Higher Categories, Homotopy Theory, and Applications

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Why Homotopy Theory and Higher Categories?

- Homotopy Theory solves topological and geometric problems with tools from algebra.
  Method:

  1. Introduce algebraic invariant
  2. Calculate invariant in simple cases
  3. Rephrase geometric problem in terms of invariant

  Example: Distinguish Surfaces using the invariant π_1 (later)

Why Homotopy Theory and Higher Categories?

- Category Theory provides an extremely useful setting to make comparisons and analogies mathematically precise. (Eilenberg–Mac Lane 1945, Brown–Porter 2006).

   1. Proofs
   2. Calculations

- Category Theory is also a great place for expressing notions of universality.

Why Homotopy Theory and Higher Categories?

- Higher Category Theory combines elements of homotopy theory and category theory for results in both fields and others.

My Main Results

- Categorical Foundations of Conformal Field Theory, with Hu and Kriz

1. Captured the algebraic structure on worldsheets using 2-category theory: gluing, disjoint union, unit, requisite symmetries, coherences, and coherence diagrams
2. Rigorized Segal's definition of conformal field theory as a map of this structure
3. Factored Lattice Field Theories through Open Abelian Varieties using this notion of map

My Main Results

- Homotopy Theory of *n*-Fold Categories, with Paoli and Pronk

    1. Constructed Model Structures encoding every reasonable notion of weak equivalence in dimension 2

    2. Constructed a Model Structure on **nFoldCat** which is Quillen equivalent to **Top**

- Mathematical Music Theory, with Crans and Satyendra

Overview

1. Comparisons and Analogies: Natural Numbers and Vector Spaces
2. Analogy: Gluing Topological Spaces and Gluing Groups
3. Analogy: Gluing a Surface and Trace
4. Conformal Field Theory
5. Analogy: Homotopy Theory for Topological Spaces, Homotopy Theory for Simplicial Sets
6. Life without Elements
7. Homotopy Theory for n-fold Categories
8. Perspectives for Future Work
9. Summary

Comparisons and Analogies:
Natural Numbers and Vector Spaces

Natural Numbers : Fin. Dim. Real Vector Spaces

The Natural Numbers $= \mathbb{N} = \{0, 1, 2, \dots\}$

- Comparison is Order: $\leq$
  transitivity $1 \leq 2$ and $2 \leq 3$ $\Rightarrow 1 \leq 3$ reflexivity $m \leq m$
- Addition: 2+2=4 $0 + m = m$
- Multiplication: $2 \times 3 = 6$ $1 \times m = m$

Fin. Dim. Real Vector Spaces, e.g. $\mathbb{R}^m$, set of solutions to $f' - f = 0, \ \dots$

- Comparison is Injective Linear Map
  transitivity $U \xrightarrow{f} V \xrightarrow{g} W$ injective linear maps
  $\Rightarrow U \xrightarrow{g \circ f} W$ is injective linear map
  reflexivity $U \xrightarrow{1_U} U$
- Direct Sum: $\mathbb{R}^2 \oplus \mathbb{R}^2 \cong \mathbb{R}^4$ $\{0\} \oplus \mathbb{R}^m \cong \mathbb{R}^m$
- Tensor Product: $\mathbb{R}^2 \otimes \mathbb{R}^3 \cong \mathbb{R}^6$ $\mathbb{R}^1 \otimes \mathbb{R}^m \cong \mathbb{R}^m$

Dimension Makes Analogy Precise

$$dim : \begin{pmatrix} \text{Fin. Dim. Real Vector Spaces} \\ \text{and inj. lin. maps} \end{pmatrix} \longrightarrow (\mathbb{N}, \leq)$$

$$U \xrightarrow{f} V \qquad\qquad dim\ U \xrightarrow[\leq]{dim\ f} dim\ V$$

Transitivity preserved: If $U \xrightarrow{f} V \xrightarrow{g} W$, then

$$dim\ g \circ f = (dim\ g) \circ (dim\ f)$$

$$dim\ U \leq\ dim\ W \qquad \text{is} \qquad dim\ U \leq dim\ V \leq dim\ W$$

Reflexivity preserved:

$$dim\ (U \xrightarrow{1_U} U) \qquad \text{is} \qquad dim\ U = dim\ U$$

Categories and Functors

Definition

A category $\mathbf{L}$ *consists of a class of objects* $A, B, C, \dots$ *and a set of morphisms* $Hom(A, B)$ *for any two objects* A *and* B*, as well as a composition*

$$Hom(B, C) \times Hom(A, B) \longrightarrow Hom(A, C)$$

$$(g, f) \longmapsto g \circ f$$

which is associative and unital.
A functor $F : \mathbf{L} \longrightarrow \mathbf{M}$ *consists of assignments*

$$F : Obj\, \mathbf{L} \longrightarrow Obj\, \mathbf{M}$$

$$F_{A,B} : Hom_{\mathbf{L}}(A, B) \longrightarrow Hom_{\mathbf{M}}(FA, FB)$$

compatible with composition and units.

