Team:

Waves and Modes

Part I.   Standing Waves

Whenever a *wave* (sound, heat, light, ...) is confined to a *finite region of space* (string, pipe, cavity, ...) , something remarkable happens – the space fills up with a spectrum of vibrating patterns called "standing waves".   Confining a wave *"quantizes"* the frequency.

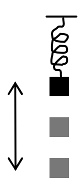
Standing waves explain the production of sound by *musical instruments* and the existence of *stationary states* (energy levels) in atoms and molecules. Standing waves are set up on a guitar string when plucked, on a violin string when bowed, and on a piano string when struck. They are set up in the air inside an organ pipe, a flute, or a saxophone. They are set up on the plastic membrane of a drumhead, the metal disk of a cymbal, and the metal bar of a xylophone. They are set up in the "electron cloud" of an atom and the "quark cloud" of a proton.

Standing waves are produced when you ring a bell, drop a coin, blow across an empty soda bottle, sing in a shower stall, or splash water in a bathtub. Standing waves exist in your mouth cavity when you speak and in your ear canal when you hear. Electromagnetic standing waves fill a laser cavity and a microwave oven. Quantum-mechanical standing waves fill the space inside carbon nanotubes and quantum computers.

A.   Modes

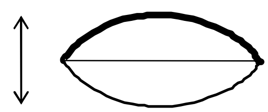
A mass on a spring has *one* natural frequency at which it freely oscillates up and down. A stretched string with fixed ends can oscillate up and down with a whole *spectrum* of frequencies and *patterns* of vibration.

Mass on Spring
(mass m, spring constant k)

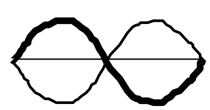


$$f_1 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

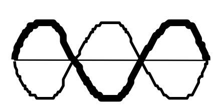
String with Fixed Ends
(length L, tension F, mass density μ)



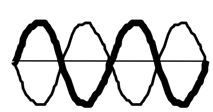
$$f_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}}$$ *Fundamental*



$f_2 = 2f_1$ 2nd *harmonic*

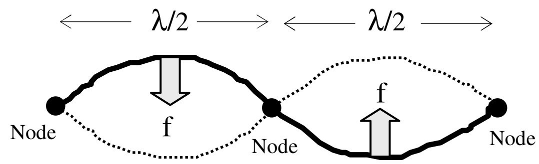


$f_3 = 3f_1$ 3rd *harmonic*



$f_4 = 4f_1$ 4th *harmonic*

These special *"Modes of Vibration"* of a string are called **STANDING WAVES** or **NORMAL MODES**. The word *"standing wave"* comes from the fact that each normal mode has *"wave"* properties (wavelength λ , frequency f), but the wave pattern (sinusoidal shape) does not travel left or right through space – it *"stands"* still. Each segment ($\lambda/2$ arc) in the wave pattern simply oscillates up and down. During its up-down motion, each segment sweeps out a *"loop"*.



A standing wave is a system of *fixed nodes* (separated by $\lambda/2$) and *vibrating loops* (frequency f). In short, a standing wave is a "flip-flopping" sine curve.

All points on the string oscillate at the *same frequency* but with different amplitudes. Points that do not move (zero amplitude of oscillation) are called *nodes*. Points where the amplitude is maximum are called *antinodes*. The mathematical equation of a standing wave is $y(x,t) = \sin(2\pi x/\lambda)\cos(2\pi ft)$. The "shape" term $\sin(2\pi x/\lambda)$ describes the sinusoidal shape of the wave pattern of wavelength λ. The "flip-flop" term $\cos(2\pi ft)$ describes the up-down oscillatory motion of each wave segment at frequency f. Each mode is characterized by a different λ and f.

