Euler Characteristics of Categories and Homotopy Colimits

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Outline

I. Introduction.

Introduction

The most basic invariant of a finite CW-complex is the Euler characteristic.

$$\chi\colon \text{finite } CW\text{-complexes} \longrightarrow \mathbb{R}$$

Remarkable connections to geometry:

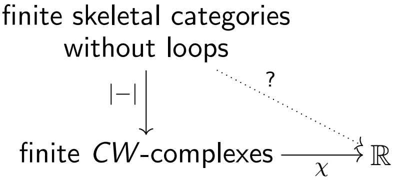
- $\chi(\text{compact connected orientable surface}) = 2 - 2 \cdot \text{genus}$,
- Theorem of Gauss-Bonnet

$$\chi(M) = \frac{1}{2\pi} \int_M \text{curvature dA}$$

for M any compact 2-dimensional Riemannian manifold.

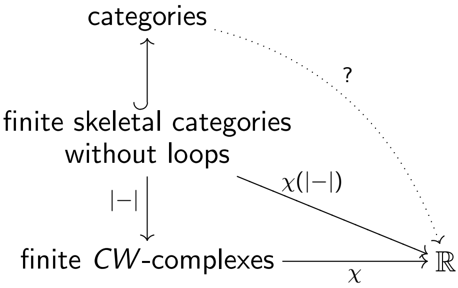
Introduction

Problem: meaningfully define χ purely in terms of the combinatorial models



Introduction

More generally:



Trivial Example Presents Challenges

$\Gamma = \widehat{\mathbb{Z}_2}$, that is, Γ has one object $*$ and $\mathrm{mor}_\Gamma(*,*) = \mathbb{Z}_2$.
$|\widehat{\mathbb{Z}_2}|$ = geometric realization of nerve of $\widehat{\mathbb{Z}_2}$

$$\begin{aligned}
\text{0-cells of } |\widehat{\mathbb{Z}_2}| &= \mathrm{ob}(\widehat{\mathbb{Z}_2}) = \{*\} \\
\text{1-cells of } |\widehat{\mathbb{Z}_2}| &= \text{non-identity maps} = \{* \to *\} \\
\text{2-cells of } |\widehat{\mathbb{Z}_2}| &= \text{paths of 2 non-id maps} = \{* \to * \to *\} \\
\text{etc.} &= \text{etc.} \\
\chi(|\widehat{\mathbb{Z}_2}|) &= \sum_{n \geq 0} (-1)^n \mathrm{card}(n\text{-cells of } |\widehat{\mathbb{Z}_2}|) \\
&= \sum_{n \geq 0} (-1)^n \overset{Leinster-Berger}{=\!=\!=\!=\!=} \frac{1}{1-(-1)} = \frac{1}{2}.
\end{aligned}$$

Desiderata for Invariants

Desiderata for $\chi, \chi^{(2)}$: categories $\to \mathbb{R}$

1. Geometric relevance
2. Compatibility with
   - equivalence of categories
   - coverings of groupoids: if $p\colon \mathcal{E} \to \mathcal{B}$, then $\chi^{(2)}(\mathcal{E}) = n \cdot \chi^{(2)}(\mathcal{B})$
   - isofibrations: if $f\colon \mathcal{E} \to \mathcal{B}$, then $\chi^{(2)}(\mathcal{E}) = \chi^{(2)}(f^{-1}(b_0)) \cdot \chi^{(2)}(\mathcal{B})$
   - finite products
   - finite coproducts
   - "pushouts" (Inclusion-Exclusion Principle) $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$
   - homotopy colimits.

Our work achieves this.

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Pushouts in **Cat** and χ

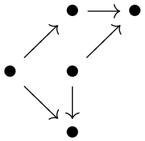
$$\begin{array}{ccc} \{0,1\} & \longrightarrow & \{0 \to 1\} \\ \downarrow & \text{pushout} & \downarrow \\ \{*\} & \longrightarrow & \widehat{\mathbb{N}} \end{array} \qquad \chi(|\widehat{\mathbb{N}}|) = \chi(S^1) = 0 = 1 + 1 - 2 \;✓$$

$$\begin{array}{ccc} \{0,1\} & \longrightarrow & \{*'\} \\ \downarrow & \text{pushout} & \downarrow \\ \{*\} & \longrightarrow & \{*\} \end{array} \qquad \chi(\{*\}) = 1 \neq 1 + 1 - 2 \;✗$$

Colimits are not homotopy invariant, cannot expect compatibility of χ with pushouts.