Categories and Functors

Example

The Natural Numbers $\mathbb{N}$ *form a category, with one morphism* $m \longrightarrow n$ *iff* $m \leq n$.

Example

Finite Dimensional Vectors Spaces and Injective Linear Maps form a category.

Example

Dimension is a functor.

$$dim : \left(\begin{array}{c} \text{Fin. Dim. Real Vector Spaces} \\ \text{and inj. lin. maps} \end{array} \right) \longrightarrow (\mathbb{N}, \leq)$$

$$U \xrightarrow{f} V \qquad\qquad dim\ U \xrightarrow[\leq]{dim\ f} dim\ V$$

Commutative Diagrams

We say that a diagram $\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array}$ in a category is

commutative if $k \circ f = g \circ j$.

Example

The diagram $\begin{array}{ccc} \mathbb{R} & \xrightarrow{+2} & \mathbb{R} \\ {\scriptstyle +3}\downarrow & & \downarrow{\scriptstyle +3} \\ \mathbb{R} & \xrightarrow[+2]{} & \mathbb{R} \end{array}$ *is commutative because for every real number* x*, we have* $x + 2 + 3 = x + 3 + 2$.

Monoidal Categories

$(\mathbb{N}, \times)$ is a monoid:

1. Associativity: $\ell \times (m \times n) = (\ell \times m) \times n$
2. Unit: $1 \times m = m = m \times 1$.

(Fin. Dim. Real Vector Spaces, $\otimes$) is a monoidal category:

1. Associativity: $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$
2. Unit: $\mathbb{R} \otimes U \cong U \cong U \otimes \mathbb{R}$
3. Coherence diagrams:

$$\begin{array}{ccccc}
((T \otimes U) \otimes V) \otimes W & \xrightarrow{\cong} & (T \otimes (U \otimes V)) \otimes W & \xrightarrow{\cong} & T \otimes ((U \otimes V) \otimes W) \\
\cong \downarrow & & & & \downarrow \cong \\
(T \otimes U) \otimes (V \otimes W) & & \xrightarrow[\cong]{\hspace{6em}} & & T \otimes (U \otimes (V \otimes W))
\end{array}$$

$$\begin{array}{ccc}
T \otimes (\mathbb{R} \otimes U) & \xrightarrow{\cong} & (T \otimes \mathbb{R}) \otimes U \\
& \cong \searrow \quad \swarrow \cong & \\
& T \otimes U &
\end{array}$$

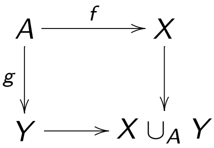
Analogy: Gluing Topological Spaces and Gluing Groups

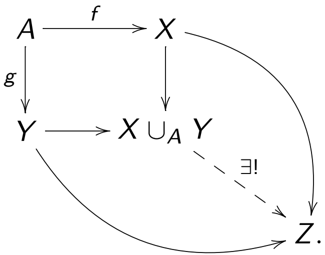
Gluing Topological Spaces

Suppose A, X, Y are topological spaces, and $f : A \longrightarrow X$, $g : A \longrightarrow Y$ are continuous maps. Then the space $X \cup_A Y$ is *X glued to Y along A*

$$X \cup_A Y = (X \coprod Y) \quad / \quad (f(a) \sim g(a)).$$

This is the *pushout*



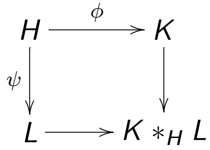


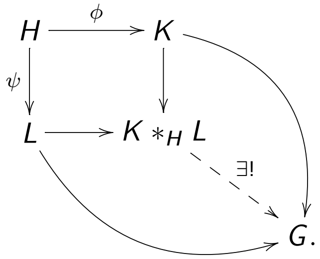
Gluing Groups

Suppose H, K, L are groups, and $\phi : H \longrightarrow K$, $\psi : H \longrightarrow L$ are group homomorphisms. Then the group $K *_H L$ is K *glued to* L *along* H

$$K *_H L = (K * L) \quad / \quad (\phi(h)\psi(h)^{-1} = e).$$

This is the *pushout*





The First Homotopy Group $\pi_1(X)$

Let X be a topological space with basepoint $* \in X$. A based loop in X is a continuous map $f : S^1 \longrightarrow X$ such that $f(1) = *$.

