Double Categories and Pseudo Algebras

CT2006

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Paper and Slides at

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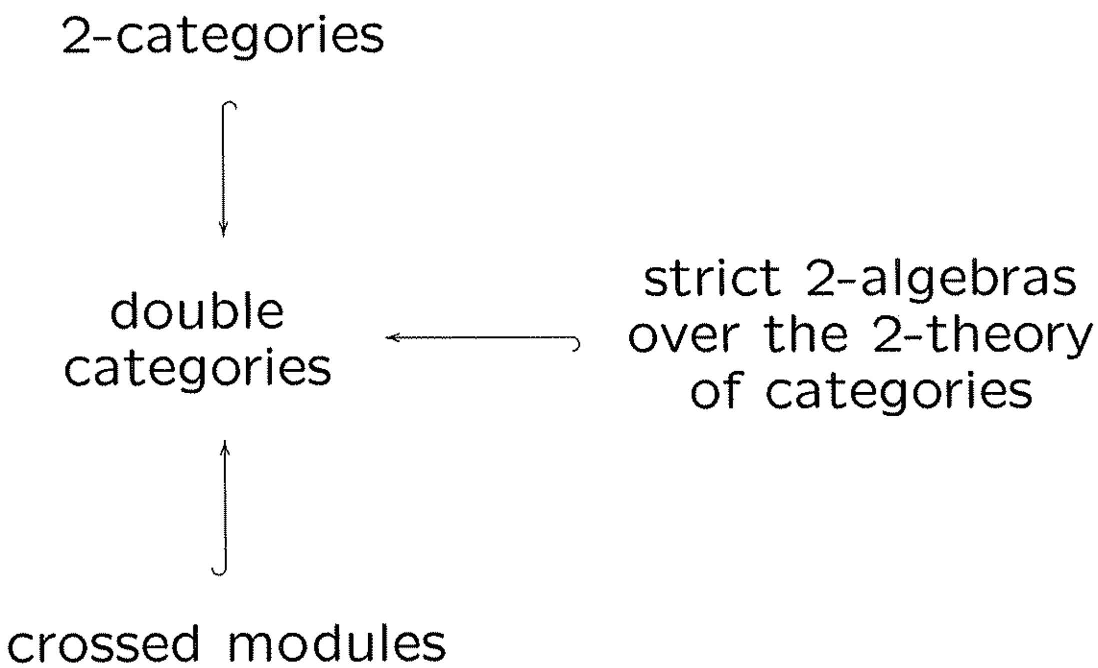
Overview

History: Pseudo Algebras in Conformal Field Theory

Talk:



Pseudo Algebras in Conformal Field Theory

Pseudo algebras over 2-theories were introduced by Hu and Kriz in 2004 to give a rigorous definition of conformal field theory as outlined by Graeme Segal in the 80's.

Advantages:

- Description of coherence isos and coherence diagrams
- Includes morphisms of worldsheets
- Includes self gluings
- Treats modular functors (i.e. anomalies)
- Also works for closed/open CFT (worldsheets with corners).

Pseudo Algebras in Conformal Field Theory

The 2-theory of *commutative monoids with cancellation* is relevant for CFT.

A conformal field theory and modular functor are morphisms of pseudo algebras over this 2-theory.

Example of Pseudo Algebra over 2-theory

A *worldsheet* is a real, compact, not necessarily connected, two dimensional, smooth manifold with complex structure and real analytically parametrized boundary components.

$I :=$ symmetric monoidal category of finite sets and bijections under $\coprod$
$X_{A,B} :=$ category of worldsheets with inbound boundary components labelled by A and outbound boundary components labelled by B.

$X : I^2 \to Cat$ is a 2-functor.

There are operations of disjoint union, gluing (cancellation), and unit

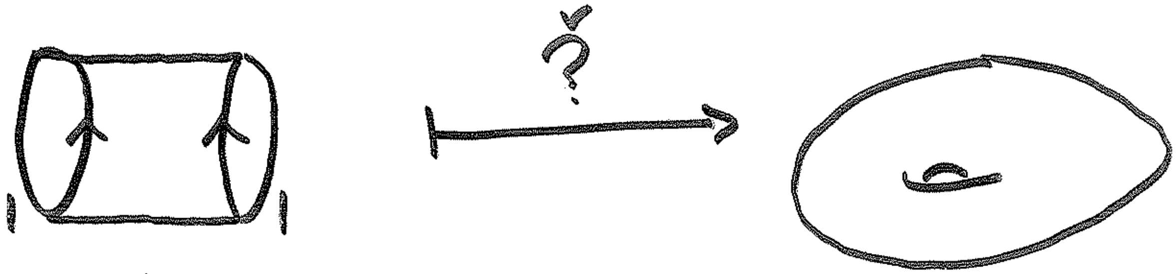
$$+ : X_{A,B} \times X_{C,D} \to X_{A+C,B+D}$$

