Beethoven and the Torus

Thomas Fiore

fioret@umich.edu

http://www.math.lsa.umich.edu/~fioret/

Consider This

- Mathematics and Music are intricately related.

- Natural scientists use mathematics to describe the physical world and to make predictions about it. Often, they use equations to do so.

- Can we use mathematics to describe music and make predictions about it? Can we find an equation to do this?

No!

Introduction

This talk will focus on

- mathematical tools of music theorists:
  transposition,
  inversion,
  the PLR group,
  and its graph

- in the context of my collaboration with Ramon Satyendra.

What is Music Theory?

- David Hume: impressions become tangible and form ideas.

- Music theory supplies us with conceptual categories to organize and understand music.

- In other words, music theory provides us with the means to find a good way of hearing a work of music.

Who Needs Music Theory?

- Composers
- Performers
- Listeners

Arithmetic Modulo 12

Think of a clock with 0 in the 12 o'clock position.

1+2=3 mod 12

11 +1 =0 mod 12

11+2 = 1 mod 12

11+ 5 = 4 mod 12

$\mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

The Integer Model of Pitch

$$
\begin{aligned}
C &= 0 \\
C\sharp = D\flat &= 1 \\
D &= 2 \\
D\sharp = E\flat &= 3 \\
E &= 4 \\
F &= 5 \\
F\sharp = G\flat &= 6 \\
G &= 7 \\
G\sharp = A\flat &= 8 \\
A &= 9 \\
A\sharp = B\flat &= 10 \\
B &= 11
\end{aligned}
$$

Bach's Fugue in d-Minor

Subject Q in Measure 1

$$\langle D, E, F, G, E, F, D, C\sharp, D, B\flat, G, A\rangle$$

$$\langle 2, 4, 5, 7, 4, 5, 2, 1, 2, 10, 7, 9\rangle$$

Form of Subject in Measure 3

$$\langle A, B, C, D, B, C, A, G\sharp, A, F, D, E\rangle$$

$$\langle 9, 11, 0, 2, 11, 0, 9, 8, 9, 5, 2, 4\rangle$$

The function $T_7 : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ is defined by $T_7(x) = x + 7$.

$$T_7 Q = \text{Measure 3}$$

Transposition by n is $T_n(x) = x + n$

Inversion about n is $I_n(x) = -x + n$

Major and Minor Chords

These are all obtained by transposing and inverting the C major chord $\langle 0,4,7\rangle$. Let S denote the set of 24 major and minor chords, *i.e.* the set whose elements are the following.

Prime Forms	Inverted Forms
$C = \langle 0,4,7\rangle$	$\langle 0,8,5\rangle = f$
$C\sharp = D\flat = \langle 1,5,8\rangle$	$\langle 1,9,6\rangle = f\sharp = g\flat$
$D = \langle 2,6,9\rangle$	$\langle 2,10,7\rangle = g$
$D\sharp = E\flat = \langle 3,7,10\rangle$	$\langle 3,11,8\rangle = g\sharp = a\flat$
$E = \langle 4,8,11\rangle$	$\langle 4,0,9\rangle = a$
$F = \langle 5,9,0\rangle$	$\langle 5,1,10\rangle = a\sharp = b\flat$
$F\sharp = G\flat = \langle 6,10,1\rangle$	$\langle 6,2,11\rangle = b$
$G = \langle 7,11,2\rangle$	$\langle 7,3,0\rangle = c$
$G\sharp = A\flat = \langle 8,0,3\rangle$	$\langle 8,4,1\rangle = c\sharp = d\flat$
$A = \langle 9,1,4\rangle$	$\langle 9,5,2\rangle = d$
$A\sharp = B\flat = \langle 10,2,5\rangle$	$\langle 10,6,3\rangle = d\sharp = e\flat$
$B = \langle 11,3,6\rangle$	$\langle 11,7,4\rangle = e$

Mathematical Groups

Definition 1 *A group G is a set G equipped with a function $*: G \times G \to G$ which satisfies the following axioms.*

1. *For any three elements a, b, c of G we have $(a * b) * c = a * (b * c)$, i.e. the operation $*$ is associative.*

2. *There is an element e of G such that $a * e = a = e * a$ for every element a of G, i.e. the element e is the unit of the group.*

3. *For every element a of G, there is an element a^{-1} such that $a * a^{-1} = e = a^{-1} * a$, i.e. every element a has an inverse a^{-1}.*

Examples of Mathematical Groups

Example 1 *The whole numbers* $\{\ldots, -1, 0, 1, \ldots\}$ *form a group with* $*$ *given by ordinary addition.*

Example 2 *The set* $\mathbb{Z}_{12}$ *is a group with* $*$ *defined as addition mod 12.*

Example 3 *There is a group whose elements are the 24 functions* $S \to S$ *given by* T_n *and* I_n *where* $n \in \mathbb{Z}_{12}$*. The operation* $*$ *is given by function composition.*

Example 4 *The* PLR *group will be defined next as a musical group of functions* $S \to S$*. The operation* $*$ *is given by function composition.*

The PLR Group

First we define functions $P, L, R : S \to S$.

