Music and Mathematics

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Introduction

Music and Mathematics are intricately related. Strings vibrate at certain frequencies. Sound waves can be described by mathematical equations. The cello has a particular shape in order to resonate with the strings in a mathematical fashion. The technology necessary to make a digital recording on a CD relies on mathematics. After all, mathematics is the language that physicists utilize to describe the natural world and all of these things occur in the natural world. Not only do physicists, chemists, and engineers use math to *describe* the physical world, but also to *predict* the outcome of physical processes.

Can one similarly find an "equation" to *describe* a piece of music? Or better yet, can one find an "equation" to *predict* the outcome of a piece of music? We can model sound by equations, so can we also model works of music with equations? Music is after all just many individual sounds, right? Should we invest time and money to find these equations so that all of humankind can enjoy predictable, easily described music?

The answer to all of these questions is predictable and easily described: a series of emphatic "NO's"! There is not an equation that will model all works of music and we should not spend time looking for it. Nevertheless, there are certain mathematical structures inherent in all works of music, and these mathematical structures are not given by equations. The language of mathematics is a convenient tool for comprehending and communicating this underlying structure.

In fact, one of the central concerns of music theory is to find a *good* way to hear a piece of music and to communicate that way of hearing.[1] Anyone who has ever heard Stockhausen's *Klavierstück III* (1952) knows that this is not always so easy to do! On a higher level, the eighteenth century Scottish philosopher David Hume said that the mind receives impressions and once these impressions become tangible and vivacious, they become ideas. Music theory supplies us with conceptual categories to organize and understand music. Our aural impressions become vivacious ideas by way of these conceptual categories. To find a good way of hearing a musical piece means to comprehend the music in such a way as to make it tangible.

Music theorists often draw on the formidable powers of mathematics in their creation of conceptual categories. The discrete whole numbers $\ldots, -2, -1, 0, 1, 2 \ldots$ are particularly well suited for labelling the pitches, or the keys of the piano. The area of mathematics called combinatorics enables one to count the many ways of combining pitches, *i.e.* numbers. This provides taxonomies and classifications of the various sets that arise. Group theory, another area of pure mathematics, describes the ways that sets and pitches relate and how they can be transformed from one to

[1] John Rahn. *Basic Atonal Theory.* New York: Schirmer Books, 1980. See page 1.

the other. It is in this sense that pure mathematics provides a convenient framework for the music theorist to communicate good ways of hearing a work of music.

Music theory is not just for the listener. Music theory is also useful to the composer. Bach, Mozart, and Beethoven were well versed in the music theory of their respective epochs and applied it daily. This abstract discussion has relevance for the performing artist as well. A classical pianist may play thousands upon thousands of notes in one concert, all from memory. How does a classical pianist do that? Is it necessary for the classical pianist to memorize each individual note? Of course not. Music theory provides the performing artist with an apparatus for pattern recognition, and this strengthens the musical memory. A piece of music does not consist of many individual isolated sounds, but rather several ideas woven together. One can find the thread of the musical fabric with music theory. On the other hand, this same musical memory allows the listener to conceive of a piece of music as a whole, rather than isolated individual events. Music theory is not limited to classical music. Jazz musicians also use music theory in their improvisations and compositions. Non-western music also lends itself to analysis within the framework of music theory.

In this module we investigate some of the group theoretical tools that music theorists have developed in the past 30 years to find good ways of hearing particular works of music. Group theory does *not* provide us with equations to describe a piece of music or predict its outcome. Instead, it is just one conceptual category that listeners, composers, and performers can use to make sense of a work of art and to communicate ways of hearing to others. The aural impression of a piece of music becomes an idea in the sense of Hume with this apparatus.

In the next two lectures we will study the T/I and PLR groups and use them to analyze works of music from Johann Sebastian Bach, Ludwig van Beethoven, Richard Wagner, and Paul Hindemith. This group theoretical point of view will elevate our aural impressions to the status of ideas. We will conceive of the music not as individual sounds, but at as a whole. The mathematical framework will provide us with a way of hearing the pieces and a means of communicating this hearing.

These notes were prepared for a series of three guest lectures in the undergraduate course Math 107 at the University of Michigan under the direction of Karen Rhea in Fall 2003. These lectures were part of my Fourth Year VIGRE Project aimed at introducing undergraduates to my interdisciplinary research on generalized contextual groups with Ramon Satyendra. I extend my thanks to Ramon Satyendra of the University of Michigan Music Department for helpful conversations in preparation of this module.

Lecture 1 Transposition and Inversion

1. Introduction

Some of the first mathematical tools that music students learn about are transposition and inversion. In this introductory lecture we learn about the mathematical concepts necessary to formalize these musical tools. These concepts include set, function, and modular arithmetic. Musicians are usually come into contact with transposition and inversion in the context of pitch. To bridge the gap between sound and number we will conceive of the integer model of pitch as musicians normally do. After the mathematical discussion we turn to examples from Bach and Wagner.

2. Mathematical Preliminaries

The mathematical concepts of set, function, domain, range, and modular arithmetic will be needed for our discussion.

2.1. Sets and Functions.

DEFINITION 2.1. A *set* is a collection of objects. The collection of objects is written between curly parentheses $\{\}$. The objects of the set are called *elements*. Two sets are said to be *equal* if they have the same elements.

For example, the set $\{4, 5, 10\}$ is the set consisting of the numbers $4, 5, 10$ and nothing else. The set $\{A\flat, B\flat\}$ is the set whose elements are the two pitch classes $A\flat$ and $B\flat$. The symbols $\}6, 100, 11, 5\}$ do not denote the set whose elements are the numbers $6, 100, 11, 5$ because one of the parentheses is the wrong way. The order of the elements of a set does not matter. For example,

$$\{4, 5, 10\} = \{5, 4, 10\} = \{10, 4, 5\}.$$

The sets $\{56, 70\}$ and $\{56\}$ are not equal because they do not have the same elements (the second set is missing 70).

A function gives us a rule for getting from one set to another.

DEFINITION 2.2. A *function* f from a set S to a set S' is a rule which assigns to each element of S a unique element of S'. This is usually denoted by $f : S \to S'$. This symbolism is read: "f goes from S to S'." In this situation, the set S is called the *domain* of the function f and S' is called the *range* of the function f. The *inputs* are the elements of the domain. The *outputs* are the elements of range.

For example, consider the function $f : \{1, 2, 3\} \to \{4, 5\}$ defined by

$$f(1) = 4$$

$$f(2) = 5$$

$$f(3) = 4.$$

Here the domain is the set $S = \{1, 2, 3\}$ and the range is the set $S' = \{4, 5\}$. These three equations tell us the rule that assigns an output to each input. Note that each element $1, 2, 3$ of the domain gets a unique output, *i.e.* $f(1)$ does not have two different values. In precalculus we like to say that the function passes the vertical line test. The definition

$$g(1) = 4$$
$$g(1) = 5$$
$$g(2) = 4$$
$$g(3) = 4$$

does not define a function $g : \{1, 2, 3\} \to \{4, 5\}$ because two different values 4 and 5 are assigned to 1. The definition

$$h(1) = 4$$
$$h(2) = 4$$
$$h(3) = 4$$

does define a function $h : \{1, 2, 3\} \to \{4, 5\}$ because a unique output is assigned to each input. Note that the same output is assigned to each input and 5 is not even used.

Functions can be composed, provided the domain of one is the range of the other. For example, if $j : \{4, 5\} \to \{7, 8, 9\}$ is given by the rule

$$j(4) = 9$$
$$j(5) = 8$$

and f is as above, then we get a new function $j \circ f : \{1, 2, 3\} \to \{7, 8, 9\}$ called the *composition of j with f* defined by $j \circ f(x) = j(f(x))$. In this example we have

$$j \circ f(1) = j(f(1)) = j(4) = 9$$
$$j \circ f(2) = j(f(2)) = j(5) = 8$$
$$j \circ f(3) = j(f(3)) = j(4) = 9.$$

Sometimes functions are given by formulas rather than tables. In the next section we will see some functions that are given by formulas.

