Double Categories and Pseudo Algebras

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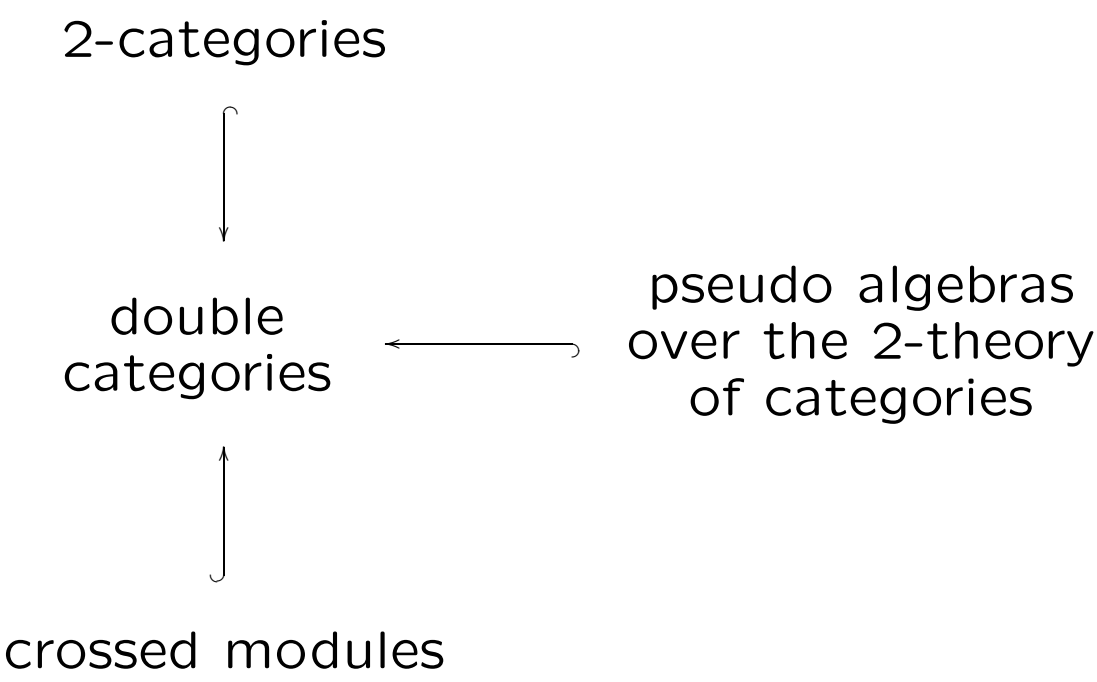
Overview


2-categories
double
categories
pseudo algebras
over the 2-theory
of categories
crossed modules

Chronology

1942-1945 Eilenberg-Mac Lane: category theory

1946, 1950 Whitehead-Mac Lane: crossed modules, homotopy 2-types

1963 Ehresmann: double categories
Lawvere: Theories

1970's R. Brown: 2-groups, crossed modules, Van Kampen Theorems

1988 Segal Bourbaki talk: a CFT "is" a cocycle for elliptic cohomology

1991 Mac Lane: coherence in CFT

2002-2005 Fiore, Hu, Kriz: pseudo algebras over theories and 2-theories as a rigorous foundation of CFT

2-Categories

Definition 1 *A 2-category* $\mathbf{C}$ *is a category enriched in categories, i.e.*

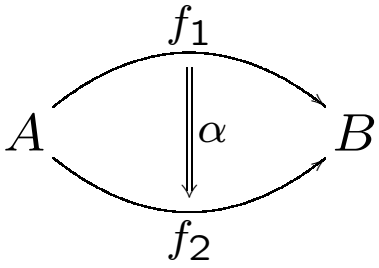
- *a set of objects* $Obj\,\mathbf{C}$
- *for each object* A *and* B *a category* $Mor_{\mathbf{C}}(A, B)$
- *composition functors*

$$Mor_{\mathbf{C}}(B, C) \times Mor_{\mathbf{C}}(A, B) \xrightarrow{\circ} Mor_{\mathbf{C}}(A, C)$$

- *identities* $1_A \in Mor_{\mathbf{C}}(A, A)$

which satisfy the usual axioms for a category.