B. Harmonics

The simplest normal mode, where the string vibrates in <u>one loop</u>, is labeled n = 1 and is called the **fundamental mode** or the **first harmonic**. The second mode (n = 2), where the string vibrates in <u>two loops</u>, is called the **second harmonic**. The n^{th} harmonic consists of n vibrating loops. The set of all normal modes {n = 1, 2, 3, 4, 5, ... } is the **harmonic spectrum**. The spectrum of natural frequencies is $\{f_1, f_2, f_3, f_4, f_5, \ldots\}$. Note that the frequency f_n of mode n is simply a <u>whole-number multiple</u> of the fundamental frequency: $f_n = nf_1$. The mode with 3 loops vibrates three times as fast as the mode with 1 loop.

Harmonics are the basis of <u>*HARMONY*</u> in music. The sectional vibrations of a string as *one whole*, *two halves*, *three thirds*, and so on, are very special because these vibrations produce musical tones that sound the "most pleasant" when sounded together, i.e. they represent the most *harmonious* combination of sounds. This explains the origin of the word "harmonic".

<u>Exercise</u>: Sketch the 6^{th} *harmonic* of the string.

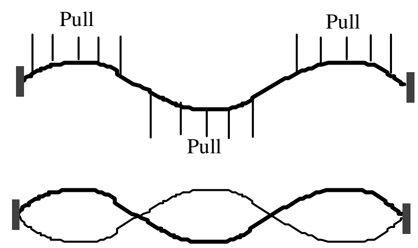
If the frequency of the 5^{th} *harmonic* is 100 *Hz*, what is the frequency of the 6^{th} *harmonic*?

If the length of the string is 3 *m* , what is the wavelength of the 6^{th} *harmonic*?

C. Creating a Mode

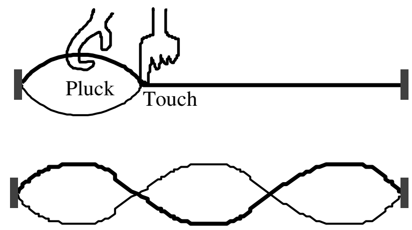
In general, when you pluck a string, you excite an infinite number of harmonic modes (we will discuss this later). How do you excite *only one* of the modes? There are three different methods:

1. <u>The Mathematician Method</u>: "Sine Curve Initial Condition"



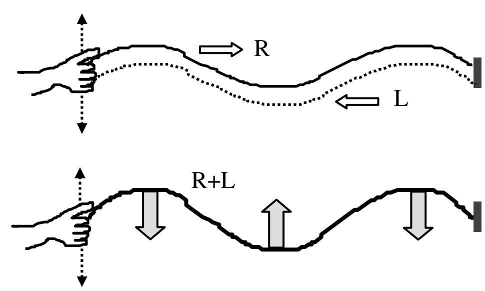
If you pull each mass element of a stretched string away from equilibrium (flat string) so that the string forms the shape of a *sine curve*, and then let go, the whole string will vibrate in one normal mode pattern. If you start with any other initial shape – one that is *not* sinusoidal – then the motion of the string will be made up of different modes. Starting out with an exact sine-curve shape is not easy to do – you need some kind of fancy contraption.

2. <u>The Musician Method</u>: "Touch and Pluck"



Guitar players and violin players do this all the time. They gently touch the string at one point (where you want the node to be) and pluck the string at another point (antinode) to make a "loop". The oscillation (up and down motion) of the plucked loop will *"drive"* the rest of the string to form additional equal-size loops which oscillate up and down at the same frequency as the driving loop.

3. <u>The Physicist Method</u>: "Resonance"



If you gently shake (vibrate) the end of a string up and down, a wave will travel to the right (R), hit the fixed end, and reflect back to the left (L). If you shake at just the right "resonance" frequency – one that matches one of the natural frequencies of the string – then the two *traveling waves* (R and L) will combine to produce *a standing wave* of large amplitude: R + L = STANDING WAVE.

