Overview

We describe two musical actions of the dihedral group of order 24 on the major and minor triads.

1. Through transpositions and inversions.

2. Through the neo-Riemannian PLR-group.

These two actions are *dual*. 
Our Focus

Mathematical tools in neo-Riemannian Music Theory:

- The $\mathbb{Z}_{12}$ Model of Pitch Class
- Transposition and Inversion
- The Dihedral Group
- Centralizers
- The neo-Riemannian PLR-group
- Its associated graphs.
This talk summarizes:


Available at http://www-personal.umd.umich.edu/~tmfiore/ or your library.
I. Translating Music to Algebra
We have a bijection between the set of pitch classes and $\mathbb{Z}_{12}$.

**Figure:** The musical clock.
Transposition

The bijective function

\[ T_n : \mathbb{Z}_{12} \rightarrow \mathbb{Z}_{12} \]

\[ T_n(x) := x + n \]

is called \textit{transposition} by musicians, or \textit{translation} by mathematicians.

\[ T_{-4}(7) = 7 - 4 = 3 \]

\[ T_{-3}(5) = 5 - 3 = 2 \]
Inversion

The bijective function

\[ I_n : \mathbb{Z}_{12} \rightarrow \mathbb{Z}_{12} \]

\[ I_n(x) := -x + n \]

is called *inversion* by musicians, or *reflection* by mathematicians.

\[ I_0(0) = -0 = 0 \]

\[ I_0(7) = -7 = 5 \]

<table>
<thead>
<tr>
<th></th>
<th>\langle C, G \rangle</th>
<th>\langle C, F \rangle</th>
</tr>
</thead>
<tbody>
<tr>
<td>\langle 0, 7 \rangle</td>
<td>\langle 0, 7 \rangle</td>
<td>\langle I_0(0), I_0(7) \rangle</td>
</tr>
</tbody>
</table>
Altogether, these transpositions and inversions form the \( T/I \)-group.

This is the group of symmetries of the 12-gon, the dihedral group of order 24.

**Figure:** The musical clock.
Major and minor triads are very common in Western music.

\[C\text{-major} = \langle C, E, G \rangle\]
\[= \langle 0, 4, 7 \rangle\]

\[c\text{-minor} = \langle G, E^\flat, C \rangle\]
\[= \langle 7, 3, 0 \rangle\]

The set \( S \) of consonant triads

<table>
<thead>
<tr>
<th>Major Triads</th>
<th>Minor Triads</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C = \langle 0, 4, 7 \rangle )</td>
<td>( 0, 8, 5 ) = ( f )</td>
</tr>
<tr>
<td>( C^# = D_b = \langle 1, 5, 8 \rangle )</td>
<td>( 1, 9, 6 ) = ( f^# = g^\flat )</td>
</tr>
<tr>
<td>( D = \langle 2, 6, 9 \rangle )</td>
<td>( 2, 10, 7 ) = ( g )</td>
</tr>
<tr>
<td>( D^# = E_b = \langle 3, 7, 10 \rangle )</td>
<td>( 3, 11, 8 ) = ( g^# = a^\flat )</td>
</tr>
<tr>
<td>( E = \langle 4, 8, 11 \rangle )</td>
<td>( 4, 0, 9 ) = ( a )</td>
</tr>
<tr>
<td>( F = \langle 5, 9, 0 \rangle )</td>
<td>( 5, 1, 10 ) = ( a^# = b^\flat )</td>
</tr>
<tr>
<td>( F^# = G_b = \langle 6, 10, 1 \rangle )</td>
<td>( 6, 2, 11 ) = ( b )</td>
</tr>
<tr>
<td>( G = \langle 7, 11, 2 \rangle )</td>
<td>( 7, 3, 0 ) = ( c )</td>
</tr>
<tr>
<td>( G^# = A_b = \langle 8, 0, 3 \rangle )</td>
<td>( 8, 4, 1 ) = ( c^# = d^\flat )</td>
</tr>
<tr>
<td>( A = \langle 9, 1, 4 \rangle )</td>
<td>( 9, 5, 2 ) = ( d )</td>
</tr>
<tr>
<td>( A^# = B_b = \langle 10, 2, 5 \rangle )</td>
<td>( 10, 6, 3 ) = ( d^# = e^\flat )</td>
</tr>
<tr>
<td>( B = \langle 11, 3, 6 \rangle )</td>
<td>( 11, 7, 4 ) = ( e )</td>
</tr>
</tbody>
</table>
The $T/I$-group acts on the set $S$ of major and minor triads componentwise.

$$T_1\langle 0, 4, 7 \rangle = \langle T_10, T_14, T_17 \rangle = \langle 1, 5, 8 \rangle$$

$$I_0\langle 0, 4, 7 \rangle = \langle I_00, I_04, I_07 \rangle = \langle 0, 8, 5 \rangle$$

**Figure:** $I_0$ applied to a C-major triad yields an f-minor triad.
II. Some Group Theory
The **dihedral group of order 24** is the group of symmetries of a regular 12-gon.