Homotopy Pushouts in **Cat** and χ

$$\begin{array}{ccc} \{0,1\} & \longrightarrow & \{0 \to 1\} \\ \downarrow & & \\ \{*\} & & \end{array}$$

Homotopy p.o. is 

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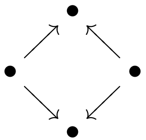
Homotopy p.o. is 

In both cases, $\chi = 0 = 1 + 1 - 2$. ✔

Main Theorem of this Talk

Theorem (Fiore-Lück-Sauer)

Let $\mathcal{C}\colon \mathcal{I} \to \mathbf{Cat}$ be a pseudo functor such that

- *$\mathcal{I}$ is directly finite: $ab = \mathrm{id} \Rightarrow ba = \mathrm{id}$;*
- *$\mathcal{I}$ admits a finite $\mathcal{I}$-CW-model, $\Lambda_n :=$ the finite set of n-cells $\lambda = \mathrm{mor}(?, i_\lambda) \times D^n$;*
- *each $\mathcal{C}(i)$ is of type (FP_R).*

Then:

$$\chi(\mathrm{hocolim}_{\mathcal{I}}\, \mathcal{C}; R) = \sum_{n \geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} \chi(\mathcal{C}(i_\lambda); R).$$

Similar formulas hold for the L^2-Euler characteristic, the functorial characteristics, and the finiteness obstruction.

II. Finiteness Obstructions and Euler Characteristics for Categories.

Modules and the Projective Class Group

R= an associative commutative ring with 1
Γ= a small category

An $R\Gamma$-*module* is a functor $M \colon \Gamma^{\mathrm{op}} \to R\text{-}\mathrm{MOD}$.

$K_0(R\Gamma)$:=*projective class group* =

$$\mathbb{Z}\{\text{iso classes of finitely generated projective } R\Gamma\text{-modules}\}$$

modulo the relation $[P_0] - [P_1] + [P_2] = 0$ for every exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective $R\Gamma$-modules.

Type (FP) and Finiteness Obstruction

Γ is *of type* (FP_R) if there is a finite projective $R\Gamma$-resolution $P_* \to \underline{R}$. In this case, the finiteness obstruction is

$$o(\Gamma; R) := \sum_{n \geq 0} (-1)^n \cdot [P_n] \in K_0(R\Gamma).$$

Remark

Suppose G is a finitely presented group of type $(FP_{\mathbb{Z}})$. Then $o(\widehat{G}; \mathbb{Z}) = o^{Wall}(BG; \mathbb{Z})$.

Examples of Type (FP)

Example

Suppose Γ is a finite category in which every endo is an iso, that is, Γ is an EI-category. If $|\mathrm{aut}_\Gamma(x)| \in R^\times$ for all $x \in \mathrm{ob}(\Gamma)$, then Γ is of type (FP_R).

Thus finite groupoids, finite posets, finite transport groupoids, and orbit categories of finite groups are all of type ($FP_\mathbb{Q}$).

Splitting Theorem of Lück

Theorem

If Γ is an EI-category, then

$$K_0(R\Gamma) \xrightarrow{S} \bigoplus_{\overline{x} \in \mathrm{iso}(\Gamma)} K_0(R\,\mathrm{aut}_\Gamma(x))$$

is an isomorphism, where $S_x(M)$ is the quotient of the R-module $M(x)$ by the R-submodule generated by all images of $M(u)$ for all non-invertible morphisms $u\colon x \to y$ in Γ.

Euler Characteristic

Definition

Suppose that Γ is of type (FP_R) and $P_* \to \underline{R}$ is a finite projective $R\Gamma$-resolution. The Euler characteristic of Γ with coefficients in R is

$$\chi(\Gamma; R) := \sum_{\overline{x} \in \operatorname{iso}(\Gamma)} \sum_{n \geq 0} (-1)^n \operatorname{rk}_R \left(S_x P_n \otimes_{R \operatorname{aut}_\Gamma(x)} R \right).$$

Example

$\mathcal{G}$ *finite groupoid* $\Rightarrow \chi(\mathcal{G}; \mathbb{Q}) = |\operatorname{iso}(\mathcal{G})|$.

L^2-Euler Characteristic

Definition

Suppose that Γ is of type (L^2) and $P_* \to \underline{\mathbb{C}}$ is a (not necessarily finite) projective $\mathbb{C}\Gamma$-resolution. The L^2-Euler characteristic of Γ is

$$\chi^{(2)}(\Gamma) := \sum_{\overline{x} \in \mathrm{iso}(\Gamma)} \sum_{n \geq 0} (-1)^n \dim_{\mathcal{N}(x)} H_n(S_x P_* \otimes_{\mathbb{C}\,\mathrm{aut}_\Gamma(x)} \mathcal{N}(x))$$

where $\mathcal{N}(x) = \mathcal{B}\big(l^2(\mathrm{aut}_\Gamma(x))\big)^{\mathrm{aut}_\Gamma(x)}$ is the group von Neumann algebra of $\mathrm{aut}_\Gamma(x)$.

