The Homotopy Theory of $n$-Fold Categories

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When do we consider two categories $A$ and $B$ the same?

Two different possibilities:

1. If there is a functor $F : A \to B$ such that $NF : NA \to NB$ is a weak homotopy equivalence. (Thomason 1980)

2. If there is a fully faithful and essentially surjective functor $F : A \to B$. (Joyal–Tierney 1991)

$2) \Rightarrow 1)$
A **2-category** is like an ordinary category except a 2-category has *Hom-categories*. Example: **Top**.

A **double category** is like an ordinary category except a double category has a *category of objects* and a *category of morphisms*. Example: Bimodules.

Recent examples show 2-categories are not enough, we need double categories.
Motivation: Why consider model structures on $\text{DblCat}$ and $\text{nFoldCat}$?

Model categories have found great utility in comparing notions of $(\infty, 1)$-category.

**Theorem** (Bergner, Joyal–Tierney, Rezk, Toën,...) The following model categories are Quillen equivalent: simplicial categories, Segal categories, complete Segal spaces, and quasicategories.

So we can expect model structures to also be of use in an investigation of iterated internalizations.
Definition (Ehresmann 1963)

A double category $\mathbb{D}$ is an internal category $(\mathbb{D}_0, \mathbb{D}_1)$ in $\text{Cat}$. 
A double category $\mathcal{D}$ consists of
a set of objects,
a set of horizontal morphisms,
a set of vertical morphisms, and
a set of squares with source and target as follows

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{j} & & \downarrow{k} \\
C & & D
\end{array}
\quad \begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{j} & & \downarrow{k} \\
C & \xrightarrow{g} & D
\end{array}
\]

and compositions and units that satisfy the usual axioms and the interchange law.
Examples of Double Categories

1. Any 2-category is a double category with trivial vertical morphisms.

2. Compact closed 1-manifolds, 2-cobordisms, diffeomorphisms of 1-manifolds, diffeomorphisms of 2-cobordisms compatible with boundary diffeomorphisms.

3. Rings, bimodules, ring maps, and twisted maps.

4. Topological spaces, parametrized spectra, continuous maps, and squares like in 3.
$N: \text{DblCat} \to [\Delta^{\text{op}} \times \Delta^{\text{op}}, \text{Set}]$

$(N\mathbb{D})_{j,k} = j \times k$ — matrices of composable squares in $\mathbb{D}$

$N$ admits a left adjoint $c$ called *double categorification*. 
Model Structures for Higher Categories in Low Dimensions
A *model category* is a complete and cocomplete category $C$ equipped with three subcategories:
1. weak equivalences
2. fibrations
3. cofibrations
which satisfy various axioms.
Notably: given a commutative diagram

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & & \downarrow \\
B & \longrightarrow & Y
\end{array}
$$

in which at least one of $i$ or $p$ is a weak equivalence, then there exists a lift $h: B \longrightarrow X$.

**Example** The category $\textbf{Top}$ with $\pi_\ast$-isomorphisms and Serre fibrations is a model category.
Model Structures on $\textbf{Cat}$

**Theorem (Thomason 1980)**

There is a model structure on $\textbf{Cat}$ such that

- $F$ is a weak equivalence if and only if $\text{Ex}^2\text{NF}$ is so.
- $F$ is a fibration if and only if $\text{Ex}^2\text{NF}$ is so.

**Theorem (Joyal–Tierney 1991)**

There is a model structure on $\textbf{Cat}$ such that

- $F$ is a weak equivalence if and only if $F$ is an equivalence of categories.
- $F$ is a fibration if and only if $F$ is an isofibration.
Model Structures on $\mathbf{2-Cat}$

**Theorem (Worytkiewicz–Hess–Parent–Tonks 2007)**

There is a model structure on $\mathbf{2-Cat}$ such that

- $F$ is a weak equivalence if and only if $\text{Ex}^2 N_2 F$ is so.
- $F$ is a fibration if and only if $\text{Ex}^2 N_2 F$ is so.

**Theorem (Lack 2004)**

There is a model structure on $\mathbf{2-Cat}$ such that

- $F$ is a weak equivalence if and only if $F$ is a biequivalence of 2-categories.
- $F$ is a fibration if and only if $F$ is an equivfibration.
There exist model structures on $\textbf{DblCat}$ for each of the following types of weak equivalences.