$$\check{?} : X_{A+C,B+C} \to X_{A,B}$$

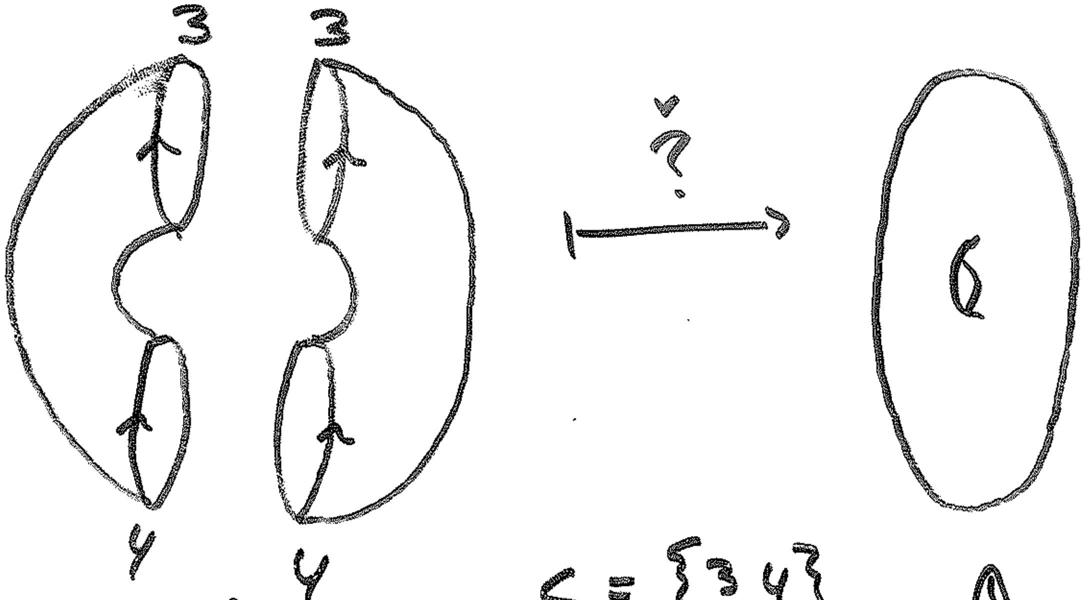
$$0 \in X_{0,0}.$$

with coherence isos and diagrams and axioms.

