HIGHER OPERADS

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This talk attempts to motivate the study of higher operads and their many variants. Some exercises and open problems in the field are presented in the text. These are notes from a talk I presented at the 2010 Graduate Student Topology and Geometry Conference in Ann Arbor, Michigan. I thank Nina White and Michelle Lee for their terrific organization of a very fun conference. Special thanks go to Kyle Ormsby for carefully and efficiently typesetting the first draft of these notes, on which I then elaborated. Any errors are of course mine alone.

1. Topological motivation: How is ΩX a homotopy associative space?

We begin with a based space (X, x_0), and consider the space of loops in X based at x_0, namely

$$\Omega X := \{f : [0,1] \to X \mid f \text{ continuous and } f(0) = x_0 = f(1)\}.$$

Concatenation of based loops provides ΩX with a multiplication: for $f, g \in \Omega X$, we have

$$(f * g)(t) := \begin{cases} f(2t) & \text{if } 0 \le t \le 1/2, \\ g(2t-1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

Note that the multiplication is not strictly associative, i.e., $(f * g) * h \neq f * (g * h)$. By the standard argument, though,

$$(f * g) * h \simeq f * (g * h), \tag{1}$$

i.e. the multiplication on ΩX is *associative up to homotopy*. To picture the homotopy in (1), one can draw the unit square with a line connecting $\frac{1}{4}$ on the bottom edge to $\frac{1}{2}$ on the top edge, and a line connecting $\frac{1}{2}$ on the bottom edge to $\frac{3}{4}$ on the top edge. Then the homotopy is defined on a horizontal slice line by doing f on the segment in the left region, g on the segment in the middle region, and h on the segment in the right region. This suggests that we should also consider concatenations of three paths f,g, and h, where the three subintervals of $[0, 1]$ for f,g,h, can have various lengths.

Further suggestions for more general concatenations arise when we consider composites of 4 based loops.

$$((f*g)*h)*i \simeq (f*g)*(h*i) \simeq f*(g*(h*i)) \simeq f*((g*h)*i) \simeq (f*(g*h))*i \tag{2}$$

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The first and the last entries of (1) are also homotopic, and we have a diagram like the famous Mac Lane pentagon. But does it commute?

This leads us to consider, abstractly and simultaneously, all possible concatenations of finitely many based loops. The formal structure encoding this information is $\mathcal{C}$, the *little intervals operad*. For $n \in \mathbb{N}$, let

$$\mathcal{C}(n) = \left\{ \lambda = (\lambda_1, \dots, \lambda_n) \;\middle|\; \begin{array}{c} \lambda_i : [0,1] \to [0,1] \text{ linear embeddings such that the} \\ \text{interiors of the images of } \lambda_i \text{ and } \lambda_j \text{ are disjoint for all } i \neq j \end{array} \right\}.$$

We should think of $\mathcal{C}(n)$ as the "space of operations of n variables;" it is equipped with the following structure:

- operad composition
$$\gamma : \mathcal{C}(n) \times \mathcal{C}(k_1) \times \cdots \times \mathcal{C}(k_n) \to \mathcal{C}(k_1 + \cdots + k_n)$$
$$(\lambda, \mu^1, \dots, \mu^n) \mapsto \begin{array}{c} (\lambda_1 \circ \mu^1_1, \lambda_1 \circ \mu^1_2, \dots, \lambda_1 \circ \mu^1_{k_1}, \\ \lambda_2 \circ \mu^2_1, \lambda_2 \circ \mu^2_2, \dots, \lambda_2 \circ \mu^2_{k_2}, \\ \dots, \\ \lambda_n \circ \mu^n_1, \lambda_n \circ \mu^n_2, \dots, \lambda_n \circ \mu^n_{k_n}) \end{array}$$
(essentially insert each μ^i into λ_i),
- operad unit
$$1_{[0,1]} \in \mathcal{C}(1),$$
- permutation action of Σ_n, the permutation group on n letters, on $\mathcal{C}(n)$ which permutes the labels i of the little intervals λ_i.

The operad composition γ is associative, unital, and equivariant, i.e., $\mathcal{C}$ is a *symmetric operad* in the sense of May [9].

