# Pseudo Algebraic Structures in CFT

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# Introduction

This work is a contribution to the categorical foundations of CFT.

A CFT is a morphism of stacks of pseudo commutative monoids with cancellation. (Hu, Kriz)

Bicategorical versions of limits, colimits, and adjoints are essential to this approach.

# General Philosophy

- equalities are replaced by coherence isos

- these coherence isos satisfy coherence diagrams

- bijection of sets is replaced by equivalence of categories

## Examples

A *pseudo functor* between 2-categories preserves compositions and identities up to coherence isos, which satisfy coherence diagrams.

A *pseudo natural transformation* $\alpha : F \to G$ between pseudo functors $F$ and $G$ is an assignment $A \mapsto \alpha_A$ which is natural up to a coherence iso. This coherence iso satisfies coherence diagrams.

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \arrow[dl, Rightarrow, "\tau_f"'] \\ FB \arrow[r, "\alpha_B"'] & GB \end{tikzcd}$$

# Pseudo Limits

Let $F : \mathcal{J} \to \mathcal{C}$ be a pseudo functor. Then $L \in \mathcal{C}$ and $\pi \in PsCone(L, F)$ comprise a *pseudo limit* of $F$ if

$$Mor_{\mathcal{C}}(L', L) \to PsCone(L', F)$$

$$f \mapsto \pi \circ f$$

is an isomorphism of categories for all $L' \in \mathcal{C}$.

Variants: *weighted pseudolimits*, *bilimits*, *weighted bilimits*

**Example** 2-equalizer vs. Pseudo-equalizer

Suppose $F : \{1 \rightrightarrows 2\} \to Cat$ has image

$$A \underset{H}{\overset{G}{\rightrightarrows}} B.$$

Then $Obj$ 2-lim $F = \{a \in A : Ga = Ha\}$, while the object class of the pseudo limit $PsCone(\mathbf{1}, F)$ is

$$\{(a, b, g, h) : a \in A, b \in B, g : Ga \xrightarrow{\cong} b, h : Ha \xrightarrow{\cong} b\}.$$

Hence 2-lim $F$ may be empty while ps-lim $F$ is not.

# Theories and Algebras

A *Lawvere theory* is a category $T$ with objects $0, 1, 2, \ldots$ such that $n$ is the product of $n$-copies of 1 and each $n$ is equipped with a limiting cone.

**Example** The Endomorphism Theory

Let $X$ be a set. The *endomorphism theory* $End(X)$ has objects $0, 1, 2, \ldots$ and morphisms $Hom_{End(X)}(m, n) = Maps(X^m, X^n)$.

A *theory* is a collection of sets $T(0), T(1), \ldots$ equipped with maps

$$\gamma : T(k) \times T(n_1) \times \cdots \times T(n_k) \to T(n_1 + \cdots + n_k)$$

that are associative, unital, and are equivariant with respect to subsitution.

Translation: $T(n) = T(n, 1)$ and also $T(m, n) = \prod_{i=1}^{n} T(m)$.

A small category $X$ is a $T$-*algebra* or an *algebra over* $T$ if it is equipped with a morphism of theories $(\Phi_n : T(n) \to End(X)(n))_n$.

**Example** Groups

The theory $T$ of groups is generated by $e \in T(0), \nu \in T(1), \mu \in T(2)$ and these satisfy the usual axioms of a group. A group is an algebra over the theory of groups.

A small category $X$ is a *pseudo $T$-algebra* if it is equipped with

- structure maps $\Phi_n : T(n) \to End(X)(n)$ for each $n \geq 0$

- coherence isos for each operation of theories ($\gamma$, 1, and substitution) which satisfy

- coherence diagrams for each relation of theories (associativity, unitality, equivariance).

For example, for $w, v, u \in T(1)$ the diagram

$$\begin{tikzcd}
\Phi_1(w) \circ \Phi_1(v) \circ \Phi_1(u) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \Phi_1(w) \circ \Phi_1(v \circ u) \arrow[d, Rightarrow] \\
\Phi_1(w \circ v) \circ \Phi_1(u) \arrow[r, Rightarrow] & \Phi_1(w \circ v \circ u)
\end{tikzcd}$$

must commute.