- Let $P(x)$ be that form of opposite type as x with the first and third notes switched. For example

$$P\langle 0,4,7\rangle = \langle 7,3,0\rangle$$

$$P\langle 3,11,8\rangle = \langle 8,0,3\rangle.$$

- Let $L(x)$ be that form of opposite type as x with the second and third notes switched. For example

$$L\langle 0,4,7\rangle = \langle 11,7,4\rangle$$

$$L\langle 3,11,8\rangle = \langle 4,8,11\rangle.$$

The PLR Group

- Let $R(x)$ be that form of opposite type as x with the first and second notes switched. For example

$$R\langle 0,4,7\rangle = \langle 4,0,9\rangle$$

$$R\langle 3,11,8\rangle = \langle 11,3,6\rangle.$$

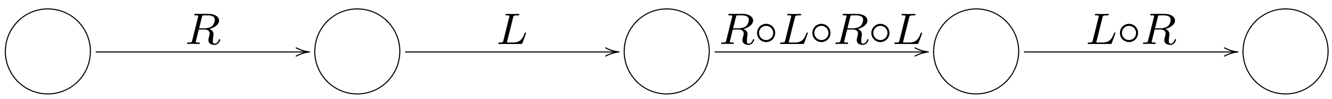
These functions are highly musical.

Definition 2 *The PLR group is the group whose set consists of all possible compositions of P,L, and R. The operation is function composition. This group is also called the neo-Riemannian group.*

The PLR Group

Theorem 1 *The neo-Riemannian PLR group has 24 elements and is dihedral.*

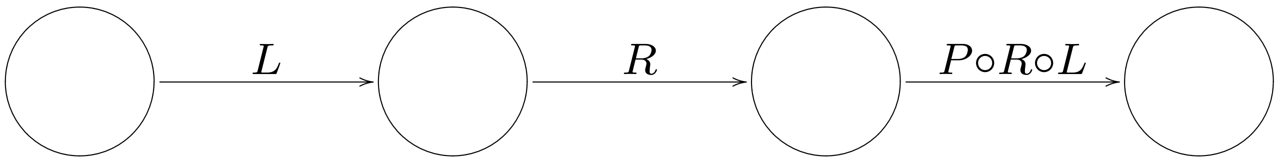
Example 5 *The Elvis Progression I-VI-IV-V-I from 50's Rock is.*



This can be found in "Stand by Me" for example.

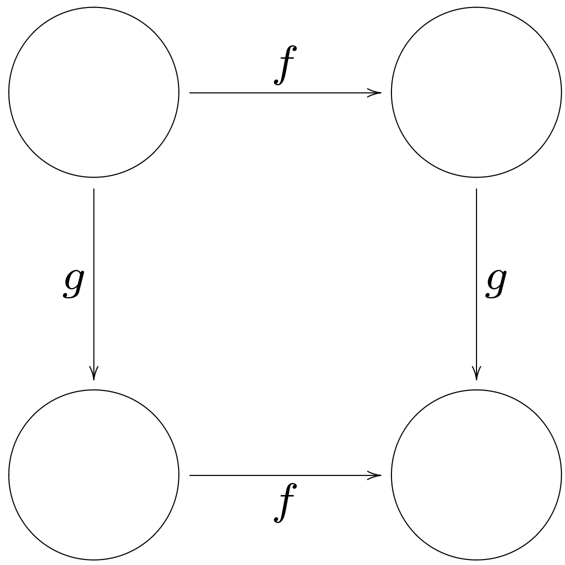
“Oh! Darling” from the Beatles

The progression $f\sharp$ minor, D major, b minor, and E major is obtained from the following application of the PLR group.



Dual Groups

The T/I group is *dual* to the PLR group in the sense that the diagram



commutes for any f in the PLR group and any g in the T/I group. Ramon Satyendra and I have generalized this beyond the set S of major and minor chords.

Topology and the Torus

Topology is a major branch of mathematics which studies qualitative questions about geometry. Two objects are qualitatively the same if one can be stretched, shrunk, or twisted into the other. Qualitative questions:

- Is the object connected?

- Does it have boundary?

- How many holes does it have?

Topology and the Torus

The triangle and circle are the same qualitatively, but they are different from the line segment.