Now note that ΩX is an *algeba over* $\mathcal{C}$. In other words, we have

$$\alpha : \mathcal{C}(n) \times (\Omega X)^n \to \Omega X$$
$$\lambda \times (f_1, \dots, f_n) \mapsto (f : [0,1] \to X)$$

where f is f_i (rescaled!) on the image of λ_i and the basepoint x_0 otherwise. Note that α is associative, unital, and equivariant; as is required to be an algebra over a symmetric operad. An equivalent formulation[1] is to say we have a symmetric operad morphism $\mathcal{C} \to \mathrm{End}(\Omega X)$. To say that ΩX is "homotopy associative" is really to say that it is an algebra over the little intervals operad $\mathcal{C}$.

In summary: we have recalled the space of a based loops ΩX in a based space X, along with its algebraic structure of concatenation of loops. The little intervals operad parametrizes all possible concatenations, and makes precise in which sense the multiplication on ΩX is *homotopy associative*, namely there is a homotopy as in (1), and these homotopies are compatible up to higher homotopies (the pentagon

[1]To see this, one uses the usual exponentiation in an appropriate category of spaces, namely $Map(X \times Y, Z) \cong Map(X, Map(Y, Z))$.

diagram commutes only up to homotopy), and these higher homotopies are compatible up to higher homotopies, and so on, ad infinitum. More, precisely, the space of based loops on a based space is an algebra over the little intervals operad $\mathcal{C}$. But could one dream that the converse is true? This is the celebrated Recognition Principle of Peter May.

Theorem 1.1 (May 1972, [9])**.** *A path connected space Y is homotopy equivalent to ΩX for some X if and only if Y is an algebra over the little intervals operad $\mathcal{C} = \mathcal{C}_1$. (A similar statement holds for n-fold loop spaces and the little n-cubes operad $\mathcal{C}_n$.)*

How is homotopy associativity related to ordinary associativity? Strictly associative spaces are controlled by the associativity operad, which in degree n is $\mathrm{Assoc}(n) = \Sigma_n$. The homotopy associativity operad $\mathcal{C}$ is in a sense an expanded replacement of the associativity operad.

Proposition 1.2. *The canonical map $\mathcal{C} \to \mathrm{Assoc}$ is a homotopy equivalence of operads.*

Proof. What do we mean by the "canonical" map $\mathcal{C} \to \mathrm{Assoc}$? Well, the linear embeddings $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{C}(n)$ have their disjoint images in $[0,1]$ ordered, as we go from left to right in $[0,1]$. This ordering determines a permutation on n-letters, which is the image of λ under the canonical map. We see that this canonical map essentially contracts each path component of $\mathcal{C}(n)$ to a point: λ and μ are in the same path component of $\mathcal{C}(n)$ if and only if we can slide the λ_i's left or right (but not past each other), and stretch or shrink them, in order to obtain the μ_i's. In this case, λ and μ determine the same permutation. Linear embeddings λ and μ determine different permutations if and only if μ cannot be obtained from λ in this way. $\square$

Exercise 1.3. *In this exercise, we learn a succinct, conceptual definition of* Set-*operad*[2] *which can then be imported to many other contexts. Let* Coll *denote the category of* collections. *Objects are sequences of sets $\{S_n\}_{n\geq 0}$, and morphisms are sequences of functions. The category* Coll *is the same as the slice category* $\mathrm{Set}/\mathbb{N}$. *Prove that the operation*

$$\otimes : \mathrm{Coll} \times \mathrm{Coll} \to \mathrm{Coll}$$

defined by

$$(S \otimes T)_\ell = \coprod_{k_1 + \cdots + k_n = \ell} S_n \times T_{k_1} \times \cdots \times T_{k_n}.$$

makes $(\mathrm{Coll}, \otimes)$ *into a monoidal category. Then prove that monoids in the monoidal category* $(\mathrm{Coll}, \otimes)$ *are precisely nonsymmetric* Set-*operads.*

[2] By Set-operad we simply mean an operad $\mathcal{O}$ in which the spaces $\mathcal{O}(n)$ are simply sets. The little intervals operad $\mathcal{C}$ is a topological operad, since each $\mathcal{C}(n)$ is a topological space.