2.2. Modular Arithmetic.

Consider the face of a clock with the numbers 0 through 11 where 0 is in the 12 o'clock position. The day starts at midnight, so we have replaced 12 by 0 on the usual face of a clock. Using the clock, we can determine what time it is 2 hours after 1. We just go clockwise 2 notches after 1 and we get 3. Similarly 5 hours after 6 is 11. But what about 1 hour after 11? Well, that is 0 because we are back at the beginning. Similarly, 2 hours after 11 is 1. This is called *arithmetic modulo 12.* Summarizing, we can write

$$1 + 2 = 3 \text{ mod } 12$$
$$6 + 5 = 11 \text{ mod } 12$$
$$11 + 1 = 0 \text{ mod } 12$$
$$11 + 2 = 1 \text{ mod } 12.$$

Whenever it is clear that one is working mod 12, we just leave off the suffix mod 12. So we have just done addition mod 12, let's consider subtraction. If we bring some of the numbers over to the right like usual arithmetic, we get

$$1 = 3 - 2$$
$$6 = 11 - 5$$
$$11 = 0 - 1$$
$$11 = 1 - 2$$

where we leave off mod 12 because it is clear by now that we are talking about arithmetic modulo 12 in this paragraph and not arithmetic modulo 7. The first two equations appear to make sense to us from usual arithmetic. But to make sense of the last two equations, we need to consider the face of the clock. If we are at 0 o'clock and go back 1 hour, then we are at 11 o'clock. Similarly, if we are at 1 o'clock and go back to hours, we just go counterclockwise two notches on the face of the clock and arrive at 11 o'clock. That is why $11 = 0 - 1$ and $11 = 1 - 2$.

Given any number, we can find out what it is mod 12 by adding or subtracting 12 enough times to get a whole number between 0 and 11. For example,

$$-12 = 0 = 12 = 24$$
$$-13 = -1 = 11 = 23$$
$$-7 = 5 = 17 = 29.$$

As a result, mathematicians and musicians use the notation

$$\mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$$

and call this set the *set of integers mod 12*.

Maybe you can guess that we are interested in integers mod 12 because there are twelve keys on the piano from middle C to the next C not counting the last C. All of this has an aural foundation. Our ears naturally hear pitches that are an octave apart, *i.e.* pitches with 12 jumps (or intervals) between them on the piano. Such pitches appear to be very similar for our ears. So in a sense, our human ears are hardwired for arithmetic mod 12!

We are also interested in *arithmetic modulo 7* because there are seven pitches in the major scale, *i.e.* seven white keys on the piano from middle C to the next C. For arithmetic modulo 7 we imagine there are seven hours in a day and that the face of the clock goes from 0 to 6 instead of 0 to 11. Arguing as in the mod 12 case, we have

$$1 + 2 = 3 \bmod 7$$
$$6 + 5 = 4 \bmod 7$$
$$6 + 1 = 0 \bmod 7$$
$$6 + 2 = 1 \bmod 7.$$

We already see a difference between mod 12 and mod 7. Notice that
6 +5 =11 mod 12 but 6+5=4 mod 7. Let's consider some examples of subtraction. If we bring some of the numbers to the right like in usual arithmetic, we get

$$1 = 3 - 2 \bmod 7$$
$$6 = 4 - 5 \bmod 7$$
$$6 = 0 - 1 \bmod 7$$
$$6 = 1 - 2 \bmod 7.$$

Subtraction can be understood by moving counterclockwise on the face of a clock with seven hours labelled $0, 1, \ldots, 6$.

Given any number, we can find out what it is mod 7 by adding or subtracting 7 enough times to get a number between 0 and 6. For example,

$$-7 = 0 \bmod 7$$

$$-9 = -2 = 5 \bmod 7$$

$$15 = 8 = 1 \bmod 7$$

$$17 = 10 = 3 \bmod 7.$$

As a result, mathematicians and musicians use the notation

$$\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$$

and call this set the *set of integers mod 7.*

Next we can talk about functions $f : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$. These are functions whose inputs are integers modulo 12 and whose outputs are also integers modulo 12. Let's consider the function $T_2 : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ defined by the formula $T_2(x) = x + 2$. Recall that we considered functions given by tables in Subsection 2.2 above. We can make a table for this function as follows.

$$T_2(0) = 0 + 2 = 2 \bmod 12$$

$$T_2(1) = 1 + 2 = 3 \bmod 12$$

$$T_2(2) = 2 + 2 = 4 \bmod 12$$

$$T_2(3) = 3 + 2 = 5 \bmod 12$$

$$T_2(4) = 4 + 2 = 6 \bmod 12$$

$$T_2(5) = 5 + 2 = 7 \bmod 12$$

$$T_2(6) = 6 + 2 = 8 \bmod 12$$

$$T_2(7) = 7 + 2 = 9 \bmod 12$$

$$T_2(8) = 8 + 2 = 10 \bmod 12$$

$$T_2(9) = 9 + 2 = 11 \bmod 12$$

$$T_2(10) = 10 + 2 = 0 \bmod 12$$

$$T_2(11) = 11 + 2 = 1 \bmod 12.$$

Another function $I_0 : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ is given by the formula $I_0(x) = -x$. For example $I_0(1) = 11$ and $I_0(6) = 6$.

This concludes the introduction of sets, functions, and modular arithmetic necessary for an understanding of transposition and inversion. Finally we can turn to some music.

3. The Integer Model of Pitch

To make use of the mathematical ideas developed in the last section, we need to translate pitch classes into numbers and introduce transpositions and inversions.[2] If you can't read music, don't panic, just use the integers modulo 12. If you can read music, then the following well established dictionary shows us how to get from pitches to integers modulo 12.

$$\begin{aligned}
&C = 0\\
&C\sharp = D\flat = 1\\
&D = 2\\
&D\sharp = E\flat = 3\\
&E = 4\\
&F = 5\\
&F\sharp = G\flat = 6\\
&G = 7\\
&G\sharp = A\flat = 8\\
&A = 9\\
&A\sharp = B\flat = 10\\
&B = 11
\end{aligned}$$

In this integer model of pitch, the C major chord $\{C, E, G\}$ is $\{0, 4, 7\}$. This C major chord is part of the main theme for Haydn's Surprise Symphony. The first part of the main theme is $\langle C, C, E, E, G, G, E, F, F, D, D, B, B, G\rangle$, which can be written $\langle 0, 0, 4, 4, 7, 7, 4, 5, 5, 2, 2, 11, 11, 7\rangle$. These angular brackets $\langle\rangle$ are often used by music theorists to emphasize that the notes occur in this order. Recall that the ordering does not matter for sets, because a set is just a collection of elements. Unordered sets, e.g. $\{0, 4, 7\}$, are sometimes called *pcsets* (pitch class sets) while ordered sets such as $\langle 0, 0, 4, 4, 7, 7, 4, 5, 5, 2, 2, 11, 11, 7\rangle$ are called *pcsegs* (pitch class segments).

Transpositions and inversions are functions $\mathbb{Z}_{12} \to \mathbb{Z}_{12}$ that are useful to every musician. There are also analogues for $\mathbb{Z}_7$. Transposition and inversion are often applied to melodies, although they can also be applied to chords. When we hear a melody consisting of several pitches, we hear the intervals between the individual notes. The relationship between these intervals is what makes a melody appealing to us. Transposition mathematically captures what musicians do all the time: the restatement of a melody at higher and lower pitch levels in a way that preserves intervals. Inversion is another way to create musical variation while preserving the intervallic sound of a melody, although it does not preserve the exact intervals.

Definition 3.1. Let n be an integer mod 12. Then the function $T_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ defined by the formula $T_n(x) = x + n \bmod 12$ is called *transposition by n.*

[2] In music theory, particularly in atonal theory, it is common to study pitch classes rather than pitches. One can see the difference between pitch classes and pitches in the following example. Middle C is a particular pitch, although the *pitch class C* refers to the aggregate of all keys on the piano with letter name C.

We already came into contact with $T_2 : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ in the previous section. Some examples for $T_5 : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ are

$$T_5(3) = 3 + 5 = 8$$
$$T_5(6) = 6 + 5 = 11$$
$$T_5(7) = 7 + 5 = 0$$
$$T_5(10) = 10 + 5 = 15 = 3$$

where we have not written mod 12 because it is clear from the context.

Definition 3.2. Let n be an integer mod 12. Then the function $I_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ defined by the formula $I_n(x) = -x + n$ is called *inversion about* n.

We already came into contact with $I_0 : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ in the previous section. Some examples for $I_7 : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ are

$$I_7(3) = -3 + 7 = 4$$
$$I_7(7) = -7 + 7 = 0$$
$$I_7(9) = -9 + 7 = -2 = 10.$$

The function I_n is called inversion about n because it looks like a reflection about n whenever one draws the number line.

