Math 254 Abstract Algebra
Homework 5, Due November 5th
Special Feature: The neo-Riemannian Group
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Read Sections 13, 14, and 15.
Do the problems below.
Section 9: 13, 10, 11, 12, 20, 21, 23, 39
Section 10: 1, 4, 12, 16, 19, 25, 34, 40
Section 11: 2
The problems 1, 2, 3, 4, 5 below about the neo-Riemannian Group.

The Structure of the neo-Riemmanian Group

Group theory has numerous applications: cryptography, algebraic coding theory, physics, number theory, topology, just to mention a few. In this course, we will consider a surprising application of group theory to music theory. Group theory provides an extremely useful language for capturing, describing, and discovering musical phenomena. We are not talking about the physics of sound production here, but instead, actual musical phenomena that occur in a work of music, like Beethoven's Ninth Symphony. In class we have already talked about one musical group: the dihedral group of order 24, denoted D_{12}. This is the group of symmetries of the regular 12-gon, and it consists of transpositions T_m and inversions I_m for $m \in \mathbb{Z}_{12}$, which transpose and invert melodies. The exercises below introduce a different musical group called the *neo-Riemannian group*, or *PLR-group*. It is a certain group of permutations on the finite set of major and minor triads. In these exercises we study its algebraic structure. Later, we'll turn to musical applications.

Don't worry if you don't read music, we'll only be using math. Here is a short description for those that do know a bit of music. Recall that we put an equivalence relation on the keys of the piano: two keys (or pitches) are equivalent if and only if they differ by a whole number of octaves. An equivalence class then consists of all keys with the same letter name. There are twelve equivalence classes. We use the following bijection as our dictionary between pitch classes and $\mathbb{Z}_{12}$.

A	$A\sharp$ $B\flat$	B	C	$C\sharp$ $D\flat$	D	$D\sharp$ $E\flat$	E	F	$F\sharp$ $G\flat$	G	$G\sharp$ $A\flat$	A
9	10	11	0	1	2	3	4	5	6	7	8	9

From now on, we *arithmetic is done modulo 12.* A *major triad* is a 3-tuple of the form $\langle m, m+4, m+7\rangle$ where $m \in \mathbb{Z}_{12}$. A *minor triad* is a 3-tuple of the form $\langle m, m+8, m+5\rangle$ where $m \in \mathbb{Z}_{12}$. (We choose this ordering just to make things easier later.) The set of all major and minor triads is listed in Figure 1. The letter next to the chord is the name of the chord, and is not a pitch class.

Let S denote the set of major and minor triads. It has 24 elements, all listed in Figure 1. Let $Sym(S)$ denote the set of permutations of S. We may consider T_m and I_m as functions $S \to S$ by letting them act componentwise, that is,

$$T_m\langle y_1, y_2, y_3\rangle = \langle T_m y_1, T_m y_2, T_m y_3\rangle$$

$$I_m\langle y_1, y_2, y_3\rangle = \langle I_m y_1, I_m y_2, I_m y_3\rangle.$$

Major Triads	Minor Triads
$C = \langle 0,4,7\rangle$	$\langle 0,8,5\rangle = f$
$C\sharp = D\flat = \langle 1,5,8\rangle$	$\langle 1,9,6\rangle = f\sharp = g\flat$
$D = \langle 2,6,9\rangle$	$\langle 2,10,7\rangle = g$
$D\sharp = E\flat = \langle 3,7,10\rangle$	$\langle 3,11,8\rangle = g\sharp = a\flat$
$E = \langle 4,8,11\rangle$	$\langle 4,0,9\rangle = a$
$F = \langle 5,9,0\rangle$	$\langle 5,1,10\rangle = a\sharp = b\flat$
$F\sharp = G\flat = \langle 6,10,1\rangle$	$\langle 6,2,11\rangle = b$
$G = \langle 7,11,2\rangle$	$\langle 7,3,0\rangle = c$
$G\sharp = A\flat = \langle 8,0,3\rangle$	$\langle 8,4,1\rangle = c\sharp = d\flat$
$A = \langle 9,1,4\rangle$	$\langle 9,5,2\rangle = d$
$A\sharp = B\flat = \langle 10,2,5\rangle$	$\langle 10,6,3\rangle = d\sharp = e\flat$
$B = \langle 11,3,6\rangle$	$\langle 11,7,4\rangle = e$

FIGURE 1. The set S of all major and minor triads.