**Example** Finite sets with $\coprod$ form a pseudo commutative monoid.

$$A \coprod B :== A \times \{1\} \cup B \times \{2\}$$

$$B \coprod A == B \times \{1\} \cup A \times \{2\}$$

**Example** Finite dimensional vector spaces with $\oplus$ and $\otimes$ form a pseudo commutative semiring.

**Example** Any symmetric monoidal category is a pseudo commutative monoid.

**Theorem 1** *(F) Let $T$ be a theory. Let $\mathcal{C}_T$ denote the 2-category of pseudo $T$-algebras. Then any pseudo functor $F : \mathcal{J} \to \mathcal{C}_T$ from a 1-category $\mathcal{J}$ to the 2-category of pseudo $T$-algebras admits all weighted pseudo limits and all weighted bicolimits.*

*Proof:* Let $L := PsCone(\mathbf{1}, U \circ F)$, where $U : \mathcal{C}_T \to Cat$ is the forgetful 2-functor. Then $L$ has a pseudo $T$-algebra structure and it is the conical pseudo limit of $F$. Theorems from Street (1980) about cotensor products imply the result for conical pseudo limits. The conical bicolimits require a different construction.

$\square$

This allows one to speak of stacks of pseudo $T$-algebras.

**Theorem 2** *(F) Let $T$ be a theory. There exists a 2-monad $C$ on $\mathcal{C}at$ such that the 2-category of strict $C$-algebras, pseudo $C$-morphisms, and 2-cells is 2-equivalent to $\mathcal{C}_T$.*

*Proof:* The 2-monad $C$ is constructed by taking the free theory on $T$ and enriching it appropriately in groupoids. $\square$

**Theorem 3** *(Blackwell, Kelly, Power 1989) Let $C$ be a 2-monad on a 2-category $\mathcal{K}$. Then the 2-category of strict $C$-algebras, pseudo $C$-morphisms, and 2-cells admits all strictly weighted pseudo limits of strict 2-functors.*

This theorem implies a special case of Theorem 1.

# Bi-adjoints

Let $\mathcal{X}, \mathcal{A}$ be 2-categories.
A *bi-adjunction* $\langle F, G, \phi \rangle : \mathcal{X} \rightharpoonup \mathcal{A}$ consists of the following data

- Pseudo functors

$$\mathcal{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{A}$$

- For all $X \in Obj\mathcal{X}$ and all $A \in Obj\mathcal{A}$ an equivalence of categories $\phi_{X,A} : Mor_{\mathcal{A}}(FX, A) \to Mor_{\mathcal{X}}(X, GA)$ which is pseudo natural in $X$ and $A$ separately.

**Theorem 4** *(Kelly 1989) (F) Let $\mathcal{X}, \mathcal{A}$ be 2-categories.*
*Let $\mathcal{X} \xleftarrow{G} \mathcal{A}$ be a pseudo functor. Then there exists a left bi-adjoint for $G$ if and only if for every object $X \in Obj\mathcal{X}$ there exists an object $R \in Obj\mathcal{A}$ and a bi-universal arrow $\eta_X : X \to G(R)$ from $X$ to $G$.*

**Theorem 5** *(F) Let $S \to T$ be a morphism of theories and let $G : \mathcal{C}_T \to \mathcal{C}_S$ be the associated forgetful 2-functor. Then there exists a left bi-adjoint $F : \mathcal{C}_S \to \mathcal{C}_T$ which is a strict 2-functor. Moreover, $\phi_{X,A}$ is strictly 2-natural in each variable.*

# 2-Theories, Algebras, and Pseudo Limits

A *2-theory* $(\Theta, T)$ *fibered over the theory* $T$ is a natural number $k$, a theory $T$, and a contravariant functor $\Theta : T \to Cat$ from the category $T$ to the 2-category $Cat$ of small categories such that

- $Obj\Theta(m) = \coprod_{n \geq 0} Hom_{T^k}(m,n)$ for all $m \geq 0$, where $T^k$ is the theory with the same objects as $T$, but with $Hom_{T^k}(m,n) = Hom_T(m,n)^k$
- If $w_1, \ldots, w_n \in Hom_{T^k}(m,1)$, then the word in $Hom_{T^k}(m,n)$ with which the $n$-tuple is identified is the product in $\Theta(m)$ of $w_1, \ldots, w_n$
- For $w \in Hom_T(m,n)$ the functor $\Theta(w) : \Theta(n) \to \Theta(m)$ is $\Theta(w)(v) = v \circ w^{\times k}$ on objects $v \in Hom_{T^k}(n,j)$.