Music theorists and composers like to transpose and invert entire pcsets or pcsegs by applying the function to each element. For example, we can transpose a C major pcset by 7 steps as in

$$T_7\{0, 4, 7\} = \{T_7(0), T_7(4), T_7(7)\} = \{0 + 7, 4 + 7, 7 + 7\} = \{7, 11, 2\}$$

by applying T_7 to each of 0,4, and 7. A musician would notice that this takes the C major chord to the G major chord. Similarly, we could invert the pcseg for the theme of Haydn's Surprise Symphony about 0, although Haydn did not do this!

$$I_0\langle 0, 0, 4, 4, 7, 7, 4, 5, 5, 2, 2, 11, 11, 7\rangle = \langle 0, 0, 8, 8, 5, 5, 8, 7, 7, 10, 10, 1, 1, 5\rangle$$

In this section we have introduced the integer model of pitch, which assigns to each of the 12 pitch classes an integer mod 12. The transpositions and inversions are functions which have inputs and outputs that are pitches. These are conceptual categories that music theorists use to find good ways of hearing pieces. Next we use them to find good ways of hearing a fugue and a prelude.

4. Fugue by Bach

Johann Sebastian Bach (1685-1750) took the art of fugue to new heights. He composed the *Well-Tempered Clavier Book I* and the *Well-Tempered Clavier Book II*, each of which contains 24 preludes and fugues. A fugue usually begins with a statement of the main theme called the *subject*. This subject returns over and over again in various voices and usually they are thread together in complex way. Every fugue has occurrences of transposition and inversion. A truly fascinating website

$$http://jan.ucc.nau.edu/\sim tas3/bachindex.html$$

on Bach describes in detail what a fugue is. Click on the link for movies on the *Well-Tempered Clavier*. There is an animation and recording for Fugue 6 in d minor of the *Well-Tempered Clavier Book I*, which we now analyze. Our analysis will be restricted to finding some transpositions and inversions, since we are only studying some of the mathematical structure. We'll leave detailed analysis to the music theorists.

The subject of the fugue is the pcseg

$$\langle D, E, F, G, E, F, D, C\sharp, D, B\flat, G, A\rangle = \langle 2, 4, 5, 7, 4, 5, 2, 1, 2, 10, 7, 9\rangle$$

which begins in measure 1 and lasts until the beginning of measure 3. Let's call this pcseg P. See the score that you got in class. Interestingly enough, this subject consists of twelve notes! In measure 3, another voice sings the melody

$$\langle A, B, C, D, B, C, A, G\sharp, A, F, D, E\rangle = \langle 9, 11, 0, 2, 11, 0, 9, 8, 9, 5, 2, 4\rangle.$$

Do you see a relationship between this pcseg and P? Notice that this pcseg is T_7P! Just try adding 7 to each element of P and you will see it. In measure 6, the subject returns in the exact same form as the introduction, just one octave lower. At measure 8, a form

$$\langle E, F, G, A, F, B\flat, G, F\sharp, G, E\flat, C\sharp, D\rangle = \langle 4, 5, 7, 9, 5, 10, 7, 6, 7, 3, 1, 2\rangle$$

of the subject enters. This one doesn't entirely match though. The first five pitches are almost T_2 of the first five pitches of P, but the next 5 pitches are T_5 of the respective pitches of P. The last pitch of the pcseg is also T_5 of the respective pcseg of P, but the eleventh pitch doesn't match. At measure 13 we have

$$\langle A, B, \mathbf{C\sharp}, D, B, \mathbf{C\sharp}, A, G\sharp, A, F, D, E\rangle = \langle 9, 11, \mathbf{1}, 2, 11, \mathbf{1}, 9, 8, 9, 5, 2, 4\rangle.$$

This is similar to T_7P as in measure 3, except for the highlighted 1's. Measures 17,18, and 21 are respectively

$$\langle A, B, C, D, B, \mathbf{C\sharp}, A, G\sharp, A, F, D, E\rangle = \langle 9, 11, 0, 2, 11, \mathbf{1}, 9, 8, 9, 5, 2, 4\rangle$$

$$\langle A, B, \mathbf{C\sharp}, D, B, C, A, G\sharp, A, F, D, E\rangle = \langle 9, 11, \mathbf{1}, 2, 11, 0, 9, 8, 9, 5, 2, 4\rangle$$

$$\langle A, B, \mathbf{C\sharp}, D, B, \mathbf{C\sharp}, A, G\sharp, A, F, D, E\rangle = \langle 9, 11, \mathbf{1}, 2, 11, \mathbf{1}, 9, 8, 9, 5, 2, 4\rangle.$$

These are also T_7P except for the highlighted 1's. The interval 7 is very important in western music and is called the perfect fifth. Here we see that transposition by a perfect fifth occurs four times before the piece is even half over. In fact, many fugues have this property. So far we have seen that transposition plays a role in this piece. But what about inversion?

Let's consider measures 14 and 22. They are respectively

$$\langle E, D, C\sharp, B, D, C\sharp, E, F, E, \mathbf{A}, \mathbf{C}, \mathbf{B\flat}\rangle = \langle 4, 2, 1, 11, 2, 1, 4, 5, 4, \mathbf{9}, \mathbf{0}, \mathbf{10}\rangle$$

$$\langle E, D, C\sharp, B, D, C\sharp, E, F, E, \mathbf{G}, \mathbf{B\flat}, \mathbf{A}\rangle = \langle 4, 2, 1, 11, 2, 1, 4, 5, 4, \mathbf{7}, \mathbf{10}, \mathbf{9}\rangle.$$

They are nearly identical, except for the last three digits. Notice also that the first two elements E, D are the same first two elements of P, just the order is switched. The last three notes of 22 are even the last three notes of P, just the order is switched. Calculating I_6P gives

$$\langle 4, 2, 1, 11, 2, 1, 4, 5, 4, 8, 11, 9\rangle$$

which is a near perfect fit with measures 14 and 22! Just the last three notes are changed to make it sound better. So we see that inversion does indeed play a role in the piece. The rest of the piece contains further transpositions and inversions of the subject.

Next time we listen to the piece, we can listen for these transposed and inverted forms of the subject. These conceptual categories make the piece more enjoyable for listeners because we come closer to understanding it. We have a good way of hearing the piece. This knowledge also makes the piece easier for performers because they recognized patterns and relationships between different parts of the piece. However, a music theorist would not be satisfied with this analysis because we have barely scratched the surface. There is much more to this fugue than a few transpositions and inversions. Nevertheless, this illustrates some of the mathematical features of the piece.

5. Tristan Prelude from Wagner

Richard Wagner (1813-1883), who was born 63 years after the death of Bach, is best known for his gigantic operas. Wagner's compositions are drastically different from Bach's. We take the prelude to the famous opera *Tristan and Isolde* as an example for transposition and inversion. This particular passage is notorious for its resistance to traditional analysis.[3] More modern methods of atonal analysis, which use transposition and inversion, are more fruitful. In this analysis we consider unordered sets, *i.e.* pcsets, although we worked with pcsegs in the previous example.[4]

Consider the piano transcription of the first few measures of the prelude. The piano transcription entitled "Wagner: Tristan Prelude" is in the packet of music I handed out in class. Let P_i denote the set of pitch classes that are heard during the circle i on the piano transcription. For example, $P_2 = \{F, B, D\sharp, A\}$. Then we notice the following pattern after looking very carefully.

$$\begin{array}{cccc}
P_1 & P_2 & P_3 & P_4 \\
P_5 & P_6 & P_7 & P_8 \\
P_9 & & P_{12} & P_{13} \\
\{0,2,5,8\} & \{0,2,6,8\} & \{0,2,6,8\} & \{0,2,5,8\}
\end{array}$$

This table means that all of the pcsets in the first column can be transposed or inverted to $\{0, 2, 5, 8\}$, all of the pcsets in the second column can be transposed or inverted to $\{0, 2, 6, 8\}$, etc. Notice that the first and last column are essentially the same, while the middle two columns are essentially the same! Here, essentially means they can be transposed or inverted to the same thing.

Notice also that everything is done according to the groups of circled notes in the music, and we almost have three groups of four, which would give us 12 again! The first and last pitches of each four note group, namely $G\sharp - B$, $B - D$, and $D - F\sharp$, also form a set that can be transposed or inverted to $\{0, 2, 5, 8\}$!

