art16]. To establish the first assertion, we express $S_M(\delta,f)-S_M^n(\delta,f)$ as the difference between

$$I_M(\delta',f)-I_M^n(\delta',f)$$

and

$$\sum_{G'\in\mathcal{E}^0_{M'}(G)}\iota_{M'}(G,G')\big(\hat{S}^{\tilde{G}'}_{\tilde{M}'}(\delta',f)-\hat{S}^{\tilde{G}',n}_{\tilde{M}'}(\delta',f)\big),$$

then the desired result follows inductively from [Art16, Corollary 10.1]. □

5. Towards a weighted stable trace formula

5.1. **Nontempered representations.** We are interested in the contribution of nontempered representations, meaning π whose local constituents π_v are not be tempered in the sense of Harish-Chandra for some v. In terms of the hypothetical global Langlands group L_F, which should arise as an extension of the global Weil group W_F [Kot84, §9], the automorphic representations of G should be parametrised by $\hat{G}$-conjugacy classes of elliptic L-homomorphisms

$$\psi: L_F\times \mathrm{SU}_2\to {}^LG$$

with bounded image, and which localise to groups L_{F_v} described by the Weil group. The discrete spectrum of G should then be partitioned into finite packets of representations $\Pi_\psi=\otimes\Pi_{\psi_v}$. If G is a classical group, Arthur has introduced an alternative group whose existence is unconditional [Art13, §1.5]. The nontempered representations will be indexed by parameters $\psi = \phi \otimes \nu$ that are non-trivial on SU_2.

We shall first abstractly define subspaces

$$\Phi_{t,\mathrm{cusp}}(G,V,\zeta) \subset \Phi_{t,2}(G,V,\zeta) \subset \Phi_{t,\mathrm{disc}}(G,V,\zeta),$$

where $\Phi_t(G,V,\zeta)$ defined in [Art02], plays the role of Langlands parameters and $\Phi_{t,\mathrm{disc}}(G,V,\zeta)$ is the subset having discrete measure, $\Phi_{t,2}(G,V,\zeta)$ is supported on representations that occur in the discrete spectrum of G, and $\Phi_{t,\mathrm{cusp}}(G,V,\zeta)$ is the subset of representations whose parameter ψ is trivial on SU_2. Namely, the complement of $\Phi_{\mathrm{cusp}}(G,V,\zeta)$ in $\Phi_2(G,V,\zeta)$ corresponds to noncuspidal characters of G; and for $M \neq G$, we have characters in the complement of $\Phi_2(M,V,\zeta)$ of $\Phi_{\mathrm{disc}}(M,V,\zeta)$ corresponding to induced representations that may or may not be cuspidal, and lie in the continuous spectrum of G. Define the complement

$$\Phi_{t,\mathrm{comp}}(G,V,\zeta) = \Phi_{t,\mathrm{disc}}(G,V,\zeta)\backslash\Phi_{t,\mathrm{cusp}}(G,V,\zeta).$$

These are the parameters that we shall be mainly interested in.

The spectral side of the stable trace formula is written as

$$\sum_{t\geq 0} S_t^G(f) = \sum_{t\geq 0}\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1}\int_{\Phi_t(M,V,\zeta)} b^M(\phi) S_M^G(\phi,f)d\phi.$$

Separating the contribution of the continuous spectrum, the spectral measure $d\phi$ associated to the remaining terms is discrete. We shall not need the definitions of $b^M(\phi)$ and $S_M^G(\phi,f)$ explicitly, and thus refer to [Art02] for the interested reader. The discrete part of the invariant trace formula is written as

$$I_{t,\mathrm{disc}}^G(f) = \sum_{M\in\mathcal{L}} |W^G(M)|^{-1} \sum_{w\in W^G(M)_{\mathrm{reg}}} |\det(w-1)|^{-1}\mathrm{tr}(M_P(w)_{\mathfrak{a}_M^G} I_{t,P}(f)),$$

which has a parallel stabilization

$$I_{t,\mathrm{disc}}^G(f) = \sum_{G'} \iota(G,G')\hat{S}_{t,\mathrm{disc}}^{G'}(f^{G'}).$$

This last identity is proven by induction, hence the stable linear form $\hat{S}_{\mathrm{disc}}^{G'}$ is not explicitly given in general. We may write the t-discrete stable part as

$$S_{t,\mathrm{disc}}^G(f) = \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\phi\in\Phi_{t,\mathrm{disc}}(M,V,\zeta)} b^M(\phi)S_M^G(\phi,f), \tag{5.1}$$

having removed the terms with continuous $d\phi$.