Two based loops f and g are homotopic if one can be continuously deformed into the other. The set of all based loops homotopic to f is called the homotopy class of f.

The first homotopy group $\pi_1(X)$ is the set of homotopy classes of based loops in X with concatenation as the group operation.

π_1 is a functor.

Examples: $\pi_1(S^2) = \{e\}$ $\pi_1(S^1) = \mathbb{Z}$

Seifert-Van Kampen Theorem

Theorem

Suppose X is a connected space, U, V, and $U \cap V$ are path-connected open subspaces of X, and $X = U \cup V$. Choose a basepoint in $U \cap V$. Then

$$\pi_1(X) \cong \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V).$$

Example

The fundamental group of a compact oriented genus g surface is

$$\langle a_1, b_1, \ldots, a_g, b_g | [a_1, b_1] \cdots [a_g, b_g] \rangle.$$

Topological problem solved: we can distinguish compact oriented surfaces with different genera.

Next Topic

Cancellation: Gluing a Surface is like Trace

Cancellation

$$
\begin{aligned}
trace : V^* \otimes V &\longrightarrow \mathbb{R} \\
\phi \otimes v &\mapsto \phi(v)
\end{aligned}
$$

$$
\begin{array}{ccc}
V^{*\otimes\{1,2\}} \otimes V^{\otimes\{1,3\}} & \longrightarrow & V^{*\otimes\{2\}} \otimes V^{\otimes\{3\}} \\
\| & & \| \\
V_1^* \otimes V_2^* \otimes V_1 \otimes V_3 & \longrightarrow & V_2^* \otimes V_3 \\
\phi_1 \otimes \phi_2 \otimes v_1 \otimes v_3 & \longmapsto & \phi_1(v_1) \cdot \phi_2 \otimes v_3
\end{array}
$$

Similarly, a surface with inbounds labelled by 1,2 and outbounds labelled by 1,3 glues to give a new surface with inbound 2 and outbound 3.

Next Topic

Conformal Field Theory

Why Conformal Field Theory?

Mathematicians are interested in CFT because of its relationship to

- the representation theory of $\mathit{Diff}^+(S^1)$
- the representation theory of loop groups $\mathit{Maps}(S^1, G)$
- a geometric definition of elliptic cohomology.

Worldsheets

A *worldsheet* x is a compact 2-dimensional real manifold with boundary equipped with

1. a complex structure
2. a real analytic parametrization $f_k : S^1 \longrightarrow k$ of each boundary component k.

Example

Any annulus with real analytically parametrized boundary components.

Pseudo Algebraic Structure of Worldsheets using 2-Theories

The worldsheets form a pseudo commutative monoid with cancellation.

$$I := \text{ category of finite sets and bijections}$$

For finite sets, a, b let $X_{a,b}$ denote the category of worldsheets equipped with bijections

$$a \leftrightarrow \text{ set of inbound components}$$

$$b \leftrightarrow \text{ set of outbound components.}$$

$$X : I^2 \longrightarrow Cat$$

$$(a, b) \longmapsto X_{a,b}$$

Pseudo Algebraic Structure of Worldsheets using 2-Theories

This 2-functor $X : I^2 \to Cat$ has the structure of a pseudo commutative monoid with cancellation:

- $(I, \coprod)$ is "like" a commutative monoid
- There are operations of disjoint union, gluing (cancellation), and unit

$$+_{a,b,c,d} : X_{a,b} \times X_{c,d} \to X_{a+c,b+d}$$

$$\check{?}_{a,b,c} : X_{a+c,b+c} \to X_{a,b}$$

$$0 \in X_{0,0}.$$

- These operations satisfy certain axioms up to coherence isomorphisms and these coherence isomorphisms satisfy coherence diagrams, all determined by the 2-theory formalism.

A *conformal field theory is a morphism of such structure.*

Theorem about Pseudo Algebras

Theorem (Fiore)

Let T be a Lawvere theory. The 2-category of pseudo T-algebras admits weighted pseudo limits and weighted bicolimits.