For objects $w_1, \dots, w_n, w \in Mor_{T^k}(m, 1)$ $\subseteq Obj\Theta(m)$ we set

$$\Theta(w; w_1, \dots, w_n) := Mor_{\Theta(m)}(\prod_{i=1}^{n} w_i, w).$$

**Example** Let $I$ be a 1-category and $X : I^k \to Cat$ a strict 2-functor. Then there is a 2-functor $End(X) : End(I)^{op} \to Cat$ called the *endomorphism 2-theory on $X$* which is defined analogously to the endomorphism theory on a set.

A *pseudo* $(\Theta, T)$*-algebra* consists of the following data:

- a small pseudo $T$-algebra $I$ with structure maps denoted $\Phi : T(n) \to Functors(I^n, I)$

- a strict 2-functor $X : I^k \to Cat$

- set maps $\phi : \Theta(w; w_1, \ldots, w_n) \to End(X)(\Phi(w); \Phi(w_1), \ldots, \Phi(w_n))$, where $\Phi(w)$ means to apply $\Phi$ to each component of $w$ to make $I^k$ into the product pseudo algebra of $k$ copies of $I$.

- a coherence iso for each operation of a 2-theory satisfying

- coherence diagrams for each relation of 2-theories.

**Example** The 2-theory of *commutative monoids with cancellation* (CMC's) is generated by 3 operations. The underlying theory $T$ is commutative monoids and k=2. The operations are described below in terms of an algebra $X : I^2 \to Cat$ rather than in terms of the abstract 2-theory.

$$+ : X_{a,b} \times X_{c,d} \to X_{a+c,b+d}$$

$$0 \in X_{0,0}$$

$$\check{?} : X_{a+c,b+c} \to X_{a,b}$$

Axioms:
$+$ is commutative, associative, and unital
$\check{?}$ is transitive and distributive.

**Example** Rigged surfaces form a pseudo algebra over the 2-theory of CMC's.

$I :=$ Finite Sets and bijections equipped with a choice of $\coprod$
$X : I^2 \to Cat$
$X_{a,b} :=$ the category of rigged surfaces with inbound boundary components labelled by the finite set $a$ and outbound boundary components labelled by the finite set $b$
$+ : X_{a,b} \times X_{c,d} \xrightarrow{\coprod} X_{a \coprod c, b \coprod d}$
$\check{?} : X_{a \coprod c, b \coprod c} \xrightarrow{\text{glue}} X_{a,b}$

**Theorem 6** *(F) The 2-category of pseudo* $(\Theta, T)$*-algebras admits weighted pseudo limits.*

This allows one to speak of stacks of pseudo $(\Theta, T)$-algebras, in particular stacks of pseudo commutative monoids.

# Definition of CFT

An *abstract chiral conformal field theory* on a stack of pseudo commutative monoids $\mathcal{D}$ is a Hilbert space $\mathcal{H}$ and a morphism
$\phi : \mathcal{D} \to \underline{\mathcal{H}}$ of stacks of pseudo commutative monoids over the identity map of the underlying stack of pseudo commutative monoids.

This definition accounts for the coherence isomorphisms involved in disjoint union and gluing.

# Conclusion

Certain bicategorical limits, colimits, and adjoints exist in the context of pseudo algebras over theories and 2-theories. Their existence allows one to speak of stacks in the definition of CFT formulated by Hu and Kriz.

# Questions

- What are the appropriate limits in weak $n$-categories for $n \geq 3$?

- What about $n$-stacks and CFT?