Examples



Example 1 *Any category is a 2-category with discrete morphism categories.*

Example 2 *Topological spaces, continuous maps, homotopy classes of homotopies.*

Example 3 *Categories, functors, and natural transformations form the 2-category* Cat.

Example 4 *Rings, bimodules, bimodule maps form a bicategory.*

Double Categories

Definition 2 *(Ehresmann 1963) A double category $\mathbb{D}$ is an internal category in Cat.*

Definition 3 *A double category $\mathbb{D}$ consists of*
a set of objects,
a set of horizontal morphisms,
a set of vertical morphisms, and
a class of squares with source and target as follows

$$A \xrightarrow{f} B \qquad \begin{array}{c} A \\ {\scriptstyle j}\downarrow \\ C \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array}$$

and compositions and units that satisfy axioms.

Compositions and Units for Morphisms in a Double Category

Horizontal:

$$A \xrightarrow{f_1} B \xrightarrow{f_2} C = \begin{bmatrix} f_1 & f_2 \end{bmatrix} = f_2 \circ f_1$$

$$A \xrightarrow{1^h_A} A \xrightarrow{f_1} B = f_1 = A \xrightarrow{f_1} B \xrightarrow{1^h_B} B$$

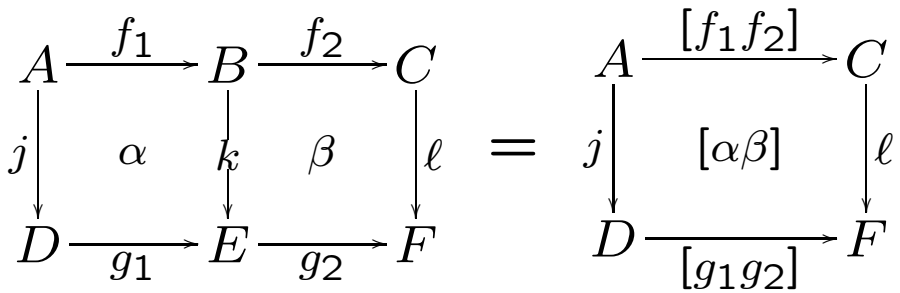
Vertical:

$$\begin{array}{c} A \\ \downarrow{\scriptstyle j_1} \\ B \\ \downarrow{\scriptstyle j_2} \\ C \end{array} = \begin{bmatrix} j_1 \\ j_2 \end{bmatrix} = j_2 \circ j_1$$

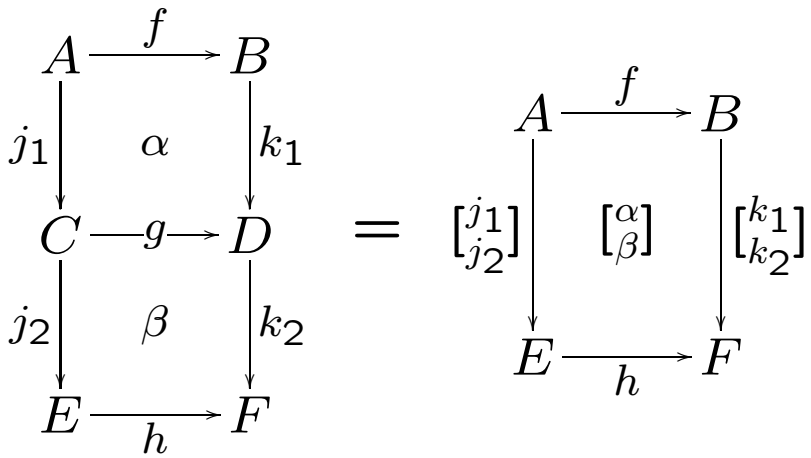
$$\begin{array}{c} A \\ \downarrow{\scriptstyle 1^v_A} \\ A \\ \downarrow{\scriptstyle j_1} \\ B \end{array} = \begin{array}{c} A \\ \downarrow{\scriptstyle j_1} \\ B \end{array} = \begin{array}{c} A \\ \downarrow{\scriptstyle j_1} \\ B \\ \downarrow{\scriptstyle 1^v_B} \\ B \end{array}$$