One expects the contribution of the tempered, cuspidal spectrum of G to the stable trace formula to be precisely the sum over $t\geq 0$ of

$$S_{t,\mathrm{cusp}}^G(f) = \sum_{\phi\in\Phi_{t,\mathrm{cusp}}(G,V,\zeta)} b^G(\phi)f^G(\phi). \tag{5.2}$$

where $f^G(\phi) = S_G^G(\phi,f)$ is the stable character associated to the cuspidal parameter ϕ. We would like to remove the contribution of the complement

$$S_{t,\mathrm{comp}}^G(f) = S_{t,\mathrm{disc}}^G(f) - S_{t,\mathrm{cusp}}^G(f)$$

to the trace formula. It is given by the sum of stable characters of nontempered automorphic representations of G

$$\sum_{\phi\in\Phi_{t,\mathrm{comp}}(G,V,\zeta)} b^G(\phi)f^G(\phi)$$

and stable characters of induced representations occurring in the discrete part of the trace formula

$$\sum_{M\in\mathcal{L}^0}|W_0^M||W_0^G|^{-1}\sum_{\phi\in\Phi_{t,\mathrm{disc}}(M,V,\zeta)} b^M(\phi)S_M^G(\phi,f).$$

5.2. **Setting up the limit.** Let us now explain the expected decomposition (1.1), referring to [Art17] for details. The sum over G' should be over a suitable generalisation of elliptic endoscopic data. In the least, it involves a quasisplit group G' over F, a split extension

$$1\to\hat{G}'\to\mathcal{G}'\to W_F\to 1$$

of the dual group $\hat{G}$ by the global Weil group W_F, and an L-embedding $\xi':\mathcal{G}'\to {}^LG$ such that

$$\left|\mathrm{Cent}(\mathcal{G}',\hat{G})/Z(\hat{G})^\Gamma\right|<\infty,$$

and auxilliary data of $\tilde{G}'$, which is a z-extension of G', with an L-embedding $\tilde{\xi}':\mathcal{G}'\to {}^L\tilde{G}'$ and an automorphic character $\tilde{\eta}'$ of an induced torus $\tilde{C}'$. The coefficient $\iota(r,G')$ should involve the dimension datum $m_{G'}(r)$ of G' with respect to r, defined to be the multiplicity of the trivial representation of $\mathcal{G}'$ in the composition $r\circ\xi'$, that is, the dimension of $\mathrm{Hom}_{\mathcal{G}'}(1,r\circ\xi')$.

Let $\tilde{Z}'$ be the preimage in $\tilde{G}'$ of the canonical image of Z in G'. Then $\tilde{\eta}'$ extends canonically to $\tilde{Z}'(\mathbf{A})/\tilde{Z}'(F)$. We write $\tilde{\zeta}'$ for the product of $\tilde{\eta}'$ with the pullback of ζ to $\tilde{Z}'$. Then the stable transfer

$$f\to f'=f^{\tilde{G}'}$$

should be a mapping from $\mathcal{H}(G_V,\zeta_V)$ to $\mathcal{SI}(\tilde{G}'_V,\tilde{\zeta}')$ such that $f'(\phi')=f^G(\tilde{\xi}'\circ\phi')$ where ϕ' ranges over bounded η'_V-equivariant Langlands parameters for $\tilde{G}_V$, which are $\mathcal{G}'_V$-orbits of L-homomorphisms $\phi'=\prod_{v\in V}\phi'_v$ where

$$\phi'_v:W_{F_v}\to\mathcal{G}'_v$$

has bounded image for every $v\in V$. We note that this presupposes the local Langlands correspondence for G' and $\tilde{G}'$. The stable transfer should be equipped with appropriate transfers as in the endoscopic setting. If furthermore f belongs to the spherical Hecke algebra of G_V, then the transfer f' is determined by its image in the spherical Hecke algebra of $\tilde{G}'_V$ under the Satake isomorphism, induced by the embedding $\tilde{\xi}'$.

The primitive linear forms $P^G_{\mathrm{cusp}}(f)$ would then be defined inductively by setting

$$P^G_{\mathrm{cusp}}(f)=S^1_{\mathrm{cusp}}(f)-\sum_{G'\neq G}\iota(1,G')\hat{P}^{\tilde{G}'}(f').$$

It should represent the contribution to the stable trace formula of tempered, cuspidal automorphic representations on G that are primitive in the sense that they are not functorial images from some smaller group. When $r\neq 1$, this represents an identity to be proved.