Gluing

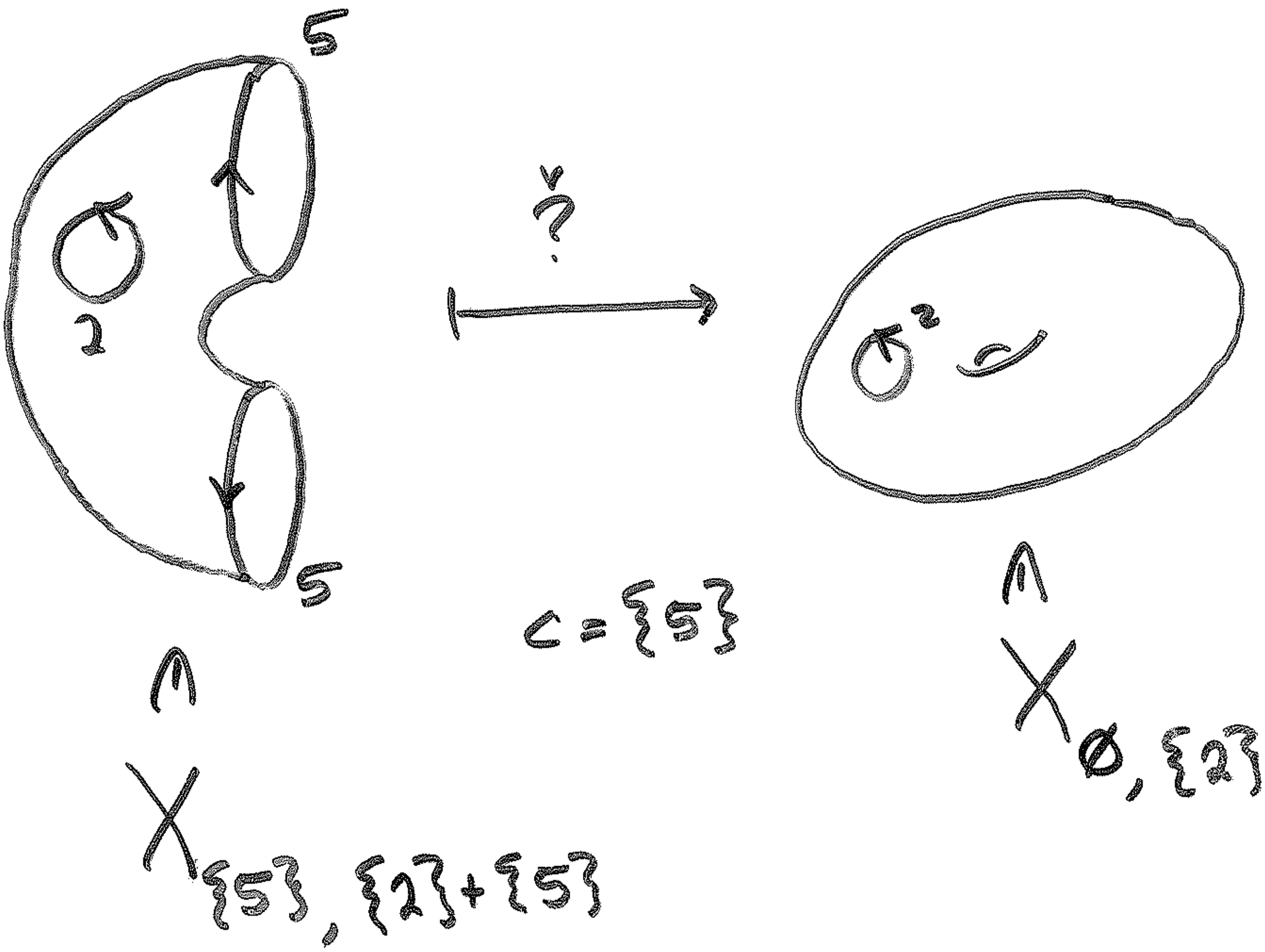


$\in$ $X_{\{1\},\{1\}}$ $\qquad c = \{1\}$ $\qquad \in$ $X_{\emptyset,\emptyset}$

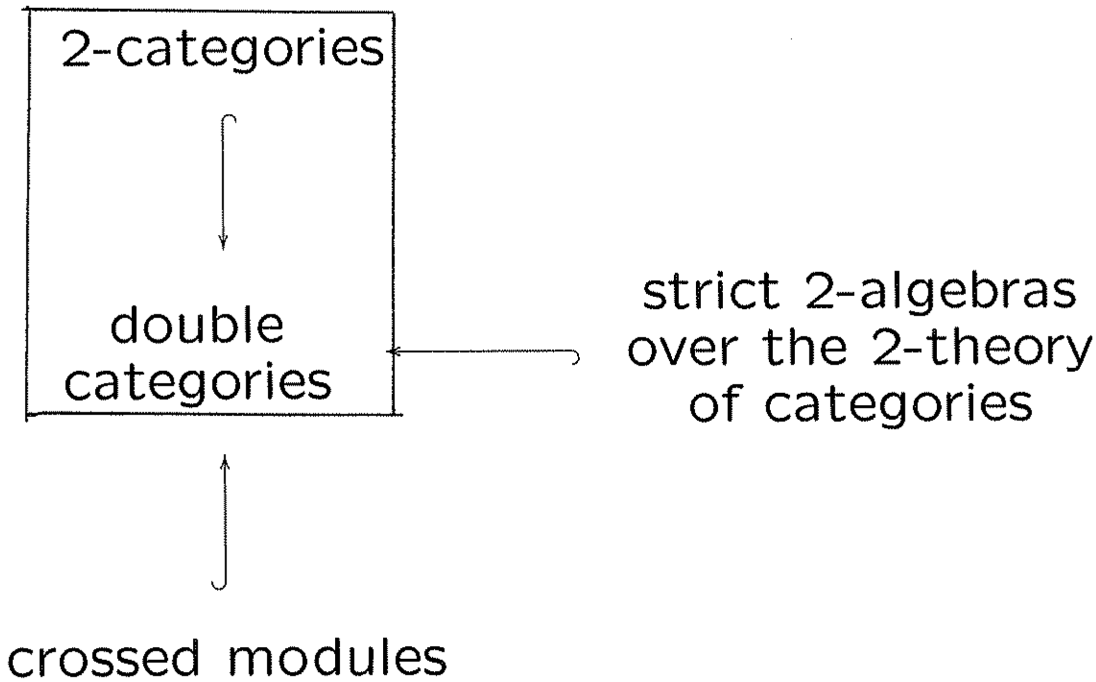


$\in$ $X_{\{3,4\},\{3,4\}}$ $\qquad c = \{3,4\}$ $\qquad \in$ $X_{\emptyset,\emptyset}$

