A Thomason Model Structure on the Category of Small n-fold Categories

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Motivation

When do we consider two categories A and B the same?
Two *different* possibilities:

1) If there is a functor $F\colon A \longrightarrow B$ such that $NF\colon NA \longrightarrow NB$ is a weak homotopy equivalence. (Thomason 1980)

2) If there is a fully faithful and essentially surjective functor $F\colon A \longrightarrow B$. (Joyal–Tierney 1991)

2) $\Rightarrow$ 1)

Motivation: 2-categories vs. Double Categories

- A **2-category** is like an ordinary category except a 2-category has *Hom-categories*. Example: **Top**.

- A **double category** is like an ordinary category except a double category has a *category of objects* and a *category of morphisms*. Example: Bimodules.

- Recent examples show 2-categories are not enough, we need double categories.

Motivation: Why consider model structures on $\mathbf{DblCat}$ and $\mathbf{nFoldCat}$?

Model categories have found great utility in comparing notions of $(\infty, 1)$-category.

Bergner, Joyal–Tierney, Rezk, Toën...:

Theorem The following model categories are Quillen equivalent: simplicial categories, Segal categories, complete Segal spaces, and quasi-categories.

So we can expect model structures to also be of use in an investigation of iterated internalizations.

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Double Categories

Definition (Ehresmann 1963) A *double category* $\mathbb{D}$ is an internal category $(\mathbb{D}_0, \mathbb{D}_1)$ in $\mathbf{Cat}$.

Definition A *double category* $\mathbb{D}$ consists of
a set of objects,
a set of horizontal morphisms,
a set of vertical morphisms, and
a set of squares with source and target as follows

$$A \xrightarrow{f} B \qquad \begin{tikzcd} A \arrow[d, "j"'] \\ C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

and compositions and units that satisfy the usual axioms and the interchange law.

Examples of Double Categories

1. Any 2-category is a double category with trivial vertical morphisms.

2. Compact closed 1-manifolds, 2-cobordisms, diffeomorphisms of 1-manifolds, diffeomorphisms of 2-cobordisms compatible with boundary diffeomorphisms

3. Rings, bimodules, ring maps, and twisted maps.

4. Topological spaces, parametrized spectra, continuous maps, and squares like in 3.

Bisimplicial Nerve of a Double Category

$$N\colon\ \mathbf{DblCat} \longrightarrow [\Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}}, \mathbf{Set}]$$

$$(N\mathbb{D})_{j,k} = j \times k - \text{matrices of composable squares in } \mathbb{D}$$

$$\begin{tikzcd}
{} \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{11}"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{12}"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{13}"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{14}"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{15}"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{16}"] & {} \arrow[d] \\
{} \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{21}"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{22}"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{23}"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{24}"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{25}"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{26}"] & {} \arrow[d] \\
{} \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{31}"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{32}"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{33}"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{34}"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{35}"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{36}"] & {} \arrow[d] \\
{} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {}
\end{tikzcd}$$

N admits a left adjoint c called *double categorification*.

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Model Categories

A *model category* is a complete and cocomplete category $\mathbf{C}$ equipped with three subcategories:

1. weak equivalences
2. fibrations
3. cofibrations

which satisfy various axioms. Most notably: given a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"', "\text{cofibration}"'] & X \arrow[d, "p", "\text{fibration}"] \\ B \arrow[r] & Y \end{tikzcd}$$

in which at least one of i or p is a weak equivalence, then there exists a lift $h\colon B \longrightarrow X$.

Example The category $\mathbf{Top}$ with π_*-isomorphisms and Serre fibrations is a model category.

Model Structures on **Cat**

Theorem (Thomason 1980)
There is a model structure on **Cat** such that

- F is a weak equivalence if and only if $Ex^2 NF$ is so.

- F is a fibration if and only if $Ex^2 NF$ is so.

Theorem (Joyal–Tierney 1991)
There is a model structure on **Cat** such that

- F is a weak equivalence if and only if F is an equivalence of categories.

- F is a fibration if and only if F is an *isofibration*.

Model Structures on **2-Cat**

Theorem (Worytkiewicz–Hess–Parent–Tonks 2007)
There is a model structure on **2-Cat** such that

- F is a weak equivalence if and only if $Ex^2 N_2 F$ is so.

- F is a fibration if and only if $Ex^2 N_2 F$ is so.

Theorem (Lack 2004)
There is a model structure on **2-Cat** such that

- F is a weak equivalence if and only if F is a biequivalence of 2-categories.

- F is a fibration if and only if F is an *equiv-fibration*.

Model Structures on **DblCat**

Theorems (Fiore–Paoli–Pronk, AGT, 2008) There exist model structures on **DblCat** for each of the following types of weak equivalences.

- F is a weak equivalence if and only if F is fully faithful and "essentially surjective."

- F is a weak equivalence if and only if F is a weak equivalence of double categories as algebras in Cat($\mathbf{Graph}$).

- F is a weak equivalence if and only if $N_h F$ is a weak equivalence in $[\Delta^{\text{op}}, \mathbf{Cat}]$.