D. Resonance

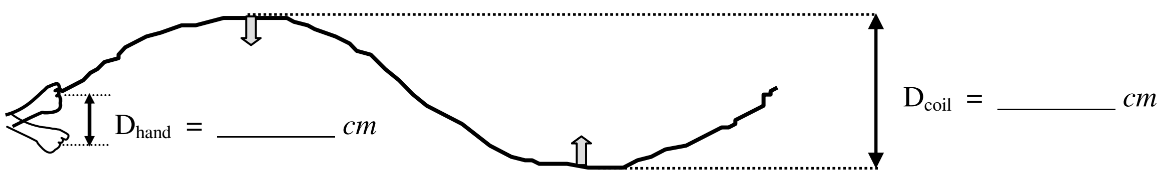
Since RESONANCE is one of the most important concepts in science, we will focus on this method. Resonance phenomena are everywhere: tuning a radio, making music, shattering a crystal glass with your voice, imaging the body with an MRI machine, picking cherries, designing lasers, engineering bridges, skyscrapers, and machine parts, etc. Consider pushing a person in a swing. If the frequency of your hand (periodic driving force) matches the natural frequency of the swing, then the swing will oscillate with large amplitude. It is a matter of <u>timing</u> , <u>not strength</u>. A sequence of *"gentle pushes"* applied at just the *"right time"* – in perfect rhythm with the swing – will cause a dramatic increase in the amplitude of the swing. A small stimulus gets amplified into a LARGE response.

<u>Experiment</u>: *Shake a Slinky , Make a Mode*

Go into the hallway. Stretch the slinky (in the air, not on the floor) so that its end-to-end length is 4 to 5 *m*. It is okay if the slinky sags a little. Keep one end fixed. <u>GENTLY</u> shake (vibrate) the other end *side-to-side* (side-to-side is easier than up-and-down). <u>DO NOT</u> shake with a large force or large amplitude. Remember that resonance is all in the *timing*, not the force! Shake at just the *right frequency* to produce the two-loop (n = 2) normal mode. When you have *"tuned into"* this n = 2 state, take special note:

You are now RESONATING with the coil. You and the coil are "one" !

Your shaking hand is perfectly in-sync with the coil's very own natural vibration. The frequency of your hand matches f_2 of the coil. Note how the distance D_{hand} that your hand moves back and forth is much smaller than the distance D_{coil} that the coil moves back and forth. The driven slinky system has the ability to *amplify* a small input (D_{hand}) into a large output (D_{coil}). This is the trademark of resonance: a weak driving force (hand moves a little) causes a powerful motion (coil moves a lot). Estimate the values of D_{hand} and D_{coil} by simply observing the motions of your hand and the coil when you are resonating with the coil. Compute the *"Amplification Factor"* for this driven coil system.



Amplification Factor : D_{coil} / D_{hand} = __________ .

Part II. Music, Guitar, & Fourier

One of the most far-reaching principles in theoretical physics is this:

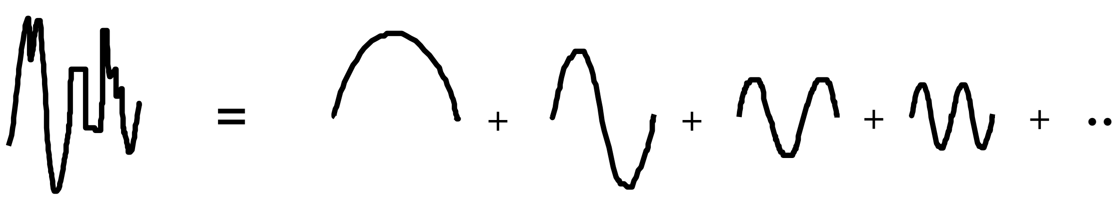
The general motion of any vibrating system can be represented as a sum of mode motions.

Reducing the complex "generic whole" into a set of simple "harmonic parts" provides deep insight into nature. Therefore, normal modes are important for two reasons:

(1) The motion of each mode is <u>SIMPLE</u> , being described by a *simple harmonic oscillator* (trig) function: ***cos(2πft)*** .