In other words, we mathematically see and musically hear a self similarity on different levels. When we listen to the piece again, we can listen for these features. The conceptual categories of transposition and inversion provide us with a good way of hearing these introductory measures to Wagner's opera *Tristan and Isolde*. Mathematics is the tool that we use to communicate this way of hearing to others.

[3]This analysis is obtained from John Rahn page 78, who in turn quotes Benjamin Boretz.

[4]What works for one piece of music may not work for another. In the Bach fugue it was better to use pcsegs because the pcsets would tell us very little in that case. However, pcsets are more appropriate for the Tristan prelude than pcsegs.

Lecture 1 Homework Problems

1. Introduction

1. Reread the introduction to this music module. Is music part of the physical world? Write a short paragraph on this topic.

2. Mathematical Preliminaries

2.1. Sets and Functions.

2. Give three examples of sets that are not listed in the text.

3. Give two functions whose domain is $\{5, 4, 7\}$ and whose range is $\{1, 2\}$. You will probably want to use tables.

4. Is there a function with $f(4) = 5$ and $f(4) = 7$?

2.2. Modular Arithmetic.

5. Do the following calculations mod 12. Your answers should be numbers between 0 and 11. The numbers 0 and 11 may also be answers!

$$7 + 5$$
$$1 + 4$$
$$8 + 8$$
$$6 + 6$$
$$9 - 7$$
$$7 - 9$$
$$2 - 8.$$

3. The Integer Model of Pitch

6. Use the integer model of pitch to rewrite the following melody in "Heavenly Aida" in Act I of Verdi's opera *Aida*: $\langle G, A, B, C, D, G, G\rangle$.

7. Use the integer model of pitch to rewrite the following melody in the "Toreador Song" in Act II of Bizet's opera *Carmen*: $\langle C, D, C, A, A, A, G, A, B\flat, A, B\flat, G, C\rangle$.

8. Calculate $T_4(3), T_1(2), T_8(7), I_4(6), I_4(8)$.

9. Transpose the melody above from "Heavenly Aida" a perfect fifth by applying T_7 to each element.

10. Invert the melody above in the "Toreador Song" about 6 by applying I_6 to each element.

11. Calculate $T_5 \circ I_3(4)$ and $I_3 \circ T_5(4)$. Are they the same?

4. Fugue by Bach

12. Look at the website on Bach listed in the text.

5. Tristan Prelude from Wagner

Challenge: The unordered pcsets P_9 and P_{13} on the piano transcription of the *Tristan* Prelude are

$$P_9 = \{C, F, G\sharp, D\}$$
$$P_{13} = \{B, D\sharp, A, F\sharp\}.$$

Translate these pcsets to integers mod 12. Find an integer n such that $I_n(P_9) = P_{13}$. You may have to change the ordering of the elements of the set.

Addendum to Lecture 1

1. Introduction

We consider a further application of transposition and inversion in music theory. Thus far we have considered only composers who have lived before the 20th century, so it is high time we consider someone who is closer to our time.

2. Fugue by Hindemith

Paul Hindemith (1895-1963) was known as a champion of contemporary music and a promoter of early music. In 1941, just 5 years before becoming a U.S. citizen, he composed *Ludus Tonalis*. This is a collection of 12 fugues with eleven interludes, framed by a prelude and a postlude. Such a collection of fugues reminds one immediately of Bach's *Well-Tempered Clavier*, although Hindemith also had more modern ideas of symmetry and symmetry breaking in mind. The title *Ludus Tonalis* means *Tonal Game* in Latin, and this can be seen in the symmetry and asymmetry of individual pieces as well as in the collection as a whole. One of the most striking features of the collection is that the postlude is exactly the same as the prelude except upside down and backwards!

Paul Hindemith also had an interesting life. An excellent website

$$http://www.hindemith.org/$$

on Hindemith describes his trips to Egypt, Turkey, and Mexico, his flight from the Nazis, and his emigration to the United States. The website also has historical photos and references to literature.

Hindemith's Fugue in G provides us with further examples of transposition and inversion, although we will encounter difficulties. This example will illustrate some of the difficulties that music theorists encounter and how they get around some of these difficulties. Hindemith's fugue will also be a warm-up for the next lecture on the PLR group. Professor Ramon Satyendra and I have recently created a theoretical apparatus to treat musical difficulties such as the one we are about to study.[5]

The Fugue in G begins with a statement of the subject as in Bach's Fugue in d-minor. The subject is

$$\langle G, G, G, G, G, G, G, C, D, G, C, F\rangle = \langle 7, 7, 7, 7, 7, 7, 7, 0, 2, 7, 0, 5\rangle$$

and consists of eleven notes and five pitches in two measures which have five beats each (all prime numbers!). The subject is very prominent because the repeated G at the beginning tells us the voice is entering. When we listen to the piece, we

[5]Thomas M. Fiore and Ramon Satyendra. "Generalized Contextual Groups." To appear in *Music Theory Online*.

can listen for that repeated staccato note and we will easily find occurrences of the subject. For example, one quickly hears that measures 3, 8, and 15 contain repeated notes which begin occurrences of the subject.

Instead of comparing occurrences of the subject as in Bach, we will look at a smaller unit. The two three-note groups at the end of the subject are also very prominent to our ears. In this analysis we will consider the relationships between these three-note groups in the various occurrences of the subject. These relationships are given by the conceptual categories transposition, inversion, and *contextual inversion*, which will enable us to find a good way of hearing the piece.

Let's consider the set S of all transposed and inverted forms of the pcseg

$$\langle G, C, D\rangle = \langle 7, 0, 2\rangle$$

which is the first occurrence of the three note pattern we want to study. Some examples of elements of S are

$$T_0\langle 7,0,2\rangle = \langle 7,0,2\rangle$$
$$T_1\langle 7,0,2\rangle = \langle 8,1,3\rangle$$
$$T_2\langle 7,0,2\rangle = \langle 9,2,4\rangle$$
$$\textit{etc.}$$
$$I_0\langle 7,0,2\rangle = \langle 5,0,10\rangle$$
$$I_1\langle 7,0,2\rangle = \langle 6,1,11\rangle$$
$$I_2\langle 7,0,2\rangle = \langle 7,2,0\rangle$$
$$\textit{etc.}$$

Notice that the elements of the set S are *ordered* sets, namely pcsegs. Two pcsegs are different if they have different orders. Thus, the pcsegs $\langle 7,0,2\rangle$ and $\langle 7,2,0\rangle$ are different elements of S.[6] The elements of S that are transpositions of $\langle 7,0,2\rangle$ are called *prime forms* and the elements of S that are inversions of $\langle 7,0,2\rangle$ are called *inverted forms.* So for example, $\langle 8,1,3\rangle$ is a prime form and $\langle 6,1,11\rangle$ is an inverted form.

There are several functions $S \to S$ that are important for our analysis. The transpositions $T_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ and inversions $I_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ induce functions $S \to S$ which we again denote by T_n and I_n. These "induced" transpositions and inversions are obtained by just applying T_n and I_n to each entry as we did above, so these are nothing new. Now we introduce a new function $J : S \to S$ as follows. We define $J(x)$ to be that form of the opposite type as x which has the same first two pitch classes as x but in the opposite order.[7] So for example $J\langle 7,0,2\rangle = \langle 0,7,5\rangle$ because $\langle 7,0,2\rangle$ and $\langle 0,7,5\rangle$ are opposite types of forms and they have the first two pitch classes in common but in switched order. Similarly, $J\langle 0,7,5\rangle = \langle 7,0,2\rangle$. The function $J : S \to S$ is an example of a *contextual inversion.* It is called contextual

[6]This is another difficulty that arises. The pcset $\{7,0,2\}$ is *inversionally symmetric*, so it is impossible to define the contextual inversion J on the unordered sets. One must define the contextual inversion J on the ordered pcsegs as we are about to do. Don't worry if you don't understand this, because you shouldn't. We thought about this problem for a while before understanding it.

[7]One might wonder why this is a well defined function, but that can be proved mathematically. You will just have to believe me that in this setup there is exactly one form of the opposite type as x that has the same first two pitch classes in common with x but in the opposite order.

because it inverts depending on the context of the first two pitch classes in the pcseg.