Topology and the Torus

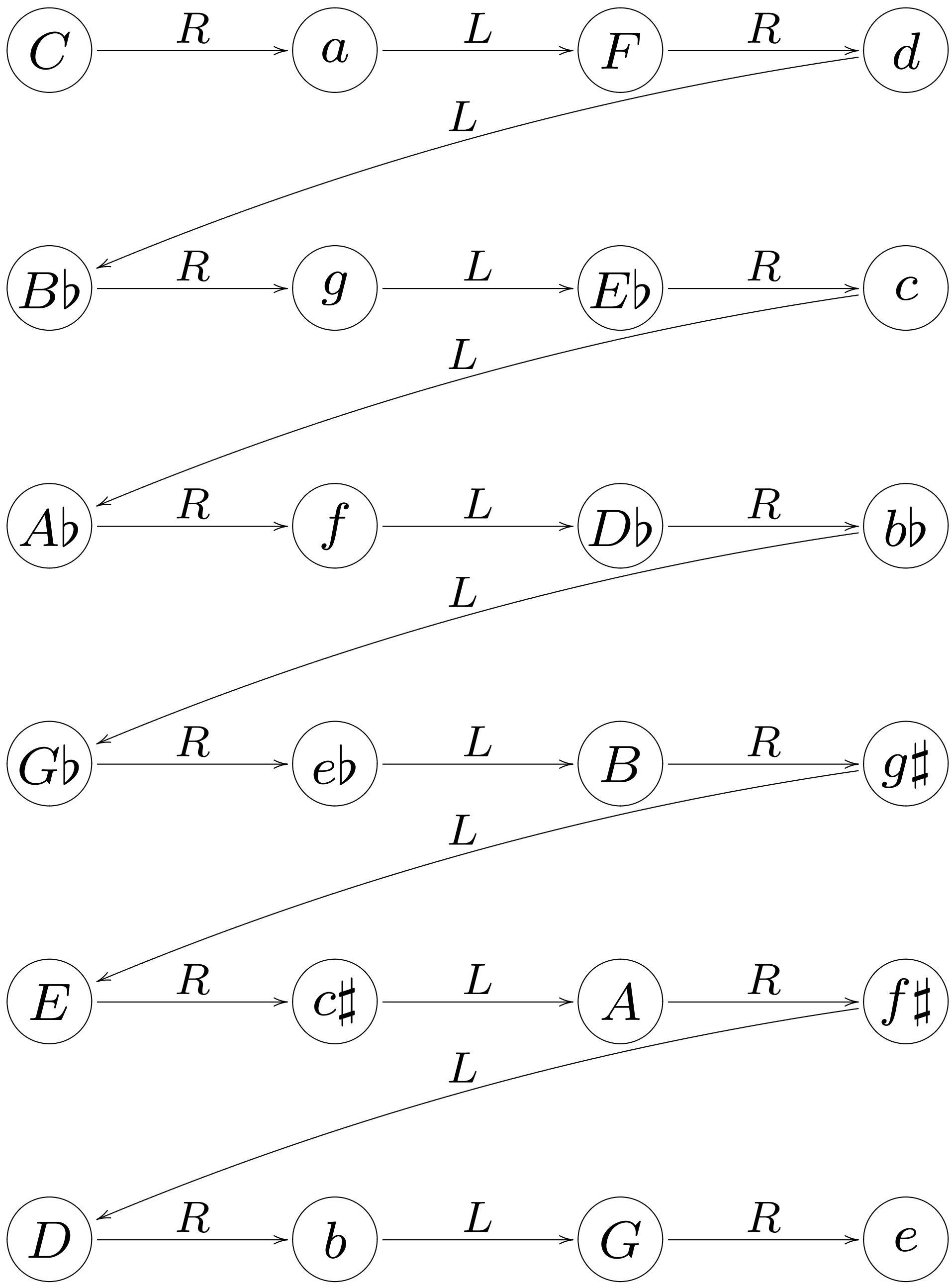
The torus is obtained from a square sheet by gluing the vertical edges and then the horizontal edges.

Topology and the Torus

Consider Douthett and Steinbach's PLR graph on the torus.

Beethoven's Ninth Symphony

The remarkable chord progression in measures 143-176 of the second movement of Beethoven's Ninth Symphony is:



Beethoven’s Ninth Symphony

This chord progression is precisely a path on Douthett and Steinbach’s torus! It nearly covers the whole torus without repeats!

Summary

- In this talk I have introduced some of the conceptual categories that music theorists use to make aural impressions into tangible ideas in the sense of Hume.

- The conceptual categories of interest to mathematicians were the PLR group, its associated graph, and the torus.

- We have used these tools to find good ways of hearing a fugue by Bach, a Beatles tune, and Beethoven's 9th Symphony.