(2) The set of modes serves as a <u>BASIS</u> for any kind of wave/vibrational motion:

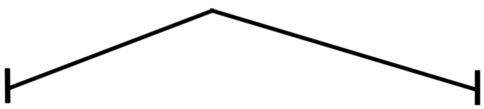
Any Motion = Sum of Mode Motions



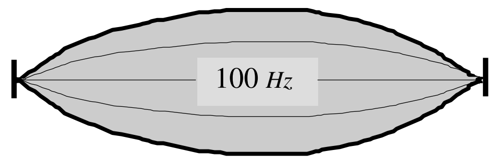
A. Plucked String = Mixture of Harmonic Parts

Suppose you pluck the "A-string" of a guitar and look closely at the shape of the vibrating string. You will see the fundamental mode (1st harmonic) "flip flopping" at the rate of about 100 vibrations per second. At such a high frequency, the string looks like one big blurred loop:

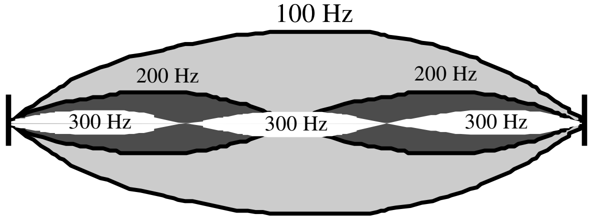
Initial Pluck
Pull string at one point and release

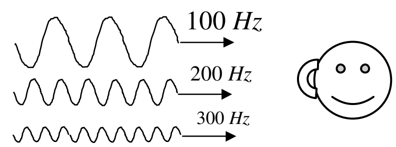


Resulting Motion
"Flip-Flopping Loop"



However, what you see is **NOT THE WHOLE PICTURE !** In reality, the string is vibrating with a <u>whole spectrum</u> of normal mode shapes and frequencies all at the <u>same time</u>: 1-loop @ 100 *Hz* , 2-loop @ 200 *Hz* , 3-loop @ 300 *Hz* , 4-loop @ 400 *Hz* , etc. The "actual" picture of a plucked string (showing only the first three harmonics) is:





Sound Waves (produced by vibrating loops colliding with air molecules)

> In the language of *Music*, when you pluck a guitar string, the resulting musical tone consists of the **<u>Fundamental Tone</u>** (frequency f) together with a whole series of higher pitched and generally fainter **<u>Overtones</u>** (2f , 3f , 4f , etc.)

It is difficult to see the 2-loop, 3-loop, 4-loop vibrations , … and to hear the corresponding overtones 2f , 3f , 4f , … because the higher-harmonic *amplitudes* are usually much smaller than the 1^{st} harmonic *amplitude*. The 1-loop vibration is the shape you tend to see (and the tone you tend to hear) because it is the "loudest" component in the harmonic mixture – the one that "shakes" the air and your eardrum the most. If you listen carefully in a quiet room (no background noise) to the plucked A-string of a guitar, you can hear the first overtone 200 *Hz* (second harmonic) which is one **"Octave"** above the fundamental tone 100 *Hz*. Some musicians can hear several overtones.

PHYSICS OF MUSIC FACT: When a guitar, piano, violin, and saxophone play the same note (middle C for example) at the same loudness, you hear four different sounds. Why ? Although the frequency of the fundamental tone (f_1 = 262 Hz) is the same, the *intensities* of the overtones (f_2 = 524 *Hz* , f_3 = 786 *Hz* , f_4 = 1048 *Hz* ,…) are different for each instrument.

Thought Experiment: A simple proof of *"Plucked String = Mixture of Harmonic Modes"*

A plucked guitar string is vibrating in the lowest two modes: 100 *Hz fundamental* and 200 *Hz overtone* (*octave*). Suppose you lightly touch this vibrating string at its exact midpoint. What happens? Draw a picture of the plucked string before and after the touch:

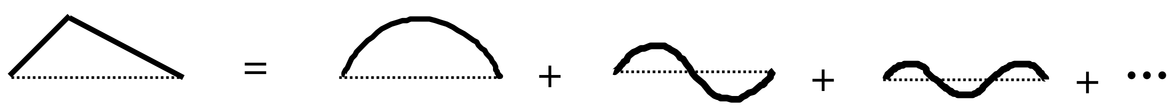
Before *After*

What happens to the fundamental loop when you touch the string?

What tone do you hear after the touch?