- $F$ is a weak equivalence if and only if $F$ is fully faithful and “essentially surjective.”
- $F$ is a weak equivalence if and only if $F$ is a weak equivalence of double categories as algebras in $\text{Cat}(\text{Graph})$.
- $F$ is a weak equivalence if and only if $N_h F$ is a weak equivalence in $[\Delta^{op}, \text{Cat}]$. 
Adjunction:

\[
\begin{array}{ccc}
\mathbf{SSet} & \perp & \mathbf{Cat} \\
\downarrow & & \downarrow \\
N & & c
\end{array}
\]

\(cX\) is the free category on the graph \((X_0, X_1)\) modulo the relation below.

\[g \circ f \sim h\] whenever \(X\) has a 2-simplex

\[
\begin{array}{ccc}
f & \sigma & g \\
\uparrow & \sigma & \downarrow \\
\downarrow & \sigma & \downarrow \\
& \sigma & \\
& h & \\
\end{array}
\]

The unit component \(\partial \Delta[3] \longrightarrow Nc(\partial \Delta[3])\) is not a weak equivalence.
The unit and counit of the adjunction

\[
\begin{array}{ccc}
\text{SSet} & \overset{\perp}{\leftrightarrow} & \text{SSet} \\
\downarrow^{\text{Ex}^2} & & \downarrow^{\text{N}} \\
\text{Cat} & \overset{c}{\leftrightarrow} & \text{Cat}
\end{array}
\]

are weak equivalences (Fritsch–Latch 1979, Thomason). So the Thomason model structure on \textbf{Cat} is Quillen equivalent to \textbf{SSet} and also \textbf{Top}.
**Definition**

An $n$-fold category is an internal category in $(n-1)\text{FoldCat}$.

**Example**

A double category is a 2-fold category.

We have a fully faithful $n$-fold nerve.

$$N : n\text{FoldCat} \rightarrow SSet^n$$

$$(N\mathbb{D})_{j_1,\ldots,j_n} = n\text{FoldCat}([j_1] \boxtimes \cdots \boxtimes [j_n], \mathbb{D}).$$

**Adjunction:**

$$\adjunction{SSet^n}{n\text{FoldCat}}{N}{c}$$
If \( Y : (\Delta^{op})^\times n \to \textbf{Set} \), then the \textit{n-fold Grothendieck construction} on \( Y \) is the \textit{n-fold category} \( \Delta \otimes n / Y \) with

\[
\text{Objects} = \{ (y, k) | k \in \Delta^\times n, y \in Y_k \}
\]

and \( n \)-cubes \( (y, k) \to (z, \ell) \) are morphisms \( \overline{f} : \overline{k} \to \overline{\ell} \) in \( \Delta^\times n \) such that

\[
\overline{f}^*(z) = y.
\]

This is the \textit{n-fold category} of multisimplices of \( Y \).
Main Theorem 1: The $n$-fold Grothendieck Construction is Homotopy Inverse to the $n$-fold Nerve

($n=1$ case was Quillen, Illusie, Waldhausen, Joyal–Tierney)

**Theorem** (Fiore–Paoli 2008)
The $n$-fold Grothendieck construction is a homotopy inverse to $n$-fold nerve. In other words, there are natural weak equivalences

\[
N(\Delta \boxtimes^n / Y) \to Y
\]

\[
\Delta \boxtimes^n / N(\mathbb{D}) \to \mathbb{D}.
\]
$\mathbf{SSet} = [\Delta^{\text{op}}, \mathbf{Set}] = \text{simplicial sets}$

$\mathbf{SSet}^n = [(\Delta^{\text{op}})^n, \mathbf{Set}] = \text{multisimplicial sets}$

The diagonal functor

$$\delta : \Delta \longrightarrow \Delta^n$$

induces $\delta^* : \mathbf{SSet}^n \longrightarrow \mathbf{SSet}$ by precomposition.

Adjunction:

$$\mathbf{SSet} \quad \perp \quad \mathbf{SSet}^n$$

$$\delta^! \quad \delta^*$$
Main Theorem 2: Thomason Structure on $n\text{FoldCat}$

Theorem (Fiore–Paoli 2008)

There is a cofibrantly generated model structure on $n\text{FoldCat}$ such that

- $F$ is a weak equivalence if and only if $\text{Ex}^2\delta^*NF$ is so.
- $F$ is a fibration if and only if $\text{Ex}^2\delta^*NF$ is so.

Further, the adjunction

\[
\begin{array}{cccccc}
\text{SSet} & \xrightarrow{\text{Sd}^2} & \text{SSet} & \xleftarrow{\text{Ex}^2} & \text{SSet} & \xrightarrow{\delta!} \text{SSet}^n & \xleftarrow{\delta^*} \text{SSet}^n & \xrightarrow{c} \text{nFoldCat} & \xleftarrow{N} & \text{nFoldCat}
\end{array}
\]

is a Quillen equivalence.