5.2.1. *A limit of trace formulas.* Now let us fix a finite dimensional complex representation r of ${}^LG$, and suppose that the finite set V of valuations v of F contains all the archimedean places of F and the places at which either G or r is ramified. Recall the partial L-function attached to a unitary irreducible representation π of $G(\mathbf{A})$,

$$L^V(s,\pi,r) = \prod_{v\notin V} \det(1 - r(c(\pi_v))q_v^{-s})^{-1}$$

where q_v is the cardinality of the residue field of F_v, and $c(\pi_v)$ the Frobenius-Hecke conjugacy class of the local constituent π_v of π. Let ϕ be the parameter for the stable global packet $\Pi_\phi = \otimes_v \Pi_{\phi_v}$ containing π.

Suppose moreover that ϕ belongs to $\Phi_{\mathrm{cusp}}(G,V,\zeta)$. Define the coefficient

$$\begin{aligned} m_r(\phi) &= -\mathrm{ord}_{s=1} L^V(s,\pi,r) \\ &= -\mathrm{Res}_{s=1}\Big(\frac{d}{ds}\log L^V(s,\pi,r)\Big), \end{aligned}$$

which is independent of the choice of π in the stable global packet Π_ϕ, and of the choice of V. It is well defined for cuspidal tempered π, in the sense that $L^V(s,\pi,r)$ is holomorphic for $\mathrm{Re}(s) > 1$. By

$$\frac{d}{ds}\log L^V(s,\pi,r) = \sum_{v\notin V}\sum_{k=1}^{\infty} \log(q_v)\mathrm{tr}(r(c(\pi_v))^k)q_v^{-ks}$$

and the usual Tauberian theorem for the last expression, the coefficient $m_r(\phi)$ is equal to

$$\lim_{N\to\infty} |V_N|^{-1} \sum_{v\in V_N} \log(q_v)\mathrm{tr}(r(c(\pi_v)) \tag{5.3}$$

where $V_N = \{v \notin V : q_v \leq N\}$. The set of coefficients $\mathrm{tr}(r(c(\pi_v))$ is bounded, since the projection of any conjugacy class $c(\pi_v)$ onto $\hat{G}$ intersects any maximal compact subgroup.

Now we return to the spectral side of the stable trace formula. We are looking for a weighted stable distribution $S^r_{\mathrm{cusp}}(f)$ that generalises the stable distribution $S_{\mathrm{cusp}}(f) = S^1_{\mathrm{cusp}}(f)$. In the absence of the Ramanujan conjecture for G we have to show that the limit (5.3) exists in a different way. For each $v \notin V$, let f^r_v be the unramified spherical function on $G(F_v)$ such that

$$\mathscr{S}(f^r_v)(c(\pi_v)) = \mathrm{tr}(r(c(\pi_v)))$$

for any unramified representation π_v of $G(F_v)$, and $\mathscr{S}$ is the (inverse) Satake transform. Note that the right-hand side is independent of the choice of π_v in Π_{ϕ_v}. Fix a function f_V in the Hecke algebra $\mathcal{H}(G,V,\zeta)$ and f^r_v in the unramified spherical algebra $\mathcal{H}(G_v,K_v)$ as above,where K_v is a fixed maximal compact subgroup of $G(F_v)$. Then choose a test function f^r in the adelic Hecke space $\mathcal{H}(G)$ whose projection to $\mathcal{H}(G,V,\zeta)$ is equal to f_V, and whose projection to $\mathcal{H}(G_v,K_v)$ is equal to f^r_v for every v not in V, thus

$$f^r = f_V \prod_{v\notin V} f^r_v.$$

We shall first define a weighted sum over cuspidal parameters in (5.2)

$$S^r_{t,\mathrm{cusp}}(f) = \sum_{\phi\in\Phi_{t,\mathrm{cusp}}(G,V,\zeta)} m_r(\phi)b^G(\phi)f^G(\phi)$$

and expanding $m_r(\phi)$, we have

$$\begin{aligned}&\lim_{N\to\infty}|V_N|^{-1}\sum_{v\in V_N}\log(q_v)\sum_{\phi\in\Phi_{t,\mathrm{cusp}}(G,V,\zeta)}b^G(\phi)f_V^G(\phi)f_v^{r,G}(\phi)\\ &=\lim_{N\to\infty}|V_N|^{-1}\sum_{v\in V_N}\log(q_v)S^1_{t,\mathrm{cusp}}(f^r).\end{aligned}$$

Note that here the interchange of the limit with the sum over ϕ is justified by the absolute convergence of $S_{t,\mathrm{disc}}(f)$ (more generally the entire spectral side of the trace formula [FLM11]) and an application of the dominated convergence theorem. It is not enough to know that the sum over ϕ is finite for a fixed test function f since we are varying the test function at each v.