Gluing



Overview


2-categories
double
categories
strict 2-algebras
over the 2-theory
of categories
crossed modules

Double Categories

Definition 1 *(Ehresmann 1963) A double category $\mathbb{D}$ is an internal category in $\mathcal{C}at$.*

Definition 2 *A double category $\mathbb{D}$ consists of
a set of objects,
a set of horizontal morphisms,
a set of vertical morphisms, and
a set of squares with source and target as follows*

$$A \xrightarrow{f} B \qquad \begin{array}{c} A \\ {\scriptstyle j}\downarrow \\ C \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array}$$

and compositions and units that satisfy axioms.

Compositions and Units for Morphisms in a Double Category

Vertical:

$$\begin{array}{c} A \\ \downarrow{\scriptstyle j_1} \\ C \\ \downarrow{\scriptstyle j_2} \\ E \end{array} = \begin{bmatrix} j_1 \\ j_2 \end{bmatrix} = j_2 \circ j_1$$

$$\begin{bmatrix} 1^v_A \\ j_1 \end{bmatrix} = \begin{array}{c} A \\ \downarrow{\scriptstyle 1^v_A} \\ A \\ \downarrow{\scriptstyle j_1} \\ C \end{array} = \begin{array}{c} A \\ \downarrow{\scriptstyle j_1} \\ C \end{array} = \begin{array}{c} A \\ \downarrow{\scriptstyle j_1} \\ C \\ \downarrow{\scriptstyle 1^v_C} \\ C \end{array} = \begin{bmatrix} j_1 \\ 1^v_C \end{bmatrix}$$

Horizontal:

$$\begin{bmatrix} f_1 & f_2 \end{bmatrix} = f_2 \circ f_1$$

$$\begin{bmatrix} 1^h_A & f_1 \end{bmatrix} = f_1 = \begin{bmatrix} f_1 & 1^h_B \end{bmatrix}$$

Compositions for Squares in a Double Category

Vertical:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle j_1}\downarrow & \alpha & \downarrow{\scriptstyle k_1} \\
C & \xrightarrow{g} & D \\
{\scriptstyle j_2}\downarrow & \beta & \downarrow{\scriptstyle k_2} \\
E & \xrightarrow[h]{} & F
\end{array}
\quad = \quad
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle \left[\begin{smallmatrix} j_1 \\ j_2 \end{smallmatrix}\right]}\downarrow & \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] & \downarrow{\scriptstyle \left[\begin{smallmatrix} k_1 \\ k_2 \end{smallmatrix}\right]} \\
E & \xrightarrow[h]{} & F
\end{array}
$$

Horizontal: similarly

Identity squares: i^v_f and i^h_j.

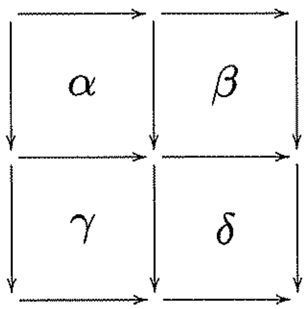
Axioms for a Double Category

All compositions are *associative* and *unital* (as above) and

$$\begin{bmatrix} i^h_{j_1} \\ i^h_{j_2} \end{bmatrix} = i^h_{\left[\begin{smallmatrix} j_1 \\ j_2 \end{smallmatrix}\right]}$$

$$\begin{bmatrix} i^v_{f_1} & i^v_{f_2} \end{bmatrix} = i^v_{[f_1 f_2]}.$$

Interchange Law:

If 

, then $\begin{bmatrix} \begin{bmatrix} \alpha & \beta \end{bmatrix} \\ \begin{bmatrix} \gamma & \delta \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \alpha \\ \gamma \end{bmatrix} & \begin{bmatrix} \beta \\ \delta \end{bmatrix} \end{bmatrix}$ and