Let's see what the musical meaning of this J is. At first it seems to be defined in an unnecessarily complicated way, but actually it is a very musical function. The output is that opposite form that is closest (but not equal) to the input in the sense that they overlap in the first two pitch classes. This is quite audible. In the subject of the fugue for example, we have two three-note groups. Guess how they are related! Well, the three note groups are

$$\langle G, C, D\rangle = \langle 7, 0, 2\rangle$$

$$\langle C, G, F\rangle = \langle 0, 7, 5\rangle$$

and they overlap by two notes. We see also that they are of opposite types, namely $\langle 7, 0, 2\rangle$ is a prime form and $\langle 0, 7, 5\rangle$ is an inverted form. So $J\langle 7, 0, 2\rangle = \langle 0, 7, 5\rangle$ and $J\langle 0, 7, 5\rangle = \langle 7, 0, 2\rangle$. But there is just one catch. When we look at the actual music, we see that the second three-note group is $\langle G, C, F\rangle$ and not $\langle C, G, F\rangle$. The order isn't exactly right. This is one of the difficulties that the music theorist encounters. Nevertheless, the unordered pcsets $\{G, C, F\}$ and $\{C, G, F\}$ are the same. In other words, J is a good enough approximation for us to use and we can control the error by just looking at the unordered sets when we compare with the actual piece.

Now for the surprising part. Consider the following diagram.

(1)

$$\begin{tikzcd}
\bigcirc \arrow[r, "J"] \arrow[d, "T_5"'] & \bigcirc \arrow[d, "T_5"] \\
\bigcirc \arrow[r, "J"'] & \bigcirc
\end{tikzcd}$$

Let's fill in the top left circle with $\langle 7, 0, 2\rangle$ and see what happens.

$$\begin{tikzcd}
\langle 7, 0, 2\rangle \arrow[r, "J"] \arrow[d, "T_5"'] & \bigcirc \arrow[d, "T_5"] \\
\bigcirc \arrow[r, "J"'] & \bigcirc
\end{tikzcd}$$

Next we apply J and T_5 to the top left entry and find the results for the next two entries.

$$\begin{tikzcd}
\langle 7,0,2\rangle \arrow[r, "J"] \arrow[d, "T_5"'] & \langle 0,7,5\rangle \arrow[d, "T_5"] \\
\langle 0,5,7\rangle \arrow[r, "J"'] & {}
\end{tikzcd}$$

There is just one circle left. We can get at it from above or from the left. From above we get $T_5\langle 0,7,5\rangle = \langle 5,0,10\rangle$. From the left we get $J\langle 0,5,7\rangle = \langle 5,0,10\rangle$. So we fill it in.

$$\begin{tikzcd}
\langle 7,0,2\rangle \arrow[r, "J"] \arrow[d, "T_5"'] & \langle 0,7,5\rangle \arrow[d, "T_5"] \\
\langle 0,5,7\rangle \arrow[r, "J"'] & \langle 5,0,10\rangle
\end{tikzcd}$$

We see that the pathway doesn't matter! But does the pathway matter if we filled in the first circle of diagram (1) with a different element of S? No! Even if we fill in the circle with another element of S the pathway still does not matter. That can be proved mathematically. Since the pathway doesn't matter, we say that diagram (1) *commutes*.

But what in the world does this commutative diagram have to do with Hindemith's fugue? Let's look at the first two instances of the subject, one starts in measure 1 and the next one starts in measure 3. Applying T_5 to the first one gives us the second one. But in this fugue it is better to look at smaller units than the subject. These smaller units are what we are focusing on. So let's look at the four three-note groups in measures 2 and 4. Those are precisely the ones we filled in the diagram!! Even the temporal aspect matches! As time ticks, we move from the upper left circle to the lower right circle in the direction of the arrows. In fact, this diagram occurs in at least four different places of the piece! Compare your score from class. The measures are

2 and 4

9 and 16

37 and 39

55 and 57.

In all cases except one the measures are just two apart. Does Hindemith break the symmetry that one time to be playful in his tonal game? We can only wonder.

In summary, we have found a good way of hearing Hindemith's Fugue in G using the conceptual categories of transposition, inversion, contextual inversion, and commutative diagrams. This analysis was different than the analysis from Bach because we looked at units smaller than the entire subject. We investigated relationships between three-note groups in the subject and its many occurrences. The contextual inversion was important for these three-note groups because all adjacent ones overlap in a very audible way. The mathematical structure we found was not given by equations, but by relationships between three-note groups given in terms of commutative diagrams. In the next lecture we will study more three-note groups, namely major and minor chords. Transposition, inversion, and contextual inversion will make another appearance. The Beatles and Beethoven are expecting us!

Addendum to Lecture 1 Homework Problems

1. Introduction

1. How many years passed between Bach's birth and Hindemith's death?

2. Fugue by Hindemith

2. In which city in Pennsylvania did Hindemith conduct a symphony in 1959? Hint: use the link on the Hindemith website for *Life*.

3. Calculate

$$J\langle 3, 8, 10\rangle$$
$$J\langle 5, 10, 0\rangle$$
$$J\langle 9, 2, 4\rangle$$
$$J\langle 8, 3, 1\rangle$$
$$J\langle 10, 5, 3\rangle$$
$$J\langle 2, 9, 7\rangle.$$

Compare the answers to the first three with the answers for the second three. Do you see a pattern? Use this pattern to figure out $J \circ J(x)$.

4. Put the pcseg $\langle 2, 7, 9\rangle$ from measure 9 of Hindemith's fugue into the upper left circle of diagram (1) and calculate what goes in the other circles. Do both paths give you the same answer?

5. Put the pcseg $\langle 10, 3, 5\rangle$ from measure 37 of Hindemith's fugue into the upper left circle of diagram (1) and calculate what goes in the other circles. Do both paths give you the same answer?

6. Put the pcseg $\langle 0, 5, 7\rangle$ from measure 55 of Hindemith's fugue into the upper left circle of diagram (1) and calculate what goes in the other circles. Do both paths give you the same answer?

Challenge: We talked about commutative diagrams in Hindemith's tonal game. Let's consider a commutative diagram in our mathematical game. Look at the title of this page and compare it to the first three section headings in the table of contents: "Lecture1 Transposition and Inversion," "Lecture 1 Homework Problems," and "Addendum to Lecture 1." Which of the following is the meaning of the title of this page?

(Addendum to) (Lecture 1 Homework Problems)

(Addendum to Lecture 1) (Homework Problems)

In other words, does the playful diagram

$$\begin{tikzcd}
\text{Lecture 1} \arrow[r] \arrow[d] & \text{Lecture 1 Homework Problems} \arrow[d] \\
\text{Addendum to Lecture 1} \arrow[r] & \text{Addendum to Lecture 1 Homework Problems}
\end{tikzcd}$$

commute? Recall that a diagram commutes when both paths give you the same result. You may interpret the horizontal arrows as assigning (Homework Problems) and the vertical arrows as adding (Addendum to).

Lecture 2 The PLR Group

1. Introduction

A *group* is a mathematical object of central importance to music theorists. A group is yet another conceptual category that music theorists draw upon in order to make music more tangible. We have been working towards the concept of a group with our numerous examples of transposition and inversion. So far, we have looked at various instances of transposition and inversion in works by Bach, Wagner, and Hindemith. Now we look one level deeper and consider how the functions transposition and inversion interact with each other. This is a big step from considering just individual instances of transposition and inversion. We will see that the collection of transpositions and inversions form a group in the mathematical sense of the word. This group is called the T/I group.

Another group of musical relevance is the PLR group. This is a set of functions whose inputs are major and minor chords and whose outputs are major and minor chords. These musical functions go back to the music theorist Hugo Riemann (1849-1919). As a result, the PLR group is sometimes called the *neo-Riemannian* group. The PLR group and the T/I group are related in many theoretically interesting ways. Nevertheless, we will focus on musical examples. If you know how to play guitar, you might know the Elvis Progression I-VI-IV-V-I from 50's rock. Any song with this progression provides us with a musical example as we'll see below. We'll also look at a song from the Beatles on the *Abbey Road* album. A more striking example however is the second movement of Beethoven's Ninth Symphony. We will see that a harmonic progression in the Ninth Symphony traces out a path on a torus!

2. Mathematical Preliminaries

In this section we introduce the mathematical concept of a group and give some examples. A group is basically a set with a way to combine elements similar to the way that one multiplies real numbers. It will be a bit formal at first, but don't let that stop you from reading! Even if you don't fully understand it, keep going! It's not meant to be easy.