Algebraically, the **dihedral group of order 24** is the group generated by two elements, $s$ and $t$, subject to the three relations

\[ s^{12} = 1, \quad t^2 = 1, \quad tst = s^{-1}. \]

The $T/I$-group is (isomorphic to) the dihedral group of order 24.
Group Actions

An *action of a group* \( G \) on a set \( S \) is a function

\[
G \times S \rightarrow S
\]

\[(g, s) \mapsto gs\]

such that \( g(hs) = (gh)s \) and \( es = s \) for all \( g, h \in G \) and all \( s \in S \). This is the same as a group homomorphism \( G \rightarrow \text{Sym}(S) \), where \( \text{Sym}(S) \) is the *symmetric group* on \( S \).

**Example**

\( S = \text{the set of major and minor triads} \)

\( G = T/I\)-group

\[
T_n(\langle x, y, z \rangle) = \langle T_n x, T_n y, T_n z \rangle \quad I_n(\langle x, y, z \rangle) = \langle I_n x, I_n y, I_n z \rangle
\]
The Orbit-Stabilizer Theorem

The orbit of an element $Y$ of a set $S$ under a group action of $G$ on $S$ consists of all those elements of $S$ to which $Y$ is moved, in other words

$$\text{orbit of } Y = \{hY \mid h \in G\}.$$ 

The stabilizer group of $Y$ consists of all those elements of $G$ which fix $Y$, namely

$$G_Y = \{h \in G \mid hY = Y\}.$$ 

Theorem (Orbit-Stabilizer Theorem)

If a finite group $G$ acts on a set $S$ and $Y \in S$, then

$$|G|/|G_Y| = |\text{orbit of } Y|.$$
## The Orbit-Stabilizer Theorem

### Example

\( S = \) the set of major and minor triads  
\( G = T/I \)-group, acting componentwise, and transitively

\[
\frac{24}{|G_Y|} = \frac{|G|}{|G_Y|} = |\text{orbit of } Y| = 24
\]

Thus \(|G_Y| = 1\) for each \( Y \) in \( S \), and the \( T/I \) group acts **simply transitively**:  
for each \( Y \) and \( Z \) in \( S \), there exists a unique \( g \in G \) such that \( gY = Z \).
Duality in the Sense of Lewin

The centralizer of a subgroup $G$ of $\text{Sym}(S)$ is the set of elements of $\text{Sym}(S)$ which commute with all elements of $G$, namely

$$C(G) = \{ \sigma \in \text{Sym}(S) \mid \sigma g = g\sigma \text{ for all } g \in G \}.$$ 

Subgroups $G_1$ and $G_2$ of $\text{Sym}(S)$ are dual if each acts simply transitively on $S$ and

$$C(G_1) = G_2 \text{ and } C(G_2) = G_1.$$
Duality in the Sense of Lewin

Example

Let $G$ be a group and consider the left and right Cayley embeddings $\lambda, \rho: G \to \text{Sym}(G)$.

$$\lambda_g(s) := gs$$
$$\rho_h(s) := sh^{-1}$$

Then $\lambda(G)$ and $\rho(G)$ act simply transitively and

$$\lambda_g \rho_h(s) = gsh^{-1} = \rho_h \lambda_g(s).$$

Further, $\lambda(G)$ and $\rho(G)$ are dual groups. In fact, all dual groups are of this form.
III. The neo-Riemannian Group
Recent work focuses on the neo-Riemannian operations $P$, $L$, and $R$.

$P$, $L$, and $R$ generate a dihedral group, called the \textit{neo-Riemannian group}. As we’ll see, this group is \textit{dual} to the $T/I$-group in the sense of Lewin.

These transformations arose in the work of the 19th century music theorist Hugo Riemann, and have a pictorial description on the \textit{Oettingen/Riemann Tonnetz}.

$P$, $L$, and $R$ are defined in terms of common tone preservation.
We consider three functions

\[ P, L, R : S \rightarrow S. \]

Let \( P(x) \) be that triad of opposite type as \( x \) with the first and third notes switched.