Compositions for Squares in a Double Category

Horizontal:

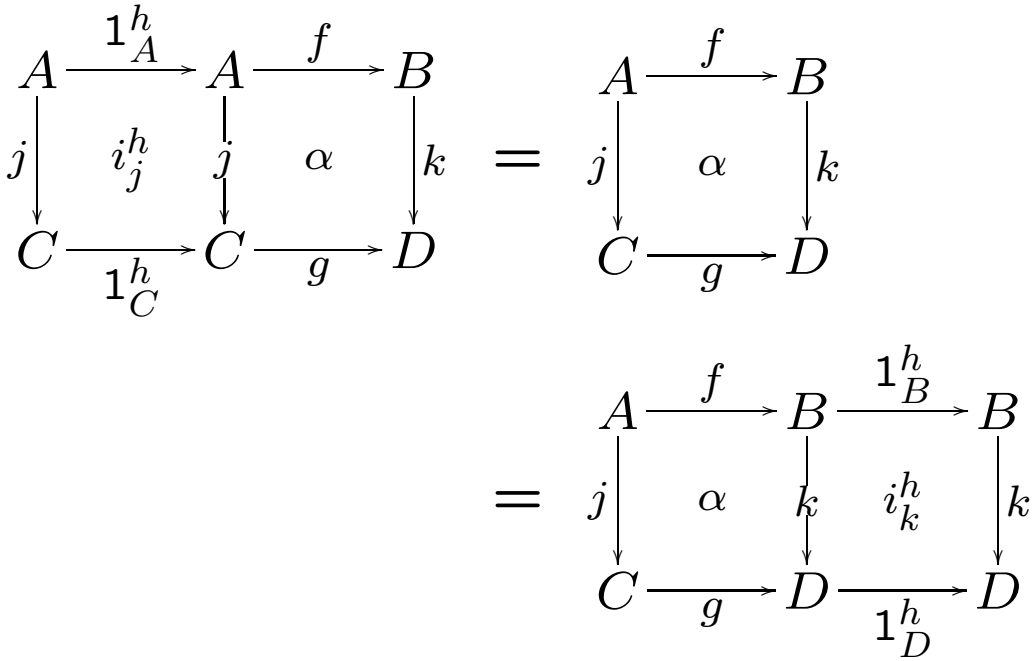


Vertical:

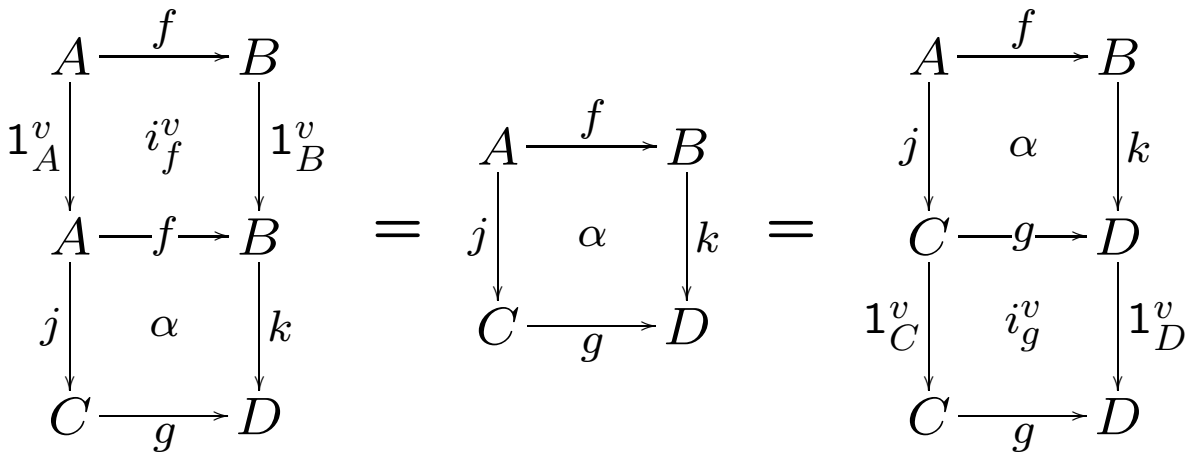


Units for Squares in a Double Category

Horizontal:



