

A Thomason Model Structure on the Category of Small n -fold Categories

AMS Special Session in Algebraic
Topology in Honor of Bill Singer

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Overview

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Motivation

When do we consider two categories A and B the same?

Two *different* possibilities:

1) If there is a functor $F: A \rightarrow B$ such that $NF: NA \rightarrow NB$ is a weak homotopy equivalence. (Thomason 1980)

2) If there is a fully faithful and essentially surjective functor $F: A \rightarrow B$. (Joyal–Tierney 1991)

2) \Rightarrow 1)

Motivation: 2-categories vs. Double Categories

- A **2-category** is like an ordinary category except a 2-category has *Hom-categories*. Example: **Top**.
- A **double category** is like an ordinary category except a double category has a *category of objects* and a *category of morphisms*. Example: Bimodules.
- Recent examples show 2-categories are not enough, we need double categories.

Motivation: Why consider model structures on \mathbf{DblCat} and $\mathbf{nFoldCat}$?

Model categories have found great utility in comparing notions of $(\infty, 1)$ -category.

Bergner, Joyal–Tierney, Rezk, Toën...:

Theorem The following model categories are Quillen equivalent: simplicial categories, Segal categories, complete Segal spaces, and quasi-categories.

So we can expect model structures to also be of use in an investigation of iterated internalizations.

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Double Categories

Definition (Ehresmann 1963) A *double category* \mathbb{D} is an internal category $(\mathbb{D}_0, \mathbb{D}_1)$ in \mathbf{Cat} .

Definition A *double category* \mathbb{D} consists of
a set of objects,
a set of horizontal morphisms,
a set of vertical morphisms, and
a set of squares with source and target as follows

$$\begin{array}{ccc} A \xrightarrow{f} B & & A \xrightarrow{f} B \\ & & \downarrow j \quad \alpha \quad \downarrow k \\ & & C \xrightarrow{g} D \end{array}$$

and compositions and units that satisfy the usual axioms and the interchange law.

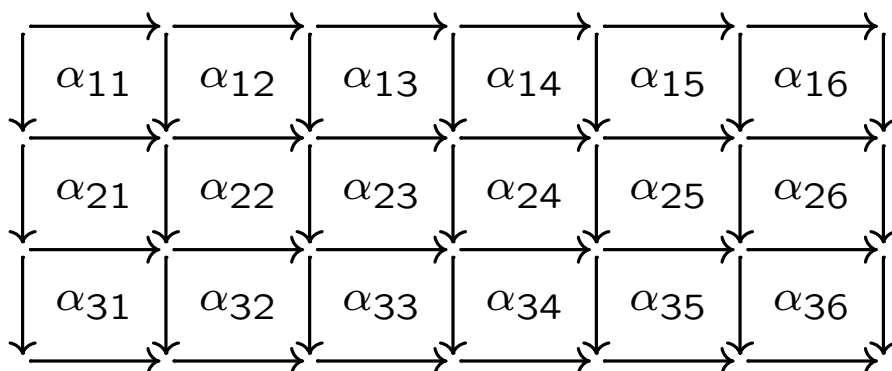
Examples of Double Categories

1. Any 2-category is a double category with trivial vertical morphisms.
2. Compact closed 1-manifolds, 2-cobordisms, diffeomorphisms of 1-manifolds, diffeomorphisms of 2-cobordisms compatible with boundary diffeomorphisms
3. Rings, bimodules, ring maps, and twisted maps.
4. Topological spaces, parametrized spectra, continuous maps, and squares like in 3.

Bisimplicial Nerve of a Double Category

$$N: \mathbf{DblCat} \rightarrow [\Delta^{\text{op}} \times \Delta^{\text{op}}, \mathbf{Set}]$$

$(N\mathbb{D})_{j,k} = j \times k$ – matrices of composable squares in \mathbb{D}



N admits a left adjoint c called *double categorification*.

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Model Categories

A *model category* is a complete and cocomplete category \mathbf{C} equipped with three subcategories:

1. weak equivalences
2. fibrations
3. cofibrations

which satisfy various axioms. Most notably: given a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \text{cofibration } i \downarrow & & \downarrow p \text{ fibration} \\ B & \longrightarrow & Y \end{array}$$

in which at least one of i or p is a weak equivalence, then there exists a lift $h: B \rightarrow X$.

Example The category \mathbf{Top} with π_* -isomorphisms and Serre fibrations is a model category.

Model Structures on **Cat**

Theorem (Thomason 1980)

There is a model structure on **Cat** such that

- F is a weak equivalence if and only if Ex^2NF is so.
- F is a fibration if and only if Ex^2NF is so.

Theorem (Joyal–Tierney 1991)

There is a model structure on **Cat** such that

- F is a weak equivalence if and only if F is an equivalence of categories.
- F is a fibration if and only if F is an *isofibration*.

Model Structures on **2-Cat**

Theorem (Worytkiewicz–Hess–Parent–Tonks 2007)

There is a model structure on **2-Cat** such that

- F is a weak equivalence if and only if Ex^2N_2F is so.
- F is a fibration if and only if Ex^2N_2F is so.

Theorem (Lack 2004)

There is a model structure on **2-Cat** such that

- F is a weak equivalence if and only if F is a biequivalence of 2-categories.
- F is a fibration if and only if F is an *equiv-fibration*.

Model Structures on **DbICat**

Theorems (Fiore–Paoli–Pronk, AGT, 2008)

There exist model structures on **DbICat** for each of the following types of weak equivalences.