For example

\[ P(0, 4, 7) = P(C{\text{-major}}) = \]

<table>
<thead>
<tr>
<th>The set ( S ) of consonant triads</th>
<th>Major Triads</th>
<th>Minor Triads</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C = \langle 0, 4, 7 \rangle )</td>
<td>( f = 0, 8, 5 )</td>
<td></td>
</tr>
<tr>
<td>( C^# = D_b = \langle 1, 5, 8 \rangle )</td>
<td>( f^# = g_b = 1, 9, 6 )</td>
<td></td>
</tr>
<tr>
<td>( D = \langle 2, 6, 9 \rangle )</td>
<td>( g = 2, 10, 7 )</td>
<td></td>
</tr>
<tr>
<td>( D^# = E_b = \langle 3, 7, 10 \rangle )</td>
<td>( g^# = a_b = 3, 11, 8 )</td>
<td></td>
</tr>
<tr>
<td>( E = \langle 4, 8, 11 \rangle )</td>
<td>( a = 4, 0, 9 )</td>
<td></td>
</tr>
<tr>
<td>( F = \langle 5, 9, 0 \rangle )</td>
<td>( a^# = b_b = 5, 1, 10 )</td>
<td></td>
</tr>
<tr>
<td>( F^# = G_b = \langle 6, 10, 1 \rangle )</td>
<td>( b = 6, 2, 11 )</td>
<td></td>
</tr>
<tr>
<td>( G = \langle 7, 11, 2 \rangle )</td>
<td>( c = 7, 3, 0 )</td>
<td></td>
</tr>
<tr>
<td>( G^# = A_b = \langle 8, 0, 3 \rangle )</td>
<td>( c^# = d_b = 8, 4, 1 )</td>
<td></td>
</tr>
<tr>
<td>( A = \langle 9, 1, 4 \rangle )</td>
<td>( d = 9, 5, 2 )</td>
<td></td>
</tr>
<tr>
<td>( A^# = B_b = \langle 10, 2, 5 \rangle )</td>
<td>( d^# = e_b = 10, 6, 3 )</td>
<td></td>
</tr>
<tr>
<td>( B = \langle 11, 3, 6 \rangle )</td>
<td>( e = 11, 7, 4 )</td>
<td></td>
</tr>
</tbody>
</table>
The neo-Riemannian Transformation $P$

We consider three functions

$P, L, R : S \rightarrow S$.

Let $P(x)$ be that triad of opposite type as $x$ with the first and third notes switched. For example

$P(\langle 0, 4, 7 \rangle) = \langle 7, 3, 0 \rangle$

$P(C\text{-major}) = c\text{-minor}$

<table>
<thead>
<tr>
<th>Major Triads</th>
<th>Minor Triads</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C = \langle 0, 4, 7 \rangle$</td>
<td>$\langle 0, 8, 5 \rangle = f$</td>
</tr>
<tr>
<td>$C# = D\flat = \langle 1, 5, 8 \rangle$</td>
<td>$\langle 1, 9, 6 \rangle = f# = g\flat$</td>
</tr>
<tr>
<td>$D = \langle 2, 6, 9 \rangle$</td>
<td>$\langle 2, 10, 7 \rangle = g$</td>
</tr>
<tr>
<td>$D# = E\flat = \langle 3, 7, 10 \rangle$</td>
<td>$\langle 3, 11, 8 \rangle = g# = a\flat$</td>
</tr>
<tr>
<td>$E = \langle 4, 8, 11 \rangle$</td>
<td>$\langle 4, 0, 9 \rangle = a$</td>
</tr>
<tr>
<td>$F = \langle 5, 9, 0 \rangle$</td>
<td>$\langle 5, 1, 10 \rangle = a# = b\flat$</td>
</tr>
<tr>
<td>$F# = G\flat = \langle 6, 10, 1 \rangle$</td>
<td>$\langle 6, 2, 11 \rangle = b$</td>
</tr>
<tr>
<td>$G = \langle 7, 11, 2 \rangle$</td>
<td>$\langle 7, 3, 0 \rangle = c$</td>
</tr>
<tr>
<td>$G# = A\flat = \langle 8, 0, 3 \rangle$</td>
<td>$\langle 8, 4, 1 \rangle = c# = d\flat$</td>
</tr>
<tr>
<td>$A = \langle 9, 1, 4 \rangle$</td>
<td>$\langle 9, 5, 2 \rangle = d$</td>
</tr>
<tr>
<td>$A# = B\flat = \langle 10, 2, 5 \rangle$</td>
<td>$\langle 10, 6, 3 \rangle = d# = e\flat$</td>
</tr>
<tr>
<td>$B = \langle 11, 3, 6 \rangle$</td>
<td>$\langle 11, 7, 4 \rangle = e$</td>
</tr>
</tbody>
</table>
The neo-Riemannian Transformations $L$ and $R$

Let $L(x)$ be that triad of opposite type as $x$ with the second and third notes switched. For example

$L\langle 0, 4, 7 \rangle = \langle 11, 7, 4 \rangle$

$L(C\text{-major}) = e\text{-minor}.$

Let $R(x)$ be that triad of opposite type as $x$ with the first and second notes switched. For example

$R\langle 0, 4, 7 \rangle = \langle 4, 0, 9 \rangle$

$R(C\text{-major}) = a\text{-minor}.$
Minimal motion of the moving voice under $P$, $L$, and $R$. 