Consider the three functions $P, L, R : S \to S$ defined by

$$P\langle y_1, y_2, y_3\rangle = I_{y_1+y_3}\langle y_1, y_2, y_3\rangle$$
$$L\langle y_1, y_2, y_3\rangle = I_{y_2+y_3}\langle y_1, y_2, y_3\rangle$$
$$R\langle y_1, y_2, y_3\rangle = I_{y_1+y_2}\langle y_1, y_2, y_3\rangle.$$

These are called *parallel, leading tone exchange,* and *relative.*

1. Calculate $P\langle 1,5,8\rangle$, $L\langle 10,6,3\rangle$, and $R\langle 9,1,4\rangle$.

If we consider major and minor as a *parity*, then there is a particularly nice verbal description of P, L, and R. The function P takes a triad to that unique triad of opposite parity which has the first component and third component switched. Thus, as unordered sets, the input and output triads overlap in two notes. For example, $P\langle 0,4,7\rangle = \langle 7,3,0\rangle$ and $P\langle 7,3,0\rangle = \langle 0,4,7\rangle$. A musician will notice that P applied to C is c, while P applied to c is C. In general, P takes a major triad to its parallel minor and a minor triad to its parallel major. A major triad and a minor triad are said to be *parallel* if they have the same letter name but are of opposite parity.

The other two functions, L and R, similarly have maximally overlapping inputs and outputs and are involutions. The function L takes a triad to that unique triad of opposite parity which has the second component and third component switched; for example $L\langle 0,4,7\rangle = \langle 11,7,4\rangle$ and $L\langle 11,7,4\rangle = \langle 0,4,7\rangle$. The function R takes a triad to that unique triad of opposite parity which has the first component and second component switched; for example $R\langle 0,4,7\rangle = \langle 4,0,9\rangle$ and $R\langle 4,0,9\rangle = \langle 0,4,7\rangle$. A musician will notice that R applied to C is a and R applied to a is C. In general, R takes a major triad to its relative minor and a minor triad to its relative major. A major triad and a minor triad are said to be *relative* if the root of the minor triad is three semitones below the root of major triad.

Check your answers to problem 1 using this description.

2. Prove that $P^2 = Id_S$, $L^2 = Id_S$, and $R^2 = Id_S$ using the formulas for P, L, and R. And also using the verbal description above.

Let $Q_i : S \to S$ denote the function defined by

$$Q_i\langle y_1, y_2, y_3\rangle := \begin{cases} \langle T_i(y_1), T_i(y_2), T_i(y_3)\rangle \text{ if } \langle y_1, y_2, y_3\rangle \text{ is major} \\ \langle T_{-i}(y_1), T_{-i}(y_2), T_{-i}(y_3)\rangle \text{ if } \langle y_1, y_2, y_3\rangle \text{ is minor} \end{cases}$$

In other words, Q_i is the transformation which transposes major triads "up" by the interval i and Q_i transposes minor triads "down" by the interval i.

3. Is Q_0 the identity? Show $Q_iQ_j = Q_{i+j}$. What is the inverse of Q_i?

4. Prove that $Q_5 = LR$. Hint: first compute $LR\langle y_1, y_2, y_3\rangle$ using the formulas above for L and R, then use the resulting formula to compute $LR\langle m, m+4, m+7\rangle$ and $LR\langle m, m+8, m+5\rangle$.

5. What is the order of Q_5? Give a proof.

The subgroup of $Sym(S)$ generated by P, L, and R is called the *neo-Riemannian group*, or *PLR-group*. We will soon see that the PLR-group is isomorphic to D_{12}. The exercises above are part of the proof. The rest we'll do next week.