Example of L^2-Euler Characteristic

Example

Let $\mathcal{G}$ *be a groupoid such that* $|\operatorname{aut}_{\mathcal{G}}(x)| < \infty$ *and* $\sum_{\overline{x} \in \operatorname{iso}(\mathcal{G})} \frac{1}{|\operatorname{aut}_{\mathcal{G}}(x)|} < \infty$.

Then $\chi^{(2)}(\mathcal{G}) = \sum_{\overline{x} \in \operatorname{iso}(\mathcal{G})} \frac{1}{|\operatorname{aut}_{\mathcal{G}}(x)|}$. *(Same as Baez-Dolan, and Leinster-Berger in finite case.)*

Comparison with Topology

Theorem

If Γ is a directly finite category of type $(FF_{\mathbb{C}})$, then

$$\chi(\Gamma;\mathbb{C}) = \chi^{(2)}(\Gamma) = \chi(B\Gamma;\mathbb{C}).$$

Example

If Γ is a finite skeletal category without loops, then it is of type $(FF_{\mathbb{C}})$, and all three invariants are equal to

$$\sum_{n\geq 0}(-1)^n c_n(\Gamma)$$

where c_n is the number of nondegenerate paths $i_0 \to i_1 \to i_2 \to \cdots \to i_n$ of n-many morphisms in Γ.

III. Classifying $\mathcal{I}$-Spaces.

$\mathcal{I}$-Spaces

$\mathcal{I}$ = a small category

An $\mathcal{I}$*-space* is a functor $X\colon \mathcal{I}^{\mathrm{op}} \to \mathrm{SPACES}$.

Example

1. $\mathrm{mor}_{\mathcal{I}}(-, i)$
2. $\mathrm{mor}_{\mathcal{I}}(-, i) \times S^{n-1}$
3. $\mathrm{mor}_{\mathcal{I}}(-, i) \times D^n$
4. *Pushouts of these*

$\mathcal{I}$-CW-complexes

An *$\mathcal{I}$-CW-complex* X is an $\mathcal{I}$-space X together with a filtration $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_n \subset \ldots \subset X = \bigcup_{n \geq 0} X_n$ such that $X = \operatorname{colim}_{n \to \infty} X_n$ and for any $n \geq 0$ the *n-skeleton* X_n is obtained from the $(n-1)$-skeleton X_{n-1} by *attaching $\mathcal{I}$-n-cells*, i.e., there exists a pushout of $\mathcal{I}$-spaces of the form

$$\begin{array}{ccc} \coprod_{\lambda \in \Lambda_n} \operatorname{mor}_{\mathcal{I}}(-, i_\lambda) \times S^{n-1} & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{\lambda \in \Lambda_n} \operatorname{mor}_{\mathcal{I}}(-, i_\lambda) \times D^n & \longrightarrow & X_n \end{array}$$

where the vertical maps are inclusions, Λ_n is an index set, and the i_λ's are objects of $\mathcal{I}$. In particular, $X_0 = \coprod_{\lambda \in \Lambda_0} \operatorname{mor}_{\mathcal{I}}(-, i_\lambda)$.

Classifying $\mathcal{I}$-Spaces

Definition

A *finite model* *for the $\mathcal{I}$-classifying space is a finite $\mathcal{I}$-CW-complex X such that $X(i)$ is contractible for each object i of $\mathcal{I}$.*

Example

$\mathcal{I} = \{k \leftarrow j \rightarrow \ell\}$ *admits a finite model*

$X_0 := \mathrm{mor}_{\mathcal{I}}(?, k) \coprod \mathrm{mor}_{\mathcal{I}}(?, \ell)$

$$\begin{array}{ccc} \mathrm{mor}_{\mathcal{I}}(-, j) \times S^0 & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ \mathrm{mor}_{\mathcal{I}}(-, j) \times D^1 & \longrightarrow & X_1 \end{array}$$

Then $X(k) = *$, $X(\ell) = *$, $X(j) = D^1 \simeq *$.

IV. Homotopy Colimit Formula and the Inclusion-Exclusion Principle.

Homotopy Colimits in **Cat**

Thomason: In **Cat**, a homotopy colimit of $\mathcal{C}\colon \mathcal{I} \to$ **Cat** is given by the Grothendieck construction.

The category $\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}$ has objects pairs (i, c), where $i \in \operatorname{ob}(\mathcal{I})$ and $c \in \operatorname{ob}\big(\mathcal{C}(i)\big)$.
A morphism from (i, c) to (j, d) is a pair (u, f), where $u\colon i \to j$ is a morphism in $\mathcal{I}$ and $f\colon \mathcal{C}(u)(c) \to d$ is a morphism in $\mathcal{C}(j)$.