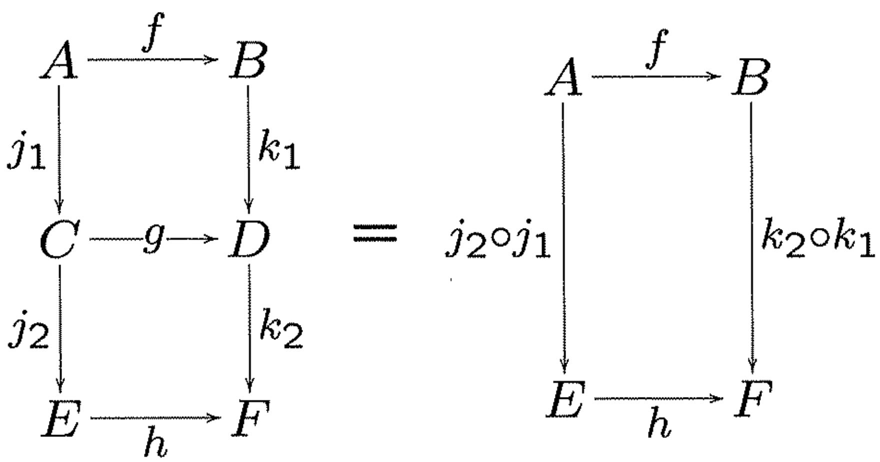
we write $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$.

Examples of Double Categories

Let I be a 1-category.

$\Box I$:= double category of commutative squares in I

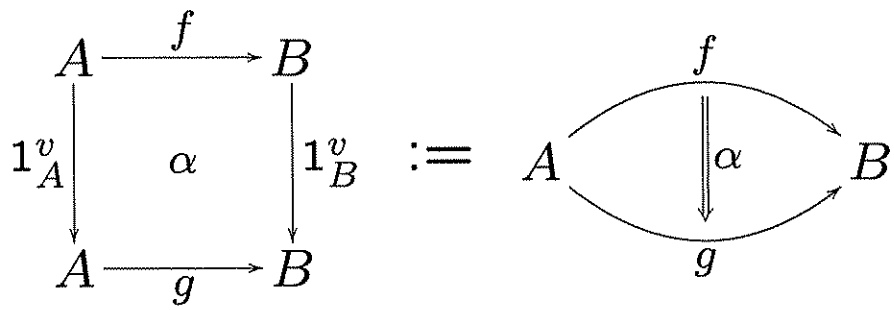
$Obj\,\Box I := Obj\,I$
$Hor\,\Box I := Mor\,I$
$Ver\,\Box I := Mor\,I$
$Sq\,\Box I :=$ commutative squares in I



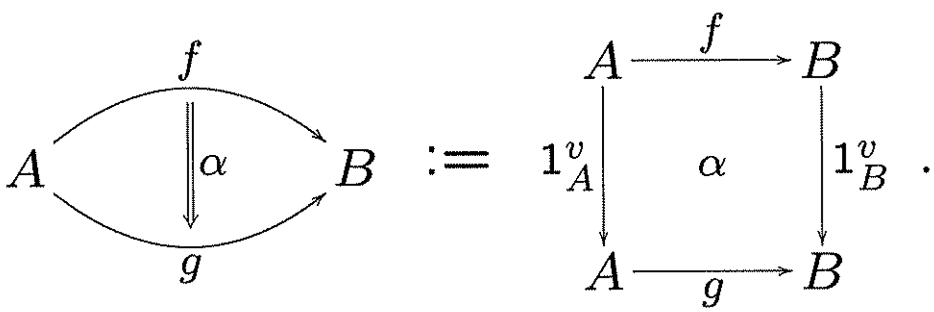
$\boxdot I$:= double category of not necessarily commutative squares in I

Examples of Double Categories

Every 2-category $\mathbf{C}$ is a double category with trivial vertical morphisms.



Definition 3 *The horizontal 2-category* $\mathbf{H}\mathbb{D}$ *of a double category* $\mathbb{D}$ *has objects* $Obj\,\mathbb{D}$, *morphisms* $Hor\,\mathbb{D}$, *and 2-cells*



Examples of Double Categories

Let $\mathbf{C}$ be a 2-category.

$\mathbb{Q}\mathbf{C}$:= Ehresmann's double category of *quintets* in $\mathbf{C}$ (1963)

$Obj\,\mathbb{Q}\mathbf{C} := Obj\,\mathbf{C}$
$Hor\,\mathbb{Q}\mathbf{C} := Mor\,\mathbf{C}$
$Ver\,\mathbb{Q}\mathbf{C} := Mor\,\mathbf{C}$

$$Sq\,\mathbb{Q}\mathbf{C} := \left\{ \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array} \;\middle|\; A \underset{g\circ j}{\overset{k\circ f}{\Downarrow \alpha}} D \right\}$$

Theorem 1 *(Grandis-Paré 2004) The functor* $\mathbb{Q} : 2\text{-}Cat \longrightarrow Dbl$ *admits a right adjoint.*