B. Fourier's Theorem. Harmonic Analysis.

How do you "really know" that a plucked string is made up of a bunch of harmonic parts:



Fourier's Theorem says that any periodic motion can be represented as a sum of simple harmonic motions. *"Fourier analysis"* or *"harmonic analysis"* refers to the decomposition of a vibration (periodic function) into its harmonic components (sine functions). Fourier analysis permeates all of science and engineering. This far-reaching mathematical tool is used in applications ranging from building a music (Moog) synthesizer, to designing circuits, to analyzing waveforms, to processing voltages in electronics, to processing light in optical systems, and to calculating a quantum jump in atomic systems.

Computer Experiment: Adding Harmonics

You will explore how just the right mixture of smooth harmonic modes (sine waves) can produce the pointed shape of a plucked string (triangular wave). Start the program Logger Pro and open the file *"Fourier"*. You will see three graphs showing the first three *base shapes* (harmonics) of a guitar string of length 100 *cm*. Note that the first harmonic y_1 is "loud", while the second and third harmonics (y_2 and y_3) are "fainter". Use the *Examine Icon* [x=?] to find the *amplitudes* (A_1 , A_2 , A_3) and the *wavelengths* (λ_1 , λ_2 , λ_3) of each harmonic.

A_1 = ____________ . A_2 = ____________ . A_3 = ____________ .

λ_1 = ____________ . λ_2 = ____________ . λ_3 = ____________ .

Use the *Examine Icon* to find the y-values of the harmonic functions for each of the x-values listed in the following table. Use your calculator to sum the y-values.

Fourier Analysis Table

Point on String x (*cm*)	1^{st} Harmonic y_1 (*cm*)	2^{nd} Harmonic y_2 (*cm*)	3^{rd} Harmonic y_3 (*cm*)	**Sum of Harmonics** $y_1 + y_2 + y_3$ (*cm*)
0				
5				
10				
15				
20				
25				
50				
70				
90				
100				

Use Graphical Analysis to plot $y_1 + y_2 + y_3$ (vertical axis) versus x (horizontal axis). Change the y-axis range to go from 0 to 50. Draw a connect-the-dots line through the points. PRINT this *Sum-Over-Harmonics* graph.

<u>Where is the Guitar String Plucked</u>?

Based on the shape of your *Sum-Over-Harmonics* graph,
what point on the string ($0 < x < 100$) is pulled and released? x = ________________ .

Fourier Essence. Note how each harmonic part (y_1, y_2, y_3) consists of sine curves (smooth and round). In sharp contrast, note how the sum of the harmonics ($y_1 + y_2 + y_3$) forms a triangle (straight lines and sharp point). This is a remarkable feature of adding sine curves:

By compounding simple sine waves, you can generate square, triangular, and sawtooth waves !

As you add more and more harmonics, the edges become sharper and the lines become straighter. Open the file *"Adding Sines"* to see the sine curves y_1, y_2, y_3 plotted on the same graph. PRINT this graph. Carefully look at y_1, y_2, y_3 to understand the *"Fourier Essence"*. Note how in a certain region all the sine curves bend in the same direction. When this happens, the *curvatures mutually reinforce each other* producing a huge curvature, that is an acute angle (sharp point). In other places, the curves bend in opposite directions – the *curvatures mutually weaken each other* – so that a straight line results.

(1) Mark these two special regions on your printed graph of the sine curves y_1, y_2, y_3.
(2) Label one region *"Reinforcing Curvatures"* and the other *"Canceling Curvatures"*.
(3) Sketch $y_1 + y_2 + y_3$ on this graph.

Part III. Discovering a Harmonic Spectrum

A. Experiment: Measuring the Spectrum

Use the slinky as before – keep one end fixed, make the length 4 to 5 *meters*, shake the other end. Record the length: L = __________ *m*. Shake at the *"right frequencies"* to create the first three harmonics n = 1 , 2 , 3 . Note: the uncertainty in measuring the integer n is exactly equal to zero: n ± 0 !