Vertical:



Axioms for a Double Category

All compositions are *associative* and *unital* (as above) and

$$\begin{bmatrix} i^h_{j_1} \\ i^h_{j_2} \end{bmatrix} = i^h_{\begin{bmatrix} j_1 \\ j_2 \end{bmatrix}}$$

$$\begin{bmatrix} i^v_{f_1} & i^v_{f_2} \end{bmatrix} = i^v_{[f_1 f_2]}.$$

Interchange Law:

If $\begin{array}{|c|c|} \hline \alpha & \beta \\ \hline \gamma & \delta \\ \hline \end{array}$, then $\begin{bmatrix} \begin{bmatrix} \alpha & \beta \end{bmatrix} \\ \begin{bmatrix} \gamma & \delta \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \alpha \\ \gamma \end{bmatrix} & \begin{bmatrix} \beta \\ \delta \end{bmatrix} \end{bmatrix}$ and

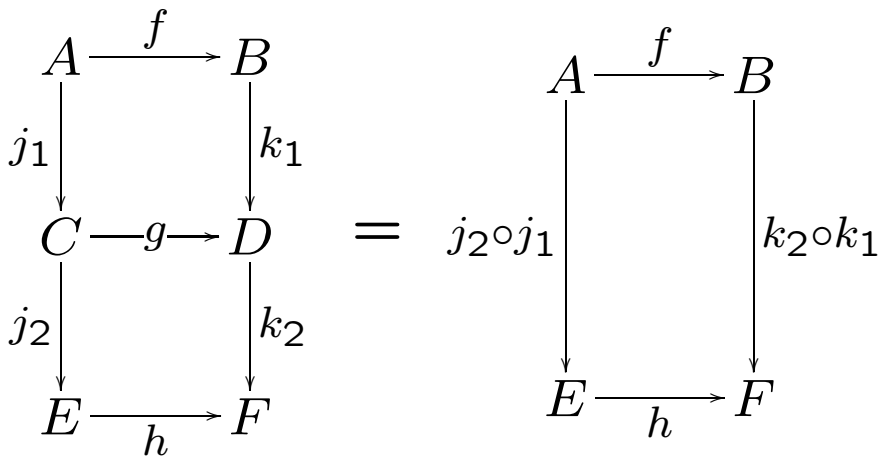
we write $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$.

Examples of Double Categories

Let I be a 1-category.

$\square I :=$ double category of commutative squares in I

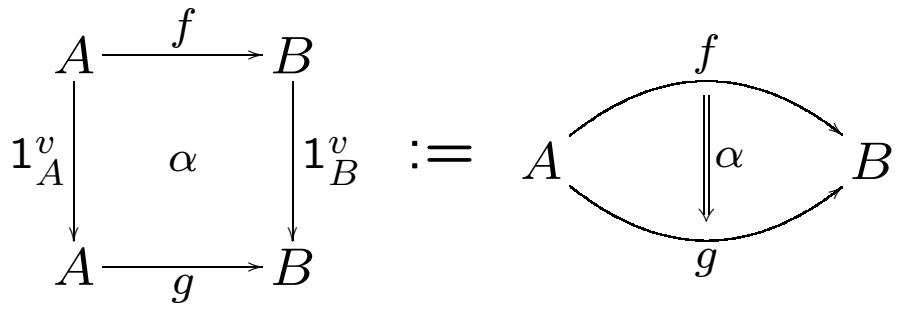
$Obj\,\square I := Obj\,I$
$Hor\,\square I := Mor\,I$
$Ver\,\square I := Mor\,I$
$Sq\,\square I :=$ commutative squares in I