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Thomason Structure on **Cat**

Adjunction:

$$\begin{tikzcd} \mathbf{SSet} \arrow[r, bend left, "c"] \arrow[r, phantom, "\perp"] & \mathbf{Cat} \arrow[l, bend left, "N"] \end{tikzcd}$$

cX is the free category on the graph (X_0, X_1) modulo the relation below.

$g \circ f \sim h$ whenever X has a 2-simplex

$$\begin{tikzcd} & \cdot \arrow[dr, "g"] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, shift left=3ex, "\sigma"] & & \cdot \end{tikzcd}$$

The unit component $\partial\Delta[3] \longrightarrow Nc(\partial\Delta[3])$ is **not** a weak equivalence.

Thomason Structure on **Cat**

The unit and counit of the adjunction

$$\begin{tikzcd} \mathbf{SSet} \arrow[r, bend left, "\mathsf{Sd}^2"] \arrow[r, phantom, "\perp"] & \mathbf{SSet} \arrow[l, bend left, "\mathsf{Ex}^2"] \arrow[r, bend left, "c"] \arrow[r, phantom, "\perp"] & \mathbf{Cat} \arrow[l, bend left, "N"] \end{tikzcd}$$

are weak equivalences (Fritsch–Latch 1979, Thomason). So the Thomason model structure on **Cat** is Quillen equivalent to **SSet** and also **Top**.

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n-fold Categories

Definition An *n-fold category* is an internal category in **(n-1)FoldCat**.

Example A double category is a *2-fold category.*

We have a fully faithful n-fold nerve.

$$N\colon \mathbf{nFoldCat} \longrightarrow \mathbf{SSet^n}$$

$$(N\mathbb{D})_{j_1,\ldots,j_n} = \mathbf{nFoldCat}([j_1] \boxtimes \cdots \boxtimes [j_n], \mathbb{D}).$$

Adjunction:

$$\begin{tikzcd} \mathbf{SSet^n} \arrow[rr, bend left, "c"] \arrow[rr, phantom, "\perp"] & & \mathbf{nFoldCat} \arrow[ll, bend left, "N"] \end{tikzcd}$$

The n-fold Grothendieck Construction

If $Y\colon (\Delta^{\mathsf{op}})^{\times n} \longrightarrow \mathbf{Set}$, then the *$n$-fold Grothendieck construction on Y* is the n-fold category $\Delta^{\boxtimes n}/Y$ with

$$\text{Objects } = \{(y, \overline{k}) | \overline{k} \in \Delta^{\times n}, y \in Y_{\overline{k}}\}$$

and n-cubes $(y, \overline{k}) \longrightarrow (z, \overline{\ell})$ are morphisms $\overline{f}\colon \overline{k} \longrightarrow \overline{\ell}$ in $\Delta^{\times n}$ such that

$$\overline{f}^*(z) = y.$$

This is the n-fold category of multisimplices of Y.

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Main Theorem 1: The n-fold Grothendieck Construction is Homotopy Inverse to the n-fold Nerve

(n=1 case was Quillen, Illusie, Waldhausen, Joyal–Tierney)

Theorem (Fiore–Paoli 2008)
The n-fold Grothendieck construction is a homotopy inverse to n-fold nerve. In other words, there are natural weak equivalences

$$N(\Delta^{\boxtimes n}/Y) \longrightarrow Y$$

$$\Delta^{\boxtimes n}/N(\mathbb{D}) \longrightarrow \mathbb{D}\,.$$

Diagonal

$\mathbf{SSet} = [\Delta^{\mathsf{op}}, \mathbf{Set}]$=simplicial sets

$\mathbf{SSet^n} = [(\Delta^{\mathsf{op}})^{\times n}, \mathbf{Set}]$=multisimplicial sets

The diagonal functor

$$\delta\colon\ \Delta \longrightarrow \Delta^n$$

$$[m] \longmapsto ([m], \ldots, [m])$$

induces $\delta^*\colon\ \mathbf{SSet^n} \longrightarrow \mathbf{SSet}$ by precomposition.

Adjunction:

$$\begin{tikzcd} \mathbf{SSet} \arrow[r, bend left, "\delta_!"] \arrow[r, phantom, "\perp"] & \mathbf{SSet^n} \arrow[l, bend left, "\delta^*"] \end{tikzcd}$$

Main Theorem 2: Thomason Structure on **nFoldCat**

Theorem (Fiore–Paoli 2008) There is a cofibrantly generated model structure on **nFoldCat** such that

- F is a weak equivalence if and only if $Ex^2\delta^*NF$ is so.

- F is a fibration if and only if $Ex^2\delta^*NF$ is so.

Further, the adjunction

$$\begin{tikzcd}
\mathbf{SSet} \arrow[r, bend left, "\mathsf{Sd}^2"] \arrow[r, phantom, "\perp"] & \mathbf{SSet} \arrow[l, bend left, "\mathsf{Ex}^2"] \arrow[r, bend left, "\delta_!"] \arrow[r, phantom, "\perp"] & \mathbf{SSet^n} \arrow[l, bend left, "\delta^*"] \arrow[r, bend left, "c"] \arrow[r, phantom, "\perp"] & \mathbf{nFoldCat} \arrow[l, bend left, "N"]
\end{tikzcd}$$

is a Quillen equivalence.