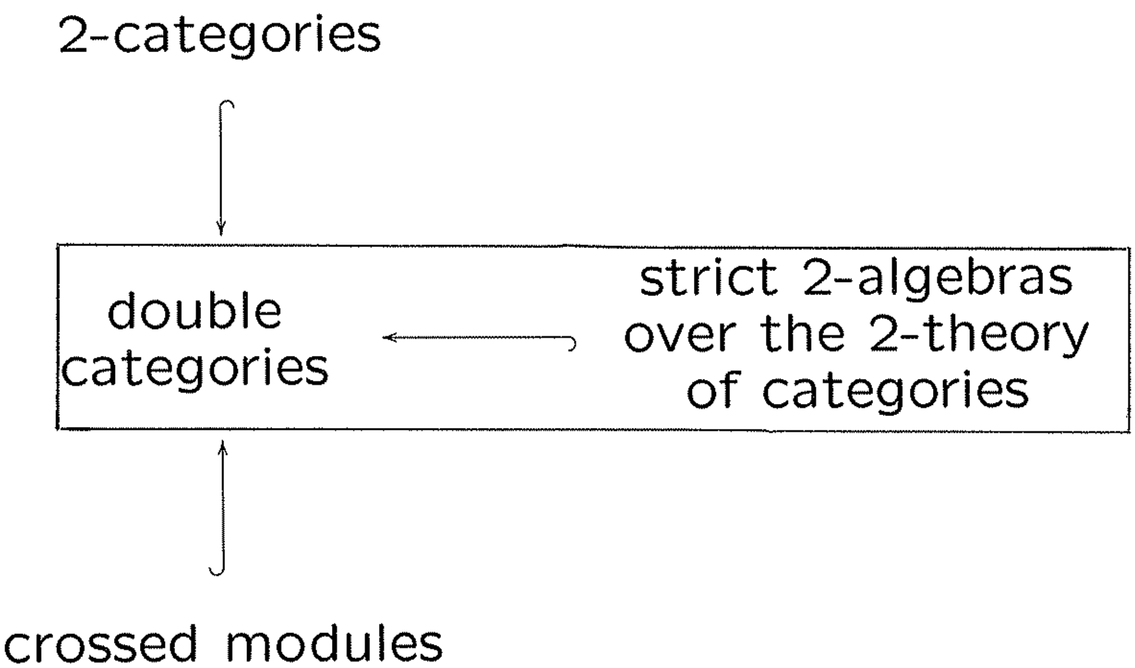
Examples of Double Categories

$\mathbb{R}$ng := pseudo double category of rings, bimodules, and twisted equivariant maps

$Obj\,\mathbb{R}$ng := rings with identity
$Hor\,\mathbb{R}$ng := bimodules
$Ver\,\mathbb{R}$ng := homomorphisms of rings
$Sq\,\mathbb{R}$ng :=

$$\left\{ \begin{array}{ccc} R & \xrightarrow{M} & S \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ T & \xrightarrow[N]{} & U \end{array} \;\middle|\; \begin{array}{l} \alpha : M \longrightarrow N \text{ group homomorphism} \\ \alpha(smr) = k(s)\alpha(m)j(r) \end{array} \right\}$$

Overview


2-categories
double categories
strict 2-algebras over the 2-theory of categories
crossed modules

Folding Structures

We introduce folding structures to compare algebras over the 2-theory of categories with double categories.

Definition 4 *A holonomy on a double category* $\mathbb{D}$ *is a 2-functor*

$$(\mathbf{V}\mathbb{D})_0 \longrightarrow \mathbf{H}\mathbb{D}$$

$$A \longmapsto \overline{A} = A$$

$$\begin{array}{c} A \\ \downarrow j \\ B \end{array} \longmapsto A \xrightarrow{\overline{j}} B \,.$$

Folding Structures

Definition 5 *A folding structure on a double category $\mathbb{D}$ consists of a holonomy $j \longmapsto \bar{j}$ and bijections*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array} \quad \overset{\Lambda}{\longleftrightarrow} \quad \begin{array}{ccc} A & \xrightarrow{[f\bar{k}]} & D \\ {\scriptstyle 1^v_A}\downarrow & \Lambda(\alpha) & \downarrow{\scriptstyle 1^v_D} \\ A & \xrightarrow[{[\bar{j}g]}]{} & D \end{array}$$

compatible with compositions and units.

A folding structure *horizontalizes* a double category.