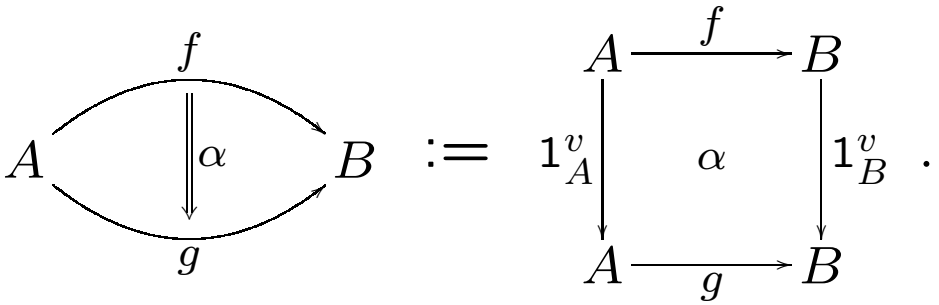
$\boxdot I :=$ double category of not necessarily commutative squares in I

Examples of Double Categories

Every 2-category $\mathbf{C}$ is a double category with trivial vertical morphisms.



Definition 4 *The horizontal 2-category* $\mathbf{H}\mathbb{D}$ *of a double category* $\mathbb{D}$ *has objects* $Obj\,\mathbb{D}$*, morphisms* $Hor\,\mathbb{D}$*, and 2-cells*



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Examples of Double Categories

Let $\mathbf{C}$ be a 2-category.

$\mathbb{Q}\mathbf{C} :=$ Ehresmann's double category of *quintets* in $\mathbf{C}$ (1963)

$Obj\,\mathbb{Q}\mathbf{C} := Obj\,\mathbf{C}$
$Hor\,\mathbb{Q}\mathbf{C} := Mor\,\mathbf{C}$
$Ver\,\mathbb{Q}\mathbf{C} := Mor\,\mathbf{C}$

$$Sq\,\mathbb{Q}\mathbf{C} := \left\{ \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array} \;\middle|\; \begin{array}{ccccc} & & k\circ f & & \\ & \nearrow & \Downarrow\alpha & \searrow & \\ A & & & & D \\ & \searrow & & \nearrow & \\ & & g\circ j & & \end{array} \right\}$$

Theorem 1 *(Grandis-Paré 2004) The functor* $\mathbb{Q} : 2\text{-}Cat \longrightarrow Dbl$ *admits a right adjoint.*

Examples of Double Categories

$\mathbb{R}$ng:= pseudo double category or rings, bimodules, and equivariant maps

$Obj\,\mathbb{R}$ng := rings with identity
$Hor\,\mathbb{R}$ng := bimodules
$Ver\,\mathbb{R}$ng := homomorphisms of rings
$Sq\,\mathbb{R}$ng :=

$$\left\{ \begin{array}{ccc} R & \xrightarrow{M} & S \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ T & \xrightarrow[N]{} & U \end{array} \;\middle|\; \begin{array}{l} \alpha : M \longrightarrow N \text{ group homomorphism} \\ \alpha(smr) = k(s)\alpha(m)j(r) \end{array} \right\}$$

Examples of Double Categories

Let C be a topological category, *i.e.* $Obj\,C$ and $Mor\,C$ are topological spaces.

$\mathbb{P}'C$:= double category of Moore paths on C.

$Obj\,\mathbb{P}'C := Obj\,C$
$Hor\,\mathbb{P}'C := Mor\,C$
$Ver\,\mathbb{P}'C := P'(Obj\,C) =$ Moore paths in $Obj\,C$
$Sq\,\mathbb{P}'C := P'(Mor\,C) =$ Moore paths in $Mor\,C$

$P'X := \{(w, s) : s \geq 0, w : [0, s] \longrightarrow X\}$

Examples of Double Categories

A *worldsheet* is a real, compact, not necessarily connected, two dimensional, smooth manifold with complex structure and real analytically parametrized boundary components.