2. Renaissance: the 1990's

In the 1990's the theory of operads and its applications experienced a Renaissance. There was much work on real geometry (compactifications of configuration spaces, the cactus operad for Chas-Sullivan product on free loop spaces, ...), mathematical physics (deformation quantization of Poisson manifolds, ...), algebra, topology, and much more. See the recent article [8] for a discussion and introduction.

Many variants and new examples were proposed and discovered. I would just like to mention two: cyclic operads and modular operads [2, 3, 4, 5]. Riemann spheres with parametrized labelled holes form a *cyclic operad* (i.e. output can be interchanged with an input). The moduli space of Riemann surfaces with holes forms a *modular operad*.

Exercise/Problem 2.1. *Find monoidal categories in which monoids are cyclic operads or modular operads.*

3. Colored operads and quasi-operads

There are still many situations which ordinary operads cannot handle. For example, say you want an algebra to be a pair consisting of a monoid and a space it acts on, or a ring R with an R-module, or say you want an algebra to be a morphism of algebras. For such situations, we have colored operads, also from the 1970's. Moerdijk and Weiss have recently developed the theory of weak colored operads, called *quasi-operads* [12, 11]. See [10] for an exposition and more explanation of some topics in this section.

Definition 3.1. A *colored operad* is a pair (C, P) consisting of a set C of colors and for each sequence $c_1, \ldots, c_n; c$ a space $P(c_1, \ldots, c_n; c)$ of operations which take inputs of "types" $c_1, \ldots, c_n$ to an output of type c. We have

- colored operad composition

$$P(c_1, \ldots, c_n; c) \times P(d_1^1, \ldots, d_{k_1}^1; c_1) \times \cdots \times P(d_1^n, \ldots, d_{k_n}^n; c_n) \to P(d_1^1, \ldots, d_{k_n}^n; c),$$

- for each $c \in C$, a colored operad unit

$$1_c \in P(c; c)$$

- for each $n \geq 1$, a Σ_n-action: $\sigma \in \Sigma_n$ gives

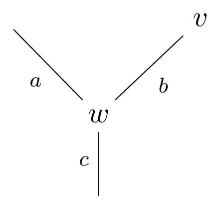
$$\sigma^* : P(c_1, \ldots, c_n; c) \to P(c_{\sigma_1}, \ldots, c_{\sigma_n}; c).$$

Definition 3.2. A *P-algebra A* consists of spaces A_c, $c \in C$ with actions

$$P(c_1, \ldots, c_n; c) \times A_{c_1} \times \cdots A_{c_n} \to A_c$$

satisfying the expected axioms.

As an example of a colored operad, any non-planar, finite, nonempty, rooted tree T generates a colored operad $\Omega(T)$ in which colors are the edges of T and the generators of operations are the vertices of T. For example, if T is the tree below,



then the operad $\Omega(T)$ has colors a, b, and c and generating operations $v \in \Omega(T)(-;b)$ and $w \in \Omega(T)(a,b;c)$. The other operations are: units $1_a, 1_b, 1_c$, composites of those already given, and permutations of all of these.

Other examples of trees are finite ordinals: any finite ordinal $[n] = \{0, 1, \ldots, n\} \in \Delta$ can be viewed as a tree with n vertices and $n+1$ edges.

Exercise 3.3. *(1) Find a (nonsymmetric) operad on two colors whose algebras are pairs (A, X) where A is a monoid and X is a space on which A acts; similarly for rings, modules.*

(2) If O is an operad, find a colored operad whose algebras are O-algebra morphisms.

(3) Find a colored operad whose algebras are ordinary operads.

(4) Find a monoidal category in which monoids are colored operads.

Remark 3.4. Non-symmetric colored operads are the same as small *multicategories*, i.e. "categories" in which arrows have a *sequence* of objects as source, as opposed to a single object as source.

Exercise 3.5. *There is a multicategory* Vect *with objects finite-dimensional real vector spaces and*

$$\mathrm{Hom}_{\mathrm{Vect}}(V_1, \ldots, V_n; V) = \{f : V_1 \otimes \cdots \otimes V_n \to V \mid f \text{ linear}\}.$$

An important consequence of the observation in Remark 3.4 is that we can take the nerve of a colored operad, since a colored operad is a symmetric multicategory.