Theorem 2 *(Brown-Mosa 1999) The notions of folding structure and connection pair are equivalent.*

Examples of Folding Structures

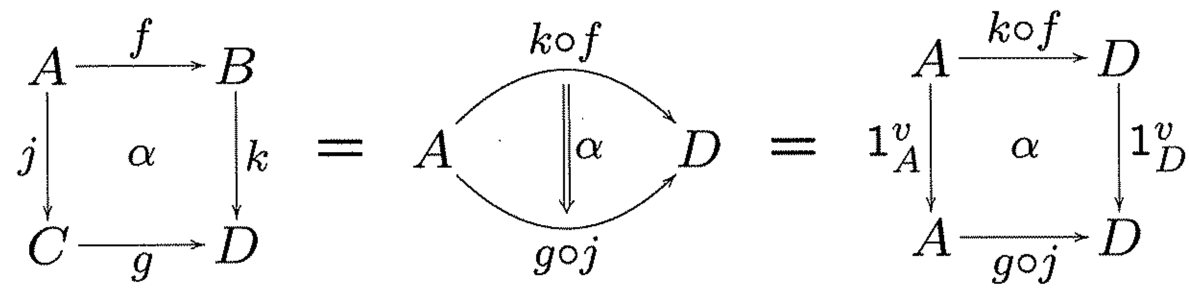
Let I be a 1-category.
$\square I$ = double category of commutative squares in I
$\boxdot I$ = double category of not necessarily commutative squares in I

Then $\square I$ and $\boxdot I$ each admit a unique folding structure.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array} \quad \xleftrightarrow{\Lambda} \quad \begin{array}{ccc} A & \xrightarrow{k\circ f} & D \\ {\scriptstyle 1^v_A}\downarrow & & \downarrow{\scriptstyle 1^v_D} \\ A & \xrightarrow[g\circ j]{} & D \end{array}$$

Examples of Folding Structures

Let $\mathbf{C}$ be a 2-category. Then $\mathbb{Q}\mathbf{C}$ admits a folding structure by definition.



Examples of Folding Structures

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$Obj\,\mathbb{R}$ng := rings with identity
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$$\left\{ \begin{array}{ccc} R & \xrightarrow{M} & S \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ T & \xrightarrow[N]{} & U \end{array} \;\middle|\; \begin{array}{l} \alpha : M \longrightarrow N \text{ group homomorphism} \\ \alpha(smr) = k(s)\alpha(m)j(r) \end{array} \right\}$$

Holonomy:

$$\overline{j} := T_j = \text{ the } (T,R)\text{-bimodule } T$$

Folding:

$$\Lambda(\alpha) : U_k \otimes_S M \Longrightarrow N \otimes_T T_j$$

$$u \otimes m \longmapsto (u \cdot \alpha(m)) \otimes 1_T$$

Comparison Theorem

Theorem 3 *(F. 2006) The 2-category of strict 2-algebras over the 2-theory of categories is 2-equivalent to the 2-category of double categories with folding structures and invertible vertical morphisms.*

The pseudo version of the theorem also holds.

Strict 2-Algebras

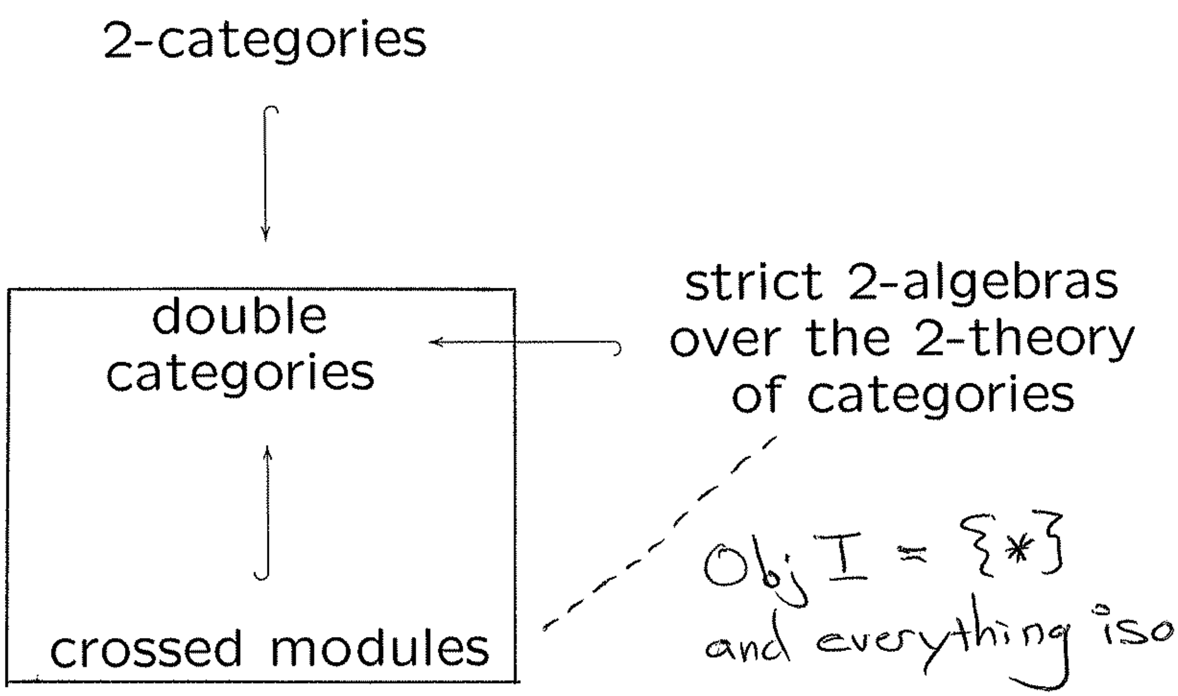
Definition 6 *A strict 2-algebra over the 2-theory of categories is a groupoid I and a strict 2-functor $X : I^2 \longrightarrow Cat$ with strictly 2-natural functors*