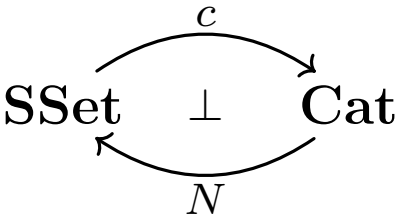
- F is a weak equivalence if and only if F is fully faithful and “essentially surjective.”
- F is a weak equivalence if and only if F is a weak equivalence of double categories as algebras in $\text{Cat}(\text{Graph})$.
- F is a weak equivalence if and only if $N_h F$ is a weak equivalence in $[\Delta^{\text{op}}, \text{Cat}]$.

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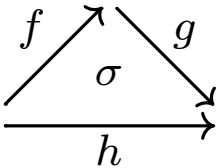
Thomason Structure on **Cat**

Adjunction:



cX is the free category on the graph (X_0, X_1) modulo the relation below.

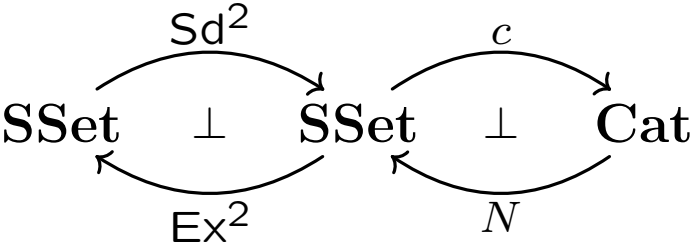
$g \circ f \sim h$ whenever X has a 2-simplex



The unit component $\partial\Delta[3] \longrightarrow Nc(\partial\Delta[3])$ is **not** a weak equivalence.

Thomason Structure on **Cat**

The unit and counit of the adjunction



are weak equivalences (Fritsch–Latch 1979, Thomason). So the Thomason model structure on **Cat** is Quillen equivalent to **SSet** and also **Top**.

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n -fold Categories

Definition An n -fold category is an internal category in $(\mathbf{n-1})\mathbf{FoldCat}$.

Example A double category is a 2-fold category.

We have a fully faithful n -fold nerve.

$$N: \mathbf{nFoldCat} \rightarrow \mathbf{SSet}^n$$

$$(N\mathbb{D})_{j_1, \dots, j_n} = \mathbf{nFoldCat}([j_1] \boxtimes \dots \boxtimes [j_n], \mathbb{D}).$$

Adjunction:

$$\begin{array}{ccc} & c & \\ \curvearrowright & & \curvearrowleft \\ \mathbf{SSet}^n & \perp & \mathbf{nFoldCat} \\ \curvearrowleft & & \curvearrowright \\ & N & \end{array}$$

The n -fold Grothendieck Construction

If $Y: (\Delta^{\text{op}})^{\times n} \rightarrow \mathbf{Set}$, then the n -fold Grothendieck construction on Y is the n -fold category $\Delta^{\boxtimes n}/Y$ with

$$\text{Objects} = \{(y, \bar{k}) \mid \bar{k} \in \Delta^{\times n}, y \in Y_{\bar{k}}\}$$

and n -cubes $(y, \bar{k}) \rightarrow (z, \bar{\ell})$ are morphisms $\bar{f}: \bar{k} \rightarrow \bar{\ell}$ in $\Delta^{\times n}$ such that

$$\bar{f}^*(z) = y.$$

This is the n -fold category of multisimplices of Y .

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Main Theorem 1: The n -fold Grothendieck Construction is Homotopy Inverse to the n -fold Nerve

($n=1$ case was Quillen, Illusie, Waldhausen, Joyal–Tierney)

Theorem (Fiore–Paoli 2008)

The n -fold Grothendieck construction is a homotopy inverse to n -fold nerve. In other words, there are natural weak equivalences

$$N(\Delta^{\boxtimes n}/Y) \rightarrow Y$$

$$\Delta^{\boxtimes n}/N(\mathbb{D}) \rightarrow \mathbb{D}.$$

Diagonal

$\mathbf{SSet} = [\Delta^{\text{op}}, \mathbf{Set}] = \text{simplicial sets}$

$\mathbf{SSet}^n = [(\Delta^{\text{op}})^{\times n}, \mathbf{Set}] = \text{multisimplicial sets}$

The diagonal functor

$$\delta: \Delta \rightarrow \Delta^n$$

$$[m] \mapsto ([m], \dots, [m])$$

induces $\delta^*: \mathbf{SSet}^n \rightarrow \mathbf{SSet}$ by precomposition.

Adjunction:

$$\begin{array}{ccc} & \delta_! & \\ \mathbf{SSet} & \begin{array}{c} \curvearrowright \\ \perp \\ \curvearrowleft \end{array} & \mathbf{SSet}^n \\ & \delta^* & \end{array}$$

Main Theorem 2: Thomason Structure on **nFoldCat**

Theorem (Fiore–Paoli 2008) There is a cofibrantly generated model structure on **nFoldCat** such that

- F is a weak equivalence if and only if $Ex^2\delta^*NF$ is so.
- F is a fibration if and only if $Ex^2\delta^*NF$ is so.

Further, the adjunction

$$\begin{array}{ccccc}
 & \xrightarrow{Sd^2} & & \xrightarrow{\delta_!} & & \xrightarrow{c} & \\
 \mathbf{SSet} & & \mathbf{SSet} & & \mathbf{SSet}^n & & \mathbf{nFoldCat} \\
 & \xleftarrow{Ex^2} & & \xleftarrow{\delta^*} & & \xleftarrow{N} & \\
 & \perp & & \perp & & \perp &
 \end{array}$$

is a Quillen equivalence.