Definition 2.1. A *group* G is a set G equipped with a function $* : G \times G \to G$ which satisfies the following axioms.

(1) For any three elements a, b, c of G we have $(a * b) * c = a * (b * c)$, *i.e.* the operation $*$ is *associative*.
(2) There is an element e of G such that $a * e = a = e * a$ for every element a of G, *i.e.* the element e is the *unit* of the group.
(3) For every element a of G, there is an element a^{-1} such that $a * a^{-1} = e = a^{-1} * a$, *i.e.* every element a has an *inverse* a^{-1}.

That is the abstract definition of a group. Now let's consider some examples of groups to see what the definition really means.

EXAMPLE 1. Let G be the set of real numbers greater than zero. Some elements are $.5, 100, \pi, 7^{100}$, and $.000000008$ for example. Next let $*$ be the usual multiplication of numbers. From school we know that the multiplication of real numbers is associative. Let $e = 1$. From school we also know that any nonzero number times 1 is just that number again, no matter if we multiply on the left or right. For example, $1 \times \pi = \pi = \pi \times 1$. So G has a unit. If a is a real number greater than 0, then we can define $a^{-1} = 1/a$ in order to satisfy the last axiom. For example,

$$\pi^{-1} \times \pi = 1 = \pi \times \pi^{-1}.$$

We have just verified the axioms (1), (2), and (3) in the definition above. Hence the set of real numbers greater than 0 equipped with $* = \times$ is a group.

EXAMPLE 2. Let G be the set of whole numbers, *i.e.* $G = \{\dots, -2, -1, 0, 1, 2, \dots\}$. Let $*$ be the usual addition of whole numbers. Then $e = 0$ defines a unit because $0 + a = a = a + 0$ for any whole number a. If we take a^{-1} to be $-a$, then one can check that G equipped with $* = +$ satisfies the axioms (1),(2),(3) above and is thus a group.

EXAMPLE 3. Suppose $S = \{1, 2, 3\}$. Then let G denote the set of invertible functions $S \to S$. Don't worry about the exact meaning of invertible. Let $*$ be the function composition described in Lecture 1. Then two functions $f : S \to S$ and $g : S \to S$ can be composed to give $g \circ f = g * f$. From school we know that function composition is associative, *i.e.* $(h \circ g) \circ f = h \circ (g \circ f)$. An example of an element of G is the unit $e : S \to S$ defined by

$$e(1) = 1$$
$$e(2) = 2$$
$$e(3) = 3.$$

Another example of an element h of G is $h : S \to S$ defined by

$$h(1) = 3$$
$$h(2) = 2$$
$$h(3) = 1.$$

Its inverse $h^{-1} : S \to S$ is h. You can check that $h \circ h(x) = x$. One can check that G satisfies the axioms (1),(2),(3) above and is thus a group.

Even if you didn't understand every step of these examples, the main point is that a group is a mathematical object that consists of set G and an operation $*$ which gives us a way to combine elements. This combination of elements is similar to the usual multiplication of numbers in the sense that it is associative, has a unit, and has inverses. The three examples above give three different examples of groups. In each example we specified a set G and an operation $*$ on that set, and then checked that it satisfied the axioms of a group.[8] The last example was a warm up for the T/I group.

[8]Note that $G, *$, and e have different meanings in each example.

EXAMPLE 4. Let S be the set of transposed and inverted forms of the C major chord $\langle 0,4,7\rangle$. The elements of S can be listed as prime forms and inverted forms. For future reference we also record the letter names of the prime and inverted forms.

Prime Forms	Inverted Forms
$C = \langle 0,4,7\rangle$	$\langle 0,8,5\rangle = f$
$C\sharp = D\flat = \langle 1,5,8\rangle$	$\langle 1,9,6\rangle = f\sharp = g\flat$
$D = \langle 2,6,9\rangle$	$\langle 2,10,7\rangle = g$
$D\sharp = E\flat = \langle 3,7,10\rangle$	$\langle 3,11,8\rangle = g\sharp = a\flat$
$E = \langle 4,8,11\rangle$	$\langle 4,0,9\rangle = a$
$F = \langle 5,9,0\rangle$	$\langle 5,1,10\rangle = a\sharp = b\flat$
$F\sharp = G\flat = \langle 6,10,1\rangle$	$\langle 6,2,11\rangle = b$
$G = \langle 7,11,2\rangle$	$\langle 7,3,0\rangle = c$
$G\sharp = A\flat = \langle 8,0,3\rangle$	$\langle 8,4,1\rangle = c\sharp = d\flat$
$A = \langle 9,1,4\rangle$	$\langle 9,5,2\rangle = d$
$A\sharp = B\flat = \langle 10,2,5\rangle$	$\langle 10,6,3\rangle = d\sharp = e\flat$
$B = \langle 11,3,6\rangle$	$\langle 11,7,4\rangle = e$

A musician might notice from this labelling that S is the set of 24 major and minor chords.[9] We use capital letters to denote the letter names of major chords and lower case letters to denote the letter names of minor chords like musicians normally do. Transposition and inversion "induce" functions $T_n : S \to S$ and $I_n : S \to S$ by applying the function to each entry of the input pcseg. Then let G consist of the 24 functions $T_n : S \to S$ and $I_n : S \to S$ where $n = 0, 1, 2, \ldots, 11$. We also let $*$ be function composition as in the previous example. It can be mathematically verified that the transposition and inversion compose according to the following rules.

$$T_m \circ T_n = T_{m+n}$$

$$T_m \circ I_n = I_{m+n}$$

$$I_m \circ T_n = I_{m-n}$$

$$I_m \circ I_n = T_{m-n}$$

Here the indices are read mod 12. We see that the result of composing transpositions and inversions is also a transposition or inversion, so that $*$ really is an operation on G. Similarly, one can verify the axioms of a group and show that G forms a group. This group is called the T/I group.

We have taken a big step now to consider how the transpositions and inversions interact with each other. They interact with each other to form a group. This is deeper than just considering individual transpositions and inversions by themselves. This abstract structure of a mathematical group is of tremendous importance because it allows us to see things in music that we otherwise would not see.

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[9]Here *chord* just means a collection of pitch classes that are played simultaneously. The major and minor chords are special chords that are very prominent in Western music. Don't worry about where they come from or why they have these letter names, you can just take this list of prime forms and inverted forms for granted.

3. The PLR Group

We now introduce the PLR group as a group of functions $S \to S$ like the T/I group in Example 4. From now on we let S denote the set of prime forms and inverted forms of the C major chord $\langle 0,4,7\rangle$ as in Example 4. First we define functions P, L, and R with domain and range S. These three functions will be contextually defined just like J in the Addendum. Let $P(x)$ be that form of opposite type as x with the first and third notes switched. For example

$$P\langle 0,4,7\rangle = \langle 7,3,0\rangle$$
$$P\langle 3,11,8\rangle = \langle 8,0,3\rangle.$$

Let $L(x)$ be that form of opposite type as x with the second and third notes switched. For example

$$L\langle 0,4,7\rangle = \langle 11,7,4\rangle$$
$$L\langle 3,11,8\rangle = \langle 4,8,11\rangle.$$

Let $R(x)$ be that form of opposite type as x with the first and second notes switched. For example

$$R\langle 0,4,7\rangle = \langle 4,0,9\rangle$$
$$R\langle 3,11,8\rangle = \langle 11,3,6\rangle.$$

We also say that these functions are contextually defined because they are *not* defined on the individual constituents of the pcseg like T_n and I_n are.

These functions are highly musical. A musician would notice that P is the function that takes a chord and maps it to its parallel minor or major, *e.g.* P applied to C major gives us c minor and P applied to c minor gives us C major. The function L is a leading tone exchange for more theoretical reasons. It takes C major to e minor for example. The function R takes a chord to its relative minor or major, for example R applied to C major is a minor and R applied to a minor is C major. These three functions are also musical in the sense that they take a chord to another one that overlaps with the original one in two notes.

Definition 3.1. The *PLR group* is the group whose set consists of all possible compositions of P,L, and R. The operation is function composition.

For example, some elements of the PLR group are, P, L, R, $L \circ R$, $R \circ L$, $P \circ L \circ P$, $L \circ L$, and $R \circ L \circ P$. At first you might think that there are infinitely many ways to combine P, L, and R. But that is not true! In fact there are only 24 elements of the PLR group. For example, the elements $L \circ L$ and $R \circ R$ and the unit are the same, namely $L \circ L(x) = x = R \circ R(x)$ for all elements x of S. It can be mathematically proven that there are only 24 elements.[10] But this is all very abstract, so we need to get down to some actual musical examples.