5.2.2. *The geometric expansion.* Expanding this, we obtain geometric terms

$$\lim_{N\to\infty}|V_N|^{-1}\sum_{v\in V_N}\log(q_v)\sum_{M\in\mathcal{L}}|W_0^G||W_0^M|^{-1}\sum_{\delta\in\Delta(M,V,\zeta)}b^M(\delta)S_M(\delta,f^r). \tag{5.4}$$

We caution the reader that this is not yet a genuine trace formula, as we would have first to remove the contribution of the nontempered representations, otherwise the relevant expressions (5.3) will not converge. It is generally expected that this contribution will be cancelled by the elliptic terms, for example by an application of a Poisson summation formula discussed above. Nonetheless, we can still study the orbital integrals independent of this problem. The dual expansion would also contain the subtracted spectral terms for which $d\phi$ is continuous.

The outer two sums of (5.4) are finite. This leads us to define a family of linear forms

$$S^r_{M,N}(\delta,f_V)=|V_N|^{-1}\sum_{v\in V_N}\log(q_v)S_M(\delta,f_Vf_v^r),$$

and, if we are able interchange the limit with the sum over δ, it will be appropriate to define

$$S^r_M(\delta,f_V)=\lim_{N\to\infty}S^r_{M,N}(\delta,f_V) \tag{5.5}$$

whereby the geometric side can then be expressed as

$$\sum_{M\in\mathcal{L}}|W_0^G||W_0^M|^{-1}\sum_{\delta\in\Delta(M,V,\zeta)}b^M(\delta)S^r_M(\delta,f). \tag{5.6}$$

Our goal then is to show that this expression is well-defined.

The result of [Art76, Corollary 7.4] provides the existence of a family of continuous seminorms ν_n on $\mathcal{C}(G(F_v))$, which we recall is defined to be the subspace of smooth functions f on $G(F_v)$ such that

$$\nu_n(f)=\sup_{x\in G}\left(|f(g_1,x,g_2)|\Xi(x)^{-1}(1+\sigma(x))^n\right)$$

is bounded for all $g_1,g_2\in\mathfrak{g}$ and $n\in\mathbf{R}$. Here σ is defined to be $||H||$ if $x=k_1(\exp H)k_2$ for k_1,k_2 in a maximal compact K and H in the Lie algebra of a fixed maximal special subgroup of G. Also, $\Xi(x)$ is the Harish-Chandra Ξ function and $f(g_1,x,g_2)$ denotes the action of the differential operators g_1 and g_2 on $f(x)$ [HC66]. The seminorms $\nu_n(f)$ thus topologise $\mathcal{C}(G(F_v))$, and are used to bound the weighted orbital integrals.

Theorem 5.1. *Fix a representation r, and suppose that for all n we have*

$$\nu_n(f_v^r) < C_n$$

for some constant $C_n > 0$ and for all $v \notin V$. Then the distribution $S_M^r(\delta, f)$ exists for any $f \in \mathcal{H}(G, V, \zeta)$ and is a tempered distribution.

Proof. For simplicity we shall take ζ to be trivial, as this does affect the proof. From the absolute convergence of the geometric side of the trace formula, and the dominated convergence theorem, we can exchange the limit in (5.6) with the infinite sum over δ, so what we need to show is that the limit (5.5) exists. Since V_N does not contain V, by the splitting formula (3.10) we may separate the local distribution at f_v taking $V_1 = V$ and $V_2 = \{v\}$,

$$S_M(\delta, f_V f_v^r) = \sum_{L_1, L_2 \in \mathcal{L}(M)} \mathrm{e}_M^G(L_1, L_2) \hat{S}_M^{L_1}(\delta_V, f_V^{L_1}) \hat{S}_M^{L_2}(\delta_V, f_v^{r, L_2}).$$