$\mathbb{W}$:= pseudo double category of worldsheets

$Obj\,\mathbb{W}$:= finite sets
$Hor\,\mathbb{W}(A,B)$:= worldsheets with inbound components labelled by A and outbound components by B
$Ver\,\mathbb{W}$:= bijections of finite sets
$Sq\,\mathbb{W}$:=

$$\left\{ \begin{array}{c|l} \begin{array}{ccc} A & \xrightarrow{x} & B \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow[y]{} & D \end{array} & \begin{array}{l} \alpha : x \longrightarrow y \text{ holomorphic diffeo.} \\ \alpha \text{ compatible with } j \text{ and } k \\ \alpha \text{ preserves boundary params.} \end{array} \end{array} \right\}$$

Folding Structures

We introduce folding structures to compare algebras over the 2-theory of categories with double categories.

Definition 5 *A holonomy on a double category $\mathbb{D}$ is a 2-functor*

$$(\mathbf{V}\mathbb{D})_0 \longrightarrow \mathbf{H}\mathbb{D}$$

$$A \longmapsto \overline{A} = A$$

$$\begin{array}{c} A \\ \downarrow j \\ B \end{array} \longmapsto A \xrightarrow{\overline{j}} B$$

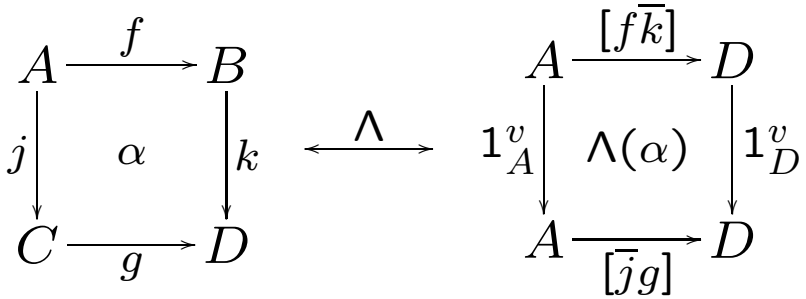
Example 5 *For a topological category C, a holonomy*

$$(\mathbf{V}\mathbb{P}'C)_0 \longrightarrow \mathbf{H}\mathbb{P}'C$$

assigns to a path of objects a morphism from the initial point to the terminal point, like in differential geometry.

Folding Structures

Definition 6 *A folding structure on a double category $\mathbb{D}$ consists of a holonomy $j \longmapsto \overline{j}$ and bijections*



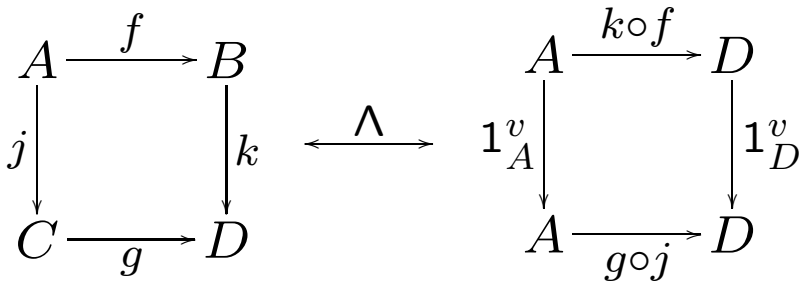
compatible with compositions and units.

A folding structure *horizontalizes* a double category.