[10]Many mathematical things can be proven about this group and the T/I group. For example, they are both isomorphic to the dihedral group of order 24. Of course I haven't told you what isomorphic or dihedral means, but philosophically it means that the T/I group and the PLR group are abstractly the same as the dihedral group! Now that is a surprise, since they first appear to be very different in their definitions. Another mathematical surprise is the following. Consider the group of all invertible functions $S \to S$. Then the T/I group is the centralizer of the PLR group in this larger group and vice-a-versa! This means philosophically that the two are "dual" in a musical sense described by David Lewin in his seminal work *Generalized Musical Intervals and Transformations*. This is very deep, and takes a long time to understand. So don't be discouraged at first!

4. Elvis and the Beatles

The Elvis Progression I-VI-IV-V-I from 50's rock can be found in many popular songs. It provides us with an example because it is basically the following diagram.

$$\bigcirc \xrightarrow{R} \bigcirc \xrightarrow{L} \bigcirc \xrightarrow{R\circ L\circ R\circ L} \bigcirc \xrightarrow{L\circ R} \bigcirc$$

If we put the C major chord $\langle 0, 4, 7\rangle$ into the left most circle and apply the functions, we get the progression C major, a minor, F major, G major, C major. This progression can be found for example in the 80's hit "Stand by Me."

The influential Beatles made their U.S. debut in 1964 on the Ed Sullivan show, just one year after the death of Paul Hindemith. From a temporal point of view, it is good to do an example from the Beatles too. The main progression of "Oh! Darling" from the album *Abbey Road* is E major, A major, E major, $f\sharp$ minor, D major, b minor, E major, b minor, E major, and A major.[11] Have a look at the handout for Lecture 2. Let's just focus on the inner four chords $f\sharp$ minor, D major, b minor, and E major. This is the progression we get when we insert $\langle 1, 9, 6\rangle = f\sharp$ minor into the first circle of the following diagram as in your homework!

$$\bigcirc \xrightarrow{L} \bigcirc \xrightarrow{R} \bigcirc \xrightarrow{P\circ R\circ L} \bigcirc$$

This shows that mathematical analysis can also be used for popular songs, not just "classical" music. Next we work our way towards the culmination of this module: Beethoven's Ninth Symphony and the path it traces out on a torus.

5. Topology and the Torus

Topology is a major branch of mathematics which studies qualitative questions about geometry rather than quantitative questions about geometry. Some qualitative questions that a topologist would ask about a geometric object are the following. Is the geometric object connected? Does it have holes? Does it have a boundary? For example, a circle and square are qualitatively the same because one can be stretched to the other.[12] Neither has a boundary and both have a hole in the center. Both are connected. A line segment with endpoints on the other hand, is qualitatively different from the circle. We cannot obtain a line segment from a circle by quantitative changes such as stretching, twisting, or shrinking. But we can obtain a line segment from a circle by the qualitative change known as cutting. A line segment with endpoints is qualitatively different from a circle because it has a boundary (two endpoints) and it has no holes. The circle and the line segment are each connected. Topology is not concerned with quantitative properties such as area, angles, and lengths, so topologists consider two objects the same if they only differ in quantitative ways. The square and the circle differ quantitatively, but are the same qualitatively. For this reason, topologists consider the square and the

[11] Actually there are some seventh chords in here and the first E major chord has an added C pitch class, but we'll just ignore that for the sake of simplicity. These seventh chords do help us make our point about overlapping chords though.

[12] Here we are considering the circle and the square without their insides. They are not shaded in. For example, the square we are talking about only consists of the four marks that make up the sides of the square.

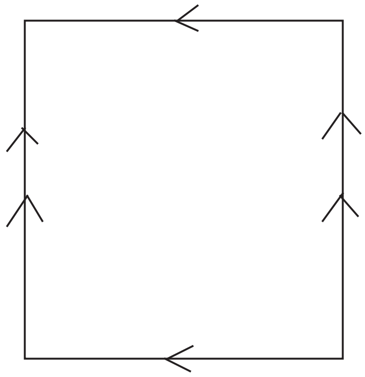


FIGURE 1. The Square Sheet.

circle to be the same. Topology is basically rubber-sheet geometry: we imagine two objects are made of rubber and consider them the same if one can be stretched, shrunk, or twisted into the shape of the other.[13]

The website

$$http://www.lehigh.edu/dmd1/public/www-data/essays.html$$

has several links to essays about the subject matter of topology. Some of the essays are more technical than others. Another great website

$$http://www.math.ohio-state.edu/\sim fiedorow/math655/yale/$$

is entitled "Math That Makes You Go Wow." This interactive website talks about philosophical, literary, musical, and artistic implications of topology.

The *torus* is an example of a mathematical object of interest to topologists. To make a torus we start off with the square sheet in Figure 1. Although it is not shaded in, we mean the whole rectangular region in Figure 1 belongs to the square sheet. The arrowheads indicate how we will glue. First we glue the horizontal lines according to the single arrowheads and obtain the cylinder in Figure 2. Next we glue the two circles at the end of the cylinder according to the double arrowheads and make the torus in Figure 3. The torus looks just like an inner tube filled with air.

The website "Math That Makes You Go Wow" mentioned above has an interactive torus. Go to the website and click on the link in Chapter 2 entitled "Orientable Surfaces: Sphere, Torus." Scroll down to the torus and move it around with your mouse to visualize it better. But what in the world does the torus have to do with the PLR group and Beethoven's Ninth Symphony?

6. Beethoven's Ninth Symphony

Ludwig van Beethoven (1770-1827) composed his Ninth Symphony during the years 1822-1824, roughly 80 years before Henri Poincaré initiated the study of

[13]Topology is different from topography, which is the study of the nature and the shape of terrain.

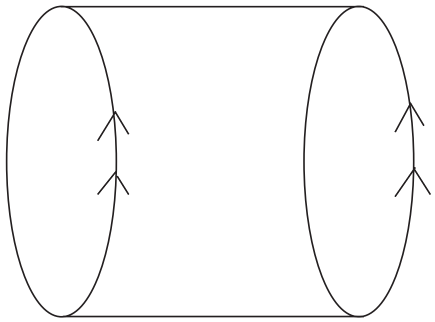


FIGURE 2. The Cylinder.

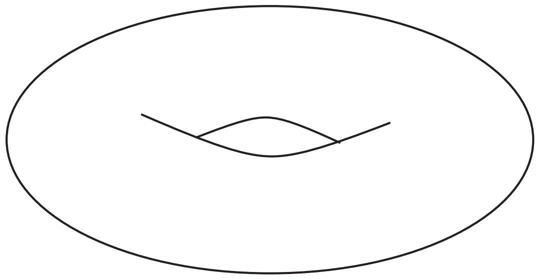


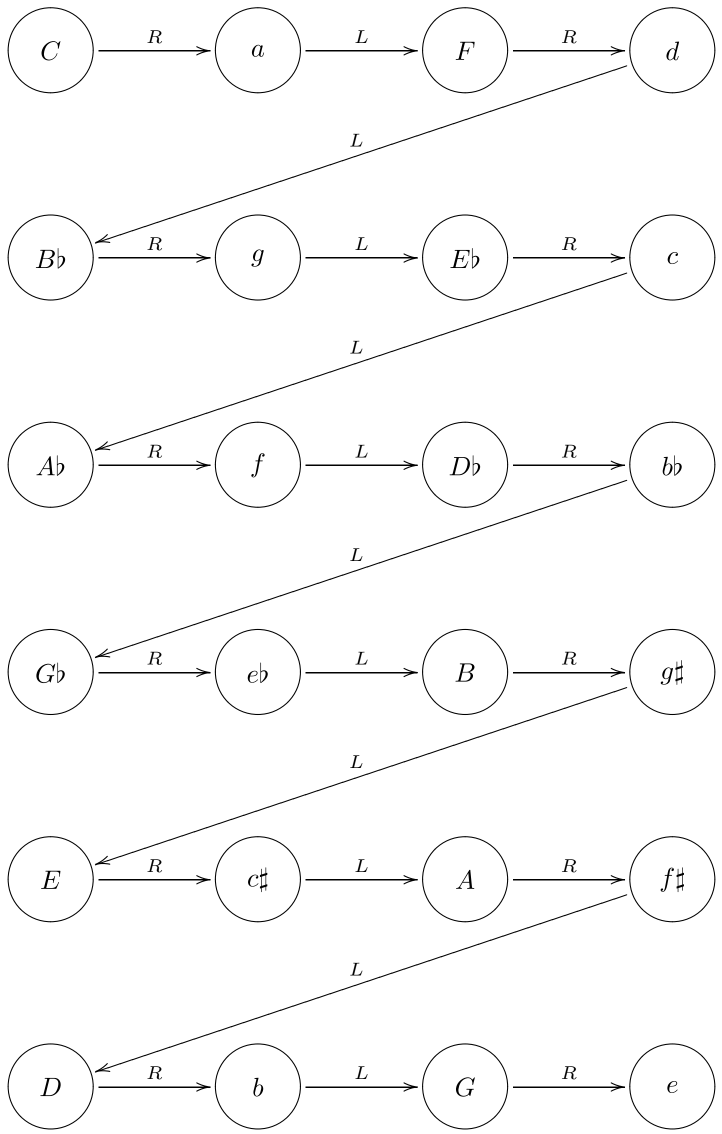
FIGURE 3. The Torus.