Measure the period of each harmonic. Note that the period of the "flip-flopping" loops on the slinky is equal to the period of your shaking hand. Use a *stopwatch* to measure the time for ten "flip flops" and then divide by ten to get the period. Sketch the shape of each mode you observed. Record your measured values of the period and the frequency of each mode.

Mode Number	Mode Shape	Period (*s*)	Frequency (*Hz*)
1			
2			
3			

Do your measured frequencies satisfy the harmonic law $f_n = nf_1$? Explain.

B. Theory: Calculating the Spectrum

So far, you have *measured* the spectrum of natural frequencies of the slinky system. You will now *calculate* this spectrum based on the theory of waves. There are three basic ingredients that make up the theory of wave motion:

Kinematics: $v = \lambda f$ (since *velocity* = *distance* over *time* = λ/T).

Dynamics: $v = (F/\mu)^{1/2}$ (from solving F = *ma* for a string with *tension* F and *mass density* μ).

Harmonics: $n(\lambda/2) = L$ (Mode n consists of *"n half-wavelengths just fitting inside length* L*"*).
For a proof of this "fitting condition", see the mode pictures on page 1.

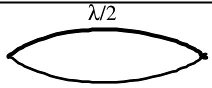
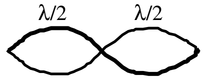
Note: Even though a standing wave does not travel, the concept of velocity still makes sense because when you shake a string, you generate a traveling wave that propagates along the string. The velocity v of this traveling wave (whose back-and-forth motion forms the standing wave) defines the "v" that appears in the standing wave equation $f_n = nv/2L$.

Derive the Frequency Formula. In the space below, combine the Kinematic relation $v = \lambda f$ with the Harmonic relation $n(\lambda/2) = L$ to find how the frequency f of mode n depends on n , v , and L.

f = ________________ .

The following table summarizes the mode physics of a vibrating string (showing only n=1,2,3).

Fill in the four blank boxes:

Mode	Wavelength (n ½-waves = string length)	Frequency $f = v/\lambda$
1st Harmonic 	**1** $\lambda/2 = L$	
2nd Harmonic 		$f_2 =$ **2** $v/2L$
3rd Harmonic	**3** $\lambda/2 = L$	

Bottom Line: The overtones are integer multiples of the fundamental: $f_n = nf_1$, where $f_1 = v/2L$. So to calculate the entire theoretical spectrum (an *infinite* number of frequencies f_n), all you need to know are the values of *two* kinematic quantities: L and v !

Record L. Length of stretched slinky on which you observed n=1,2,3: L = __________ *m* .

Find v. The velocity of a wave on a string (or coil) does not depend on the shape of the wave. All small disturbances propagate along the string at the same speed. So the best way to measure the value of v is to make a wave which is easiest to observe. The simplest kind of wave is a *single pulse*.

Stretch the coil as before so that it has the same length L. Keep *both* ends fixed. At a point near one end, pull the slinky sidewise (perpendicular to the length) and release. This will generate a *transverse traveling wave*. Observe this wave disturbance (pulse) as it travels down the slinky, reflects off the other end, and travels back to its point of origin. Use a *stopwatch* to measure the time it takes for the pulse to make four round trips, i.e. 4 down-and-back motions along the entire length of the coil. Divide by 4 to get the round-trip time t_r. Calculate the velocity v of the wave on your coil using the fact that the wave pulse travels a distance 2L (down L + back L) during the round trip time.

Round-Trip Time t_r = ____________ *s* . Wave Velocity v = ____________ *m/s* .

Calculate f. From your measured values of L and v, calculate the harmonic frequencies f_1, f_2, f_3 using the *theoretical* relation that you derived above. Show your calculation:

C. Compare Experimental and Theoretical Spectrum.

Summarize your measured and calculated values of the natural frequencies $\{f_1, f_2, f_3, ... \}$ that characterize the slinky system.

n	f_n (*experiment*)	f_n (*theory*)	% Difference
1			
2			
3			

Part IV. Precision Measurement of "Proper Tones" and Wave Speed

The system you will study consists of a string of finite length under a fixed tension. One end of the string is connected to an electric oscillator – a high-tech "shaker". Instead of *you* shaking the string by hand, the electric oscillator moves the string up and down at a frequency that you can control with *high precision*. The tension in the string is due to a precisely known mass hanging on the other end. Your *"research goal"* is to answer two basic questions:

1. What are the *natural frequencies* of this string system?
2. What is the *speed* of a wave on this system?

A. Frequency Spectrum

Just like you "tune in" to different radio stations, here you will tune in to the natural "broadcast frequencies" of the string.