It thus suffices to consider the limiting expression of the sum of local distributions

$$S_M^r(\delta_v, f_v) = \lim_{N \to \infty} |V_N|^{-1} \sum_{v \in V_N} \log(q_v) S_M(\delta_v, f_v^r),$$

and to show that the limit exists. Applying the nonarchimedean stable germ expansion (4.8), at each place $v \in V_N$ we have the summand

$$\log(q_v) \sum_{L \in \mathcal{L}(M)} \sum_{L' \in \mathcal{E}_{M'}(L)} \sum_{\sigma' \in \Sigma_{d'}(\tilde{L}', \tilde{\zeta}')} \iota_{M'}(L, L') h_{\tilde{M}'}^{\tilde{L}'}(\delta', \sigma') S_L^G(L', \sigma', f_v^r).$$

All three sums are finite and independent of the limit, so we may bring the sum over $v \in V_N$ inside,

$$|V_N|^{-1} \sum_{v \in V_N} \log(q_v) S_L^G(L', \sigma', f_v^r).$$

Once again, the inductive definition reduces the estimate for $S_L^G(L', \sigma', f_v^r)$, up to a constant depending only on δ, to the standard estimate for weighted orbital integrals,

$$J_M(\gamma, f_v^r) \leq \nu_n(f_v^r)(1 + |\log|D(\gamma)||)^p (1 + ||H_M(\gamma)||)^{-n} \tag{5.7}$$

for any $n \geq 1, p \in \mathbf{R}$, and Schwartz function f on $G(F)$ [Art99, §9]. Here γ is any representative in the stable conjugacy class of σ'. We are taking δ to be fixed, thus $|D(\gamma)|_v = 1$ except for finitely many v. The limit only affects the value of the seminorm $\nu_n(f)$, so we are reduced to showing that the partial sum

$$c|V_N|^{-1} \sum_{v \in V_N} \log(q_v) \nu_n(f_v^r).$$

converges as N tends to infinity, where $c = c_{n,\delta}$ is a constant depending only on n and δ. Now using our hypothesis on f_v^r, this is bounded above by

$$cM|V_N|^{-1} \sum_{v \in V_N} \log(q_v).$$

It then follows from the fact that the Dedekind zeta function $\zeta_F(s)$ is holomorphic for $\mathrm{Re}(s) > 1$ and has a simple pole at $s = 1$, that this sum converges as V_N tends to infinity.

Finally, we see that the bound given in [Art76, Lemma 7.2] can also be applied to the distribution $S^r(\delta, f)$, and as with [Art76, Corollary 7.3] we conclude that it is tempered in the sense that it extends to a continuous linear functional on $\mathcal{C}(G(F))$. ☐

For example, if $G = \mathrm{GL}_n$ and r is the standard representation, then for any $v \notin V$ we have that f_v^r is equal to the characteristic function of the set of matrices $\mathrm{Mat}_n(\mathcal{O})$ such that $|\det|_v = q_v$ [Ngô14, §2], satisfying the assumption of the theorem.

5.3. **Example.** We present the example of GL_2 here with r the standard representation and $F = \mathbb{Q}$. The coefficient that we will use to weight the trace formula is

$$m_r(\pi) = \operatorname*{ord}_{s=1}(L(s,\pi,r)) = \lim_{X\to\infty} \frac{1}{X} \sum_{p<X} \log(p)\mathrm{tr}(r(\pi_p)) = \lim_{X\to\infty} \frac{1}{X} \sum_{p<X} \log(p)\mathrm{tr}(\pi(f_p^r)).$$

Note that since r is the standard representation, f_p^r is simply the characteristic function of $\mathrm{GL}_2(\mathcal{O})$. As Langlands points out in [Lan04], this formula is valid only because we know that in this case the only possible place that poles that can arise is at $s = 1$.

In any case, we recall that the discrete terms in the spectral side of the trace formula are

$$R_{\mathrm{disc}}(f) = \sum_{\pi\in\Pi_{\mathrm{cusp}}(G)} \mathrm{tr}(\pi(f)) + \sum_{\chi^2=\omega} \mathrm{tr}(\chi(f)) + \frac{1}{4}\mathrm{tr}(\pi_0(f)).$$

They is equal to the geometric side subtracted by the continuous term

$$\sum_{M\in\mathcal{L}} \sum_{\gamma\in\Gamma(M)} a_M(\gamma)J_M(\gamma,f) - \frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{tr}(M(-it)M(it)\rho_{\pi,\omega}(f,it)dt.$$

We know that both sides are absolutely convergent, and in fact finite for compactly supported f. For simplicity, let us take $f \in C_c^\infty(G_V)$ and $V = \{\infty\}$, so we write $f = f_\infty$.