topology. In measures 143-176 of the second movement of the Ninth Symphony one can find the extraordinary sequence of 19 chords.

$$C, a, F, d, B\flat, g, E\flat, c, A\flat, f, D\flat, b\flat, G\flat, e\flat, B, g\sharp, E, c\sharp, A$$

Here again capital letters refer to major chords and lower case letters refer to minor chords.[14] The letter names of the chords can be converted back to numbers using the table of prime forms and inverted forms in Example 4. Note that the entire sequence can be obtained by applying to C the functions R, then L, then R, then L, and so on. In other words, we have the diagram below where the arrows are alternately labelled by R and L. Beethoven did not include the last five chords below in his composition, but we'll see why I wrote them below in a minute. Notice that all 24 major and minor chords appear below and none are repeated. This patten in itself is surprising.

[14]This sequence was first observed by Cohn in a series of articles dating back to 1991,1992, and 1997.



Let's consider the graph on the handout.[15] A *graph* is just a collection of dots called *vertices* and line segments called *edges* which connect some vertices. In this graph the vertices are labelled by the major chords and minor chords, *i.e.* elements of S. The edges are labelled by the functions R, P, and L. The edges labelled by R, P, and L are represented by dashed lines, solid lines, and dotted lines respectively. Two chords (vertices) are connected by a dashed line (*i.e.* by an edge labelled by R) if we can get from one to the other with the R function. The same goes for P and L. This graph is highly musical because the neighbors of a chord are exactly those three other chords that are maximally close to it, *i.e.* those

[15]This graph and the torus below it are a reproduction of the graph and torus in Jack Douthett and Peter Steinbach. "Parsimonious Graphs: A Study in Parsimony, Contextual Transformations, and Modes of Limited Transposition." *Journal of Music Theory* 42/2 (1998): 241-263.

three chords which overlap with the original one in two pitch classes. For example, the neighbors of D are $b, f\sharp, d$. In numbers, the neighbors of $\langle 2, 6, 9\rangle$ are

$$\langle 6, 2, 11\rangle$$
$$\langle 1, 9, 6\rangle$$
$$\langle 9, 5, 2\rangle,$$

all of which agree with $\langle 2, 6, 9\rangle$ in two pitch classes. That's not really a surprise because P, L, and R were designed to do precisely this.

What is surprising about this graph is that it makes a torus and the chord progression from Beethoven is a path on it! Let's see this. Notice that we can glue the top and the bottom because the top two rows of vertices match up with the bottom two rows. See the figures in the section on topology. Next we glue the circles on the resulting cylinder after twisting them a third of the way around. Here we are gluing those chords from the left side to those chords on the right side that are the same. For example, the $a\flat$ on the left side is glued to the $a\flat$ on the right side. The E on the left side is glued to the E on the right side and so on. We have to twist to get them to match up. So now we have our torus. To see that Beethoven's Ninth Symphony traces out a path on it, we just connect the dots which are labelled by the chord progression. Notice the pattern. If Beethoven had continued the pattern, it would trace out all of the 24 major and minor chords. Well, 19 out of 24 is pretty close though!

This example is perhaps the most striking of all our musical examples because it relates group theory, topology, and Beethoven all in one! These conceptual categories provide us with tools and a language to find an entirely new way of hearing this piece. Without mathematics we would never have heard a torus in Beethoven's Ninth Symphony. You might be interested to know that there are other examples of music on topological objects. Bach's *Musical Offering* contains a passage which is music on a Möbius strip. Schoenberg and Slonimsky also provide us with examples. See the website "Math That Makes You Go Wow" for further details.

7. Conclusion

In this module we have investigated some of the group theoretical tools that music theorists have developed to find good ways of hearing particular works of music. These tools provide us with conceptual categories to make our fleeting impressions of music into vivacious ideas. Our first conceptual categories were supplied by the transpositions and inversions. Bach's Fugue in d minor from the first book of the *Well-Tempered Clavier* had several examples of transposition and inversion. The subject made an appearance in various forms and these forms were describe by transposition and inversion. The *Tristan* Prelude in Wagner's opera *Tristan and Isolde* gave examples of transposed and inverted chords in the century after Bach's death. Hindemith's Fugue in G from the twentieth century had examples of contextual inversion within the subject. In that analysis it was fruitful to look at a unit smaller than the subject and to look for overlapping three note groups. Commutative diagrams also appeared in this context.

Our next example of a conceptual category was the concept of a group. The concept of a group is deeper than individual instances of transposition and inversion because it allows us to see a structure on the collection of transpositions and inversions. At that point, we fixed the notation S to mean the set of 24 major and minor triads and considered various invertible functions $S \to S$. Transposition and inversion induce such functions for example. The functions P, L, and R are also important functions $S \to S$. All possible compositions of these three functions give us the 24 elements of the PLR group. The PLR group makes an appearance in the Elvis Progression and in the Beatles song "Oh! Darling" from the *Abbey Road* album. The most striking of our musical examples however was in the second movement of Beethoven's Ninth Symphony. Repeated application of R and L to the C major chord generates a chord progression in measures 143-176 and this chord progression traces out a path on a torus.

I hope that the conceptual category of this module has made your impressions of mathematics, music, and music theory into vivacious ideas!

Lecture 2 Homework Problems

1. Introduction

2. Mathematical Preliminaries

1. Let $G = \mathbb{Z}_{12}$. Let $e = 0$ and $* = +$. Let's consider if this defines a group by answering the following questions about the axioms: Is it true that adding two integers mod 12 gives us another integer mod 12? Is it true that $0 + x = x = x + 0$ for any integer x mod 12? Is it true that $x - x = 0 = -x + x$ for any integer x mod 12? Try out $x = 1, 2, 3$ for example. We already know that addition is associative. Is $G = \mathbb{Z}_{12}$ a group with the above definitions?

3. The PLR Group

2. Calculate $P\langle 1, 5, 8\rangle$, $L\langle 10, 6, 3\rangle$, and $R\langle 9, 1, 4\rangle$.

4. Elvis and the Beatles

3. Insert $\langle 1, 9, 6\rangle$ into the left circle of the diagram displayed in the paragraph about "Oh! Darling" from the Beatles. Calculate the other three circles by applying the functions. Convert the pcseg numbers back to letters using the table of prime forms and inverted forms from Example 4. Does it match with the inner four chords of the chord progression for "Oh! Darling"?

5. Topology and the Torus

4. Are the triangle and the circle qualitatively the same? In other words, can we obtain one from the other by shrinking, stretching, or twisting?

5. Give one way in which a triangle and a line segment are qualitatively different. Hint: How many holes does each have?

6. Beethoven's Ninth Symphony

6. Calculate each of the following.

$$R\langle 0, 4, 7\rangle$$
$$L \circ R\langle 0, 4, 7\rangle$$
$$R \circ L \circ R\langle 0, 4, 7\rangle$$
$$L \circ R \circ L \circ R\langle 0, 4, 7\rangle$$

Next translate the results into chord names using the letters in the table in Example 4. How does this relate to the chord progression in the second movement of Beethoven's Ninth Symphony?

7. Conclusion

7. Which of our musical examples was your favorite and why? Write at least four sentences.

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