Next Topic

Homotopy Theory for Topological Spaces and Homotopy Theory for Simplicial Sets

Simplicial Sets

Let Δ be the category of nonempty finite ordinals and order preserving maps. For example, $[n] = \{0, 1, 2, \ldots, n\}$ is an object.

Definition

A ***simplicial set*** *is a functor* $\Delta^{op} \longrightarrow \mathbf{Set}$.

Example

Any simplicial complex where the vertices are linearly ordered.

Adjunction:

$$\mathbf{SSet} \underset{\mathrm{Sing}_\bullet}{\overset{|\text{-}|}{\rightleftarrows}} \mathbf{Top} \quad (\perp)$$

Homotopy Theory for **Top** and **SSet**

Weak Equivalences:

- In **Top**, f is a weak equivalence iff $\pi_n(f)$ is an isomorphism for all $n \geq 0$.
- In **SSet**, f is a weak equivalence iff $|f|$ is so.

Fibrations:

- In **Top**, f is a fibration iff in any commutative diagram of the form
$$\begin{array}{ccc} \partial D^n & \longrightarrow & A \\ \Big\downarrow & \overset{h}{\nearrow} & \Big\downarrow f \\ D^n & \longrightarrow & B \end{array}$$
, a lift h exists.
- In **SSet**, f is a fibration iff in any commutative diagram of the form
$$\begin{array}{ccc} \partial \Delta[n] & \longrightarrow & X \\ \Big\downarrow & \overset{h}{\nearrow} & \Big\downarrow f \\ \Delta[n] & \longrightarrow & Y \end{array}$$
, a lift h exists.

Analogy between **Top** and **SSet** Made Precise

Theorem (Quillen, Milnor)

The adjunction

$$\mathbf{SSet} \underset{Sing_\bullet}{\overset{|\cdot|}{\rightleftarrows}} \mathbf{Top} \qquad (\perp)$$

is a Quillen equivalence.

Later we will see a similar theorem for n-fold categories.

Life without Elements

Life without Elements

The diagrammatic approach of category theory allows us to define similar notions and do similar proofs in different contexts. For example, the Short Five Lemma in Abelian Categories.

Life without Elements

Example

An *internal category* $(\mathbb{D}_0, \mathbb{D}_1)$ *in* **Cat**, *or* *double category*, *consists of categories* $\mathbb{D}_0$ *and* $\mathbb{D}_1$ *and functors*

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\ \circ\ } \mathbb{D}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{D}_0, \qquad u : \mathbb{D}_0 \to \mathbb{D}_1$$

that satisfy the usual axioms of a category.

This definition can be iterated to obtain the notion of *n-fold category*

Example

Rings, ring homomorphisms, bimodules, and twisted bimodule homomorphisms organize themselves into a (weak) double category.

Next Topic

Homotopy Theory for *n*-fold Categories

Homotopy Theory for n-fold Categories

Theorem (Fiore–Paoli)

There is a cofibrantly generated model structure on **nFoldCat** *such that*

- F *is a weak equivalence if and only if* $Ex^2\delta^* NF$ *is so.*
- F *is a fibration if and only if* $Ex^2\delta^* NF$ *is so.*

Further, the adjunction

$$\mathbf{SSet} \underset{\mathrm{Ex}^2}{\overset{\mathrm{Sd}^2}{\rightleftarrows}} \mathbf{SSet} \underset{\delta^*}{\overset{\delta_!}{\rightleftarrows}} \mathbf{SSet^n} \underset{N}{\overset{c}{\rightleftarrows}} \mathbf{nFoldCat}$$

(each adjunction marked $\perp$)

is a Quillen equivalence.

Next Topic

Perspectives for Future Work

Future Projects

- *n*-fold analogues for Joyal's Θ
- Thomason-type structure for *n*-categories using Berger's cellular nerve
- Euler characteristics for categories (with Sauer, Lück)
- Mathematical Music Theory (with Satyendra, Chung)

Next Topic

Summary

Summary

- Homotopy theory solves geometric problems with algebraic tools. (e.g. π_1)
- Category Theory makes comparisons and analogies mathematically precise.
  - Natural numbers are like vector spaces
    $\rightsquigarrow$ monoidal categories.
  - Gluing spaces is like gluing groups
    $\rightsquigarrow$ Seifert-Van Kampen Theorem.
  - Gluing a surface is like trace
    $\rightsquigarrow$ conformal field theory.
  - Topological spaces are like simplicial sets
    $\rightsquigarrow$ homotopy theory of *n*-fold categories.
- We have developed the categorical foundations of conformal field theory.
- We have developed the homotopy theory of higher categories.