Examples of Folding Structures

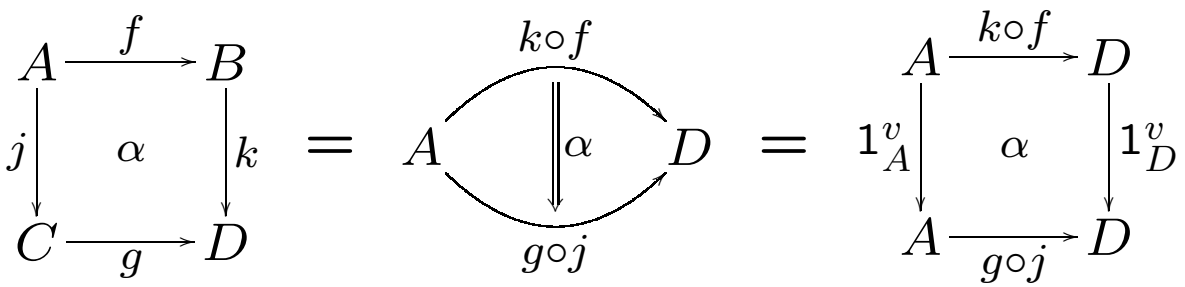
Let I be a 1-category.
$\square I$=double category of commutative squares in I
$\boxdot I$=double category of not necessarily commutative squares in I

Then $\square I$ and $\boxdot I$ each admit a unique folding structure.



Examples of Folding Structures

Let $\mathbf{C}$ be a 2-category. Then $\mathbb{Q}\mathbf{C}$ admits a folding structure by definition.



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Holonomy:

$$\overline{j} := T_j = \text{ the } (T,R)\text{-module } T$$

Folding:

$$\Lambda(\alpha) : U_k \otimes_S M \Longrightarrow N \otimes_T T_j$$

$$u \otimes m \longmapsto (u \cdot \alpha(m)) \otimes 1_T$$

Examples of Folding Structures

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Holonomy:

$$\text{bijection} \quad \longmapsto \quad \begin{array}{c} \text{labelled union of} \\ \text{infinitely thin annuli} \end{array}$$

Folding:

relabel x and y

Comparison Theorems

Observation 2 *(Brown-Mosa 1999, F. 2006) The notions of folding structure and connection pair are equivalent.*

Theorem 3 *(F. 2006) The 2-category of strict 2-algebras over the 2-theory of categories is 2-equivalent to the 2-category of double categories with folding structures and invertible vertical morphisms.*

The pseudo version of the theorem also holds.

Strict 2-Algebras

Definition 7 *A strict 2-algebra over the 2-theory of of categories is a groupoid I and a strict 2-functor $X : I^2 \longrightarrow Cat$ with strictly 2-natural functors*

$$X_{B,C} \times X_{A,B} \xrightarrow{\circ} X_{A,C}$$

$$\{*\} \xrightarrow{1_B} X_{B,B}$$

for all $A, B, C \in I$ which satisfy axioms like those of a category.

Towards Strict 2-Algebras and Crossed Modules

Consider one object cases.

groupoids	$\subseteq$ 2-groupoids	$\subseteq$ double groupoids
groups	$\subseteq$ 2-groups	$\subseteq$ double groups

Theorem 4 *(Verdier, Brown-Spencer 1976,...) 2-groups are equivalent to crossed modules.*

Theorem 5 *(Brown-Spencer 1976) Edge symmetric double groups with folding structure with trivial holonomy are equivalent to crossed modules.*

Question: What is a one object strict 2-algebra over the 2-theory of categories with everything iso?

Crossed Modules

Definition 8 *A crossed module is a group homomorphism* $\partial : H \longrightarrow G$ *with a left action*

$$G \times H \longrightarrow H$$

$$(g, \alpha) \mapsto {}^{g}\alpha$$

by automorphisms such that:

1. $\partial({}^{g}\alpha) = g\partial(\alpha)g^{-1}$ *for all* $\alpha \in H$ *and* $g \in G$

2. ${}^{\partial(\alpha)}\alpha_1 = \alpha\alpha_1\alpha^{-1}$ *for all* $\alpha, \alpha_1 \in H$.

Example 6 $H \lhd G$ *is a crossed module*

Crossed Modules

Example 7 *Let* $(X, A, *)$ *be a based pair of spaces. Then*

$$\partial : \pi_2(X, A, *) \longrightarrow \pi_1(A, *)$$

is a crossed module.