Definition 3.6 (Moerdijk-Weiss)**.** Let Ω be the category of non-planar, finite, nonempty, rooted trees. A *dendroidal set* is a functor $\Omega^{\mathrm{op}} \to \mathrm{Set}$. The *dendroidal nerve* of a colored operad P is

$$\begin{aligned} N_d(P) : \Omega^{\mathrm{op}} &\to \mathrm{Set} \\ T &\mapsto \mathrm{Hom}_{\mathrm{operad}}(\Omega(T), P), \end{aligned}$$

where the colored operad $\Omega(T)$ associated to a tree T was defined above.

Definition 3.7 (Moerdijk-Weiss)**.** A *quasi-operad* is a dendroidal set in which every inner horn has a filler.

Theorem 3.8 (Cisinski-Moerdijk [1])**.** *There is a very good model structure on the category of dendroidal sets such that*

- *the fibrant objects are quasi-operads,*
- *the cofibrations are normal monomorphisms,*
- $(\tau_d, N_d) : \mathrm{dSet} \leftrightarrow \mathrm{ColoredOperads}$ *is a Quillen adjunction,*
- *this model structure is compatible with the Joyal model structure on* sSet*,*
- *this model structure is symmetric monoidal, and the tensor product is compatible with the Boardman–Vogt tensor product of operads, and*
- *this model structure is left proper, cofibrantly generated, and combinatorial.*

Indicating the fibrant objects and cofibrations in Theorem 3.8 is sufficient for this talk, since a model structure is completely determined by its class of fibrant objects and its class of cofibrations by Proposition E.1.10 of [6].

Problem 3.9. *Find a "dendroidal geometric realization"* $|\ |_d$ *such that*

(a)

$$\begin{tikzcd} \mathrm{sSet} \arrow[r, "i_!"] \arrow[d, "|\ |"'] & \mathrm{dSet} \arrow[dl, "|\ |_d"] \\ \mathrm{Top} & \end{tikzcd}$$

commutes,

(b) $|\ |_d$ *sends* $\otimes$ *to* $\times$*, and*

(c) $|\ |_d$ *does not lose too much information.*

4. ∞-operads (Lurie DAG III)

Advantages: These are defined purely in terms of simplicial sets, the combinatorics of trees do not play a role.

Drawback: The simplicial sets that arise are complicated.

Lurie's Application: Develop E_∞-ring spectra from a quasi-category point of view.

Idea: Given a colored operad, take its category of operators, then take its nerve, and get a simplicial set over $N\Gamma$. (Here Γ is Segal's category, namely a skeleton of the category of nonempty, finite, pointed sets.) Then an ∞-operad is something like such a simplicial set over $N\Gamma$.

Let Γ be Segal's category of nonempty, finite, pointed sets. Objects are pointed sets $\langle n \rangle = \{*, 1, 2, \ldots, n\}$, where $n \geq 0$ and maps are functions preserving $*$. Let P be a colored operad. Define its "category of operators" $P^{\otimes}$ as follows. The objects of $P^{\otimes}$ are finite sequence of colors. A morphism $(X_1, \ldots, X_m) \to (Y_1, \ldots, Y_m)$ in $P^{\otimes}$ is a morphism $f : \langle m \rangle \to \langle n \rangle$ in Γ together with operations $\varphi_j : \{X_i\}_{i \in f^{-1}(j)} \to Y_j$ for each $1 \leq j \leq n$ in P. We have a forgetful functor $P^{\otimes} \to \Gamma$.

Definition 4.1 (See Definition 1.1.12 of [7])**.** An ∞-*operad* $\mathcal{O}^{\otimes}$ is a simplicial set $\mathcal{O}^{\otimes}$ equipped with a map of simplicial sets $\mathcal{O}^{\otimes} \to N\Gamma$ satisfying properties like those of $NP^{\otimes} \to N\Gamma$ described above.

Theorem 4.2 (Proposition 1.8.4 of [7])**.** *There is a good simplicial model category whose fibrant objects are essentially ∞-operads.*

Problem 4.3. *Prove that the Cisinski-Moerdijk model category on dendroidal sets is Quillen equivalent to Lurie's model category. For some discussion of an approach, see* [13]*.*

Problem 4.4. *Develop analogous pictures for cyclic operads and prove their equivalence; do the same for modular operads as well.*

Problem 4.5. *Also do the same for globular operads, thus moving Batanin's combinatorial definition of weak n-category into the simplicial world.*

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