Now we introduce the weight factor. Consider the weighted spectral sum

$$\sum_{\pi\in\Pi_{\mathrm{disc}}(G)} m_r(\pi)\mathrm{tr}(\pi(f)) = \sum_{\pi\in\Pi_{\mathrm{disc}}(G)} \left(\lim_{X\to\infty} \frac{1}{X} \sum_{p<X} \log(p)\mathrm{tr}(\pi(f_p^r))\mathrm{tr}(\pi(f_\infty)) \right).$$

Having fixed f_∞, the sum over π is nonzero for only finitely many π, so we may interchange the limit and the sum as in [Lan04, (12)],[1]

$$\lim_{X\to\infty} \frac{1}{X} \sum_{p<X} \log(p) \sum_{\pi\in\Pi_{\mathrm{disc}}(G)} \mathrm{tr}(\pi(f_p^r))\mathrm{tr}(\pi(f_\infty)) = \lim_{X\to\infty} \frac{1}{X} \sum_{p<X} \log(p) R_{\mathrm{disc}}(f_p^r f_\infty).$$

Note that the contribution of the one-dimensional spectrum is

$$\sum_{\chi^2=\omega} m_r(\chi)\mathrm{tr}(\chi(f_\infty)) = \mathrm{tr}(1(f)).$$

since $m_r(\chi)$ is 1 if χ is the trivial representation and 0 otherwise. (Moreover, $m_r(\pi_0) = 1$ as π_s is the unitary induced representation with respect to the complex parameter s.)

[1] This is slightly delicate as the sum over π is finite for each fixed p, but does vary according to p.

We now expand R_{disc} and consider the terms individually. The geometric terms are

$$\begin{aligned}&\lim_{X\to\infty}\frac{1}{X}\sum_{p<X}\log(p)\sum_{M\in\mathcal{L}}\sum_{\gamma\in\Gamma(M)}a_M(\gamma)J_M(\gamma,f_p^r f_\infty)\\&=\sum_{M\in\mathcal{L}}\sum_{\gamma\in\Gamma(M)}a_M(\gamma)\left(\lim_{X\to\infty}\frac{1}{X}\sum_{p<X}\log(p)J_M(\gamma,f_p^r f_\infty)\right),\end{aligned}$$

where the interchange of the limit and the sum over γ here is valid since the sum is finite due to the fact that $f_p^r f_\infty$ is compactly supported.[2] The splitting formula for weighted orbital integrals in this case gives

$$J_M(\gamma,f_p^r f_\infty)=J_M(\gamma_v,f_p^r)J_G(\gamma_\infty,f_\infty)+J_G(\gamma_p,f_p^r)J_M(\gamma_\infty,f_\infty).$$

We can ignore the archimedean contribution in evaluating this limit. Alternatively, we can split the orbital integral into local factors

$$J_M(\gamma,f_p^r f_\infty)=J_M(\gamma_p,f_p^r)J_M(\gamma_\infty,f_\infty)$$

Either way, we are reduced to showing that the expression

$$\lim_{X\to\infty}\frac{1}{X}\sum_{p<X}\log(p)J_M(\gamma,f_p^r)$$

exists for each M and γ. If we can show that there is a constant such that

$$J_M(\gamma,f_p^r)<C$$

for all p, it will follow that

$$\lim_{X\to\infty}\frac{1}{X}\sum_{p<X}\log(p)J_M(\gamma,f_p^r)<\lim_{X\to\infty}\frac{1}{X}\sum_{p<X}\log(p)C=C$$

by the prime number theorem. In the case we are considering, this amounts to a uniform in p bound for

$$J_M(\gamma,1_p).$$

Here we arrive at the typical bound on orbital integrals

$$J_M(\gamma,1_p)\le\nu_n(1_p)(1+|\log|D(\gamma)||)^q(1+||H_M(\gamma)||)^{-n},\qquad n\ge 1,q\in\mathbf{R},$$

but perhaps in this simple case there are easier techniques that one can apply. In particular, this would work for GL_ℓ for ℓ a prime. (Except it would still remain to treat the continuous contribution, which we have not yet done.) For any n, the seminorm

$$\nu_n(1_p)=\sup_{x\in G}\left(|1_p(x)|\Xi(x)^{-1}(1+\sigma(x))^n\right)=\sup_{x\in G}\left(\Xi(x)^{-1}(1+\sigma(x))^n\right)$$

hence is independent of p.

[2]See previous footnote

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