Theorem 6 *(Mac Lane-Whitehead 1950) Crossed modules model homotopy 2-types via this example.*

Crossed Modules with Group Action

Definition 9 *Let $\partial : H \longrightarrow G$ be a crossed module and I a group. An action of I on the crossed module $\partial : H \longrightarrow G$ consists of*

- *a left action of I on H by automorphisms written $(j, \alpha) \mapsto {}^{j}\alpha$*

- *a left action of I on G as a set written $(j, g) \mapsto jg$*

- *a right action of I on G as a set written $(g, j) \mapsto gj$*

Crossed Modules with Group Action

These actions satisfy the following axioms for all $j, k \in I, \alpha \in H, g, g_1, g_2 \in G$.

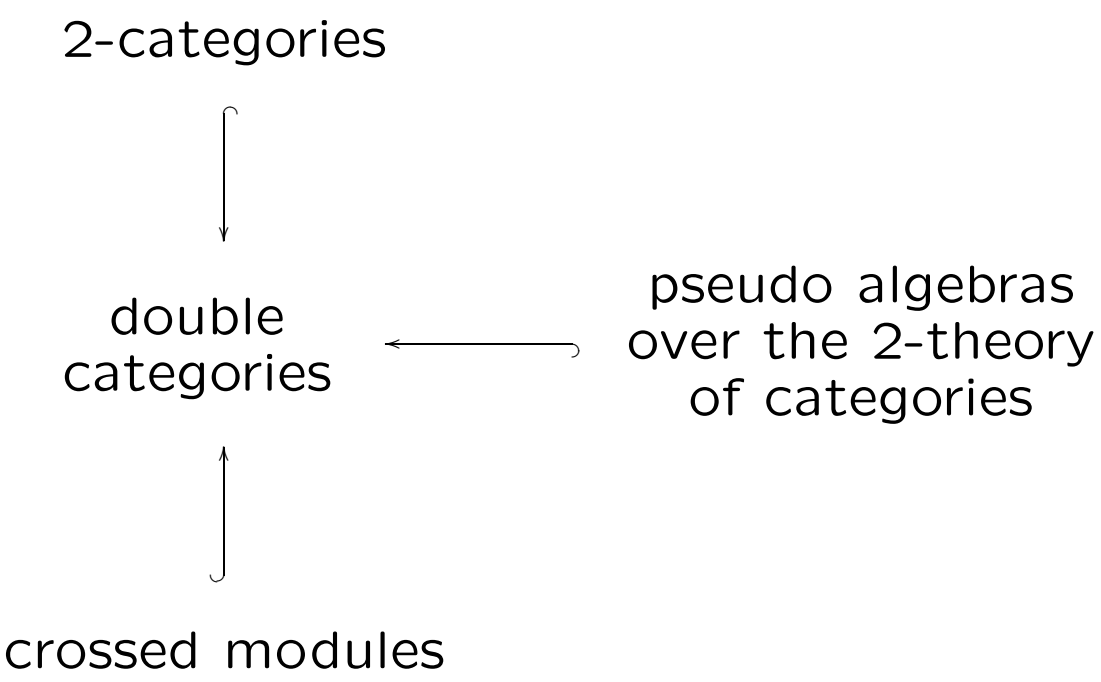
1. $(jg)k = j(gk)$
2. $(jg_1)g_2 = j(g_1g_2)$
3. $(g_1g_2)k = g_1(g_2k)$
4. $(g_1j)g_2 = g_1(jg_2)$
5. $j1_G = 1_Gj$
6. ${}^{(gk)}\alpha = {}^{g}({}^{k}\alpha)$
7. ${}^{(jg)}\alpha = {}^{j}({}^{g}\alpha)$
8. $\partial({}^{j}\alpha) = j\partial(\alpha)j^{-1}$

Strict 2-Algebras and Crossed Modules

Theorem 7 *(F. 2006) One object strict 2-algebras over the 2-theory of categories with everything iso are equivalent to crossed modules with a group action.*

In the case of a trivial group and trivial I, this says 2-groups are equivalent to crossed modules.

Conclusion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