$$X_{B,C} \times X_{A,B} \xrightarrow{\circ} X_{A,C}$$

$$\{*\} \xrightarrow{1_B} X_{B,B}$$

for all $A, B, C \in I$ which satisfy axioms like those of a category.

Overview


2-categories
double categories
crossed modules
strict 2-algebras over the 2-theory of categories
Obj I = {*}
and everything iso

Towards Strict 2-Algebras and Crossed Modules

Consider one object cases.

groupoids	$\subseteq$ 2-groupoids	$\subseteq$ double groupoids
groups	$\subseteq$ 2-groups	$\subseteq$ double groups

Theorem 4 *(Verdier, Brown-Spencer 1976,...) 2-groups are equivalent to crossed modules.*

Theorem 5 *(Brown-Spencer 1976) Edge symmetric double groups with folding structure with trivial holonomy are equivalent to crossed modules.*

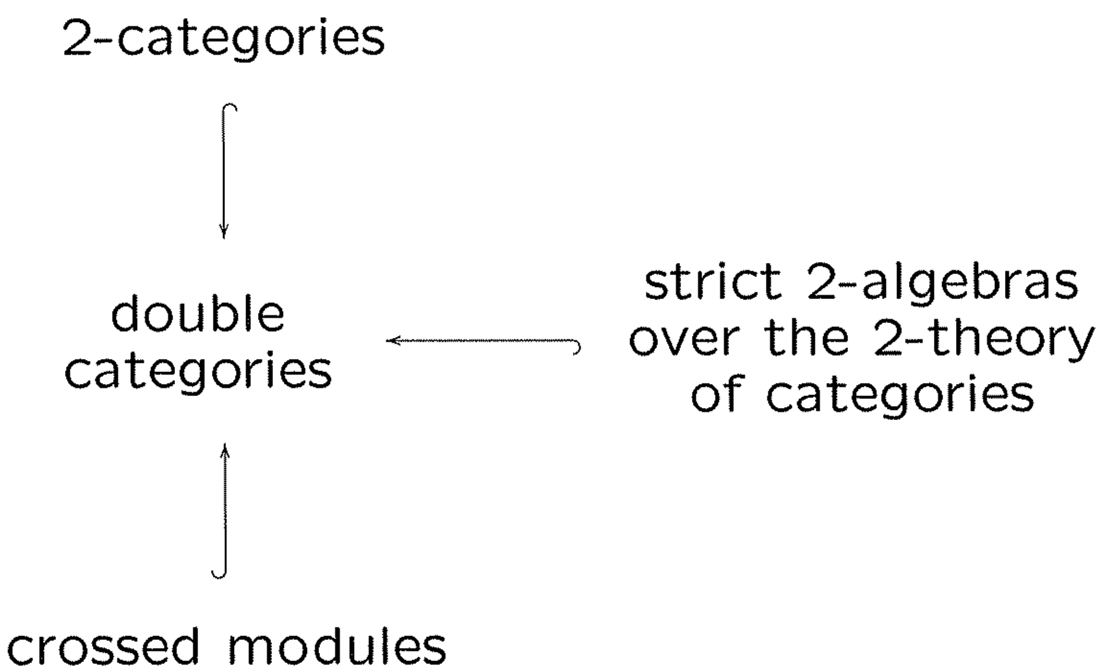
Question: What is a one object strict 2-algebra over the 2-theory of categories with everything iso?

Strict 2-Algebras and Crossed Modules

Theorem 6 *(F. 2006) One object strict 2-algebras over the 2-theory of categories with everything iso are equivalent to crossed modules with a group action.*

In the case of a trivial group and trivial I, this says 2-groups are equivalent to crossed modules.

Conclusion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