$PC = c$

$LC = e$

$RC = a$
Example: “Oh! Darling” from the Beatles

\[ f\sharp \xrightarrow{L} D \xrightarrow{R} b \xrightarrow{P \circ R \circ L} E \]

\[ E+ \quad A \quad E \]
Oh____ Darling please believe me
\[ f\sharp \quad D \]
I’ll never do you no harm
\[ b7 \quad E7 \]
Be-lieve me when I tell you
\[ b7 \quad E7 \quad A \]
I’ll never do you no harm
The neo-Riemannian PLR-Group

Definition

The neo-Riemannian PLR-group is the subgroup of Sym(S) generated by P, L, and R.

Various relations:

\[ P^2 = L^2 = R^2 = 1 \]
\[ R(LR)^3 = P \]
\[ (LR)^{12} = 1 \]
\[ ... \]
The Structure of the neo-Riemannian PLR-Group

Theorem (Lewin 80’s, Hook 2002, …)

The PLR-group is dihedral of order 24 and is generated by L and R.

Proof. (CFS) $P, L,$ and $R$ commute with $T_1$. Apply $R$ and $L$ to $C$-major to get: $C,$ $a,$ $F,$ $d,$ $B\flat,$ $g,$ $E\flat,$ $c,$ $A\flat,$ $f,$ $D\flat,$ $b\flat,$ $G\flat,$ $e,$ $E,$ $c\sharp,$ $A,$ $f\sharp,$ $D,$ $b,$ $G,$ $e,$ $C$. Thus, the 24 bijections $R,$ $L R,$ $R L R,$ $\ldots,$ $R(LR)^{11},$ and $(LR)^{12} = 1$ are distinct, the PLR-group has at least 24 elements, and that $LR$ has order 12. Further $R(LR)^3(C) = c$, and since $R(LR)^3$ has order 2 and commutes with $T_1$, we see that $R(LR)^3 = P$, and the PLR-group is generated by $L$ and $R$ alone.

If we set $s = LR$ and $t = L$, then $s^{12} = 1$, $t^2 = 1$, and $tst = s^{-1}$. Finally, the PLR-group is dihedral of order 24. QED
The neo-Riemannian PLR-Group and Duality

Corollary

The PLR-group acts simply transitively on the set of consonant triads.

Theorem (Lewin 80’s, Hook 2002, ...)

The PLR-group is dual to the $T/I$-group.

Proof. (CFS) $P, L,$ and $R$ commute with $T_1$ and $I_0$, so $PLR \leq C(T/I)$.

$$|C(T/I)|/|C(T/I)_Y| = |\text{orbit of } Y| \leq |S| = 24$$

$$24 \leq |PLR| \leq |C(T/I)| \leq 24$$

since $|C(T/I)_Y| = 1$. Thus, $PLR = C(T/I)$.

Similarly, $T/I = C(PLR)$. QED
Example of Duality: Pachelbel’s *Canon in D*, ca 1680
Example of Duality: Wagner’s *Parsifal*, 1882

(Here we consider the duality between the 24-element groups, rather than the duality between the 6-element groups in the Overview Presentation.)
IV. The Geometry of the neo-Riemannian Group
The Oettingen/Riemann Tonnetz
The Torus
The Dual Graph to the Tonnetz

Figure: Douthett and Steinbach’s Graph.
The Dual Graph to the Tonnetz

Figure: Douthett and Steinbach’s Graph.

Figure: Waller’s Torus.
Beethoven’s 9th, 2nd Mvmt, Measures 143-17 (Cohn)
We have seen:

- How to encode pitch classes as integers modulo 12, and consonant triads as 3-tuples of integers modulo 12
- How the $T/I$-group acts componentwise on consonant triads
- How the $PLR$-group acts on consonant triads
- Duality between the $T/I$-group and the $PLR$-group
- Geometric depictions on the torus and musical examples.

*But most music does not consist entirely of triads! See the extension of Fiore-Satyendra in Music Theory Online, Volume 11, Number 3, September 2005. Also, not every musical action is simply transitive. See the extension of R. Peck in the Journal of Music Theory, next talk.*