Example

1. $\mathcal{C}\colon \widehat{G} \to$ **Cat** *has homotopy colimit = homotopy orbit of G-action on $\mathcal{C}(*)$.*
2. *If $\mathcal{C}(*)$ is a set, then this gives the transport groupoid of the left G-action.*

Homotopy Colimit Formula and Incl.-Excl. Principle

Theorem (Fiore-Lück-Sauer)

*$\mathcal{C}\colon \mathcal{I} \to$ **Cat** a pseudo functor, $\mathcal{I}$ directly finite with a finite $\mathcal{I}$-CW-model, Λ_n= the finite set of $\mathcal{I}$-n-cells $\lambda = \operatorname{mor}(?, i_\lambda) \times D^n$, each $F(i)$ of type (FP_R), then*

$$\chi(\operatorname{hocolim}_{\mathcal{I}} \mathcal{C}; R) = \sum_{n \geq 0} (-1)^n \cdot \sum_{\lambda \in \Lambda_n} \chi(\mathcal{C}(i_\lambda); R).$$

Example

$\mathcal{I} = \{k \leftarrow j \rightarrow \ell\}$ admits a finite model, $\Lambda_0 = \{k, \ell\}$ and $\Lambda_1 = \{j\}$
Theorem $\Rightarrow$
χ(homotopy pushout of $\mathcal{C}$) $= \chi(\mathcal{C}(k)) + \chi(\mathcal{C}(\ell)) - \chi(\mathcal{C}(j))$.

V. Comparison with Leinster's Notions.

Comparison with Leinster's Weightings

Γ=a finite category

A *weighting* on Γ is a function $k^\bullet \colon \mathrm{ob}(\Gamma) \to \mathbb{Q}$ such that for all objects $x \in \mathrm{ob}(\Gamma)$, we have $\sum_{y \in \mathrm{ob}(\Gamma)} |\mathrm{mor}(x,y)| \cdot k^y = 1$.

Theorem (Fiore-Lück-Sauer)

$\mathcal{I}$ a finite category, X a finite model, then the function $k^\bullet \colon \mathrm{ob}(\mathcal{I}) \to \mathbb{Q}$ defined by

$$k^y := \sum_{n \geq 0} (-1)^n (\textit{number of n-cells of } X \textit{ based at } y)$$

is a weighting on $\mathcal{I}$. More generally, finite free $R\Gamma$-resolutions of $\underline{R}$ produce weightings.

Comparison with Leinster's Euler Characteristics

Definition (Leinster)

A finite category Γ has an Euler characteristic in the sense of Leinster if it admits both a weighting $k^\bullet$ and a coweighting $k_\bullet$. In this case, its Euler characteristic in the sense of Leinster is defined as

$$\chi_L(\Gamma) := \sum_{y \in \mathrm{ob}(\Gamma)} k^y = \sum_{x \in \mathrm{ob}(\Gamma)} k_x.$$

This agrees with $\chi^{(2)}$ when Γ is finite, EI, skeletal, and the left $\mathrm{aut}_\Gamma(y)$-action on $\mathrm{mor}_\Gamma(x,y)$ is free for every two objects $x, y \in \mathrm{ob}(\Gamma)$. Proof: K-theoretic Möbius inversion.

VI. Applications and Summary.

Applications

1. Let G be a group which admits a finite G-CW-model Y for the classifying space for proper G-actions. The equivariant Euler characteristic of Y is the functorial (L^2) Euler characteristic of the proper orbit category.

2. Developability of Haefliger complexes of groups:

$$\chi^{(2)}(\operatorname{hocolim}_{\mathcal{X}/G} F) = \frac{\chi^{(2)}(\mathcal{X})}{|G|} = \frac{\chi(\mathcal{X};\mathbb{C})}{|G|} = \frac{\chi(B\mathcal{X};\mathbb{C})}{|G|}.$$

Summary

- We have introduced notions of finiteness obstruction, Euler characteristic, and L^2-Euler characteristic for wide classes of categories, including certain infinite ones.
- Origins lie in the homological algebra of modules over categories and modules over group von Neumann algebras.
- These notions are compatible with: equivalences of categories, coverings, fibrations, finite products, finite coproducts, homotopy colimits.
- In the case of groups, the L^2-Euler characteristic agrees with the classical L^2-Euler characteristic of groups.
- The notions are geometric: agree with $\chi(B\Gamma)$ or equivariant Euler characteristic in certain cases.
- The notions are combinatorial: have K-theoretic Möbius inversion.