Hang 150 *grams* from the end of the string.

Set up the standing waves $n = 1, 2, 3, 4, 5$ by driving the string with the electric oscillator using the following procedure.

Procedure for Resonating with the String: Start with the frequency of the oscillator equal to 1.0 *Hz*. Set the amplitude of the oscillator equal to one-half its maximum value. Look at the end of the string attached to the oscillator. It is going up and down once every second – just like your hand shaking the slinky! Now slowly increase the frequency until the first ($n = 1$) resonance is achieved where the string forms one big "flip-flopping" loop. If the oscillator makes a rattle sound, then decrease the amplitude. Continue to increase the frequency and successively *"tune into"* each of the overtones $n = 2, 3, 4, 5$. Record your measured values of f_1, f_2, f_3, f_4, f_5 :

n	1	2	3	4	5
f_n (*Hz*)					

Do these natural frequencies (*proper tones*) of the string system form a *harmonic spectrum*? *Explain*.

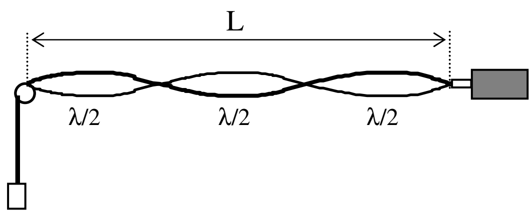
B. Wave Speed

The wave speed on the string is too fast to measure with a stopwatch. "Flick" the string and try to observe the traveling pulse. Good Luck. Setting up standing waves is the gold-standard technique to measure the speed of any kind of wave – from sound in air (300 *m/s*) to light in a vacuum (300,000,000 *m/s*). When you set up a standing wave, you are in effect "freezing" (taking a snapshot of) the speeding wave so you can easily observe the *shape* of the wave and measure the *wavelength*.

To get an accurate value of the wave speed v, find v using *two* different methods:

1. Space - Time Method: Measure λ and f for each mode n. Compute $v = \lambda f$.

You have already measured f for each n. To find λ for each n , measure the length L of your string (between it's two "fixed" endpoints) and then compute λ from the Fitting Condition: *mode* n *consists of n half-waves stuffed inside* L. The picture below shows the n = 3 fit.



L = ____________ *m* .

n	λ (*m*)	f (*Hz*)	$v = \lambda f$ (*m/s*)
1			
2			
3			
4			
5			

average v = ____________ *m/s.*

2. Force - Mass Method: Measure F and μ of string. Compute $v = (F/\mu)^{1/2}$.

The *tension* F in the string is equal to the weight of the hanging mass. To find the *mass density* μ of the string ($\mu \equiv$ *mass* per unit *length*), use the sample string on the back table. Do not remove the string attached to your oscillator and weight.

Note: The sample string came from the same spool of string as the actual string and therefore has the same value of μ. The sample string may have a larger *mass* and a larger *length* than the actual string , but the *mass-per-length* ratio is the same.

Tension: F = ________ *N* .

Mass density: $\mu \equiv$ mass / length = (*kg*) / (*m*) = _______________ *kg/m.*

Wave Speed: $v = (F/\mu)^{1/2}$ = ______________ *m/s* .

Compare your two results for v [*space-time* λf and *force-mass* $(F/\mu)^{1/2}$]:

Part V. Design Project: Create a 3-Loop Flip-Flop @ 57 Hz

The system consists of your stretched string, electric oscillator, and hanging weight.
The application requires that you set up a *3-loop* standing wave pattern on the string that vibrates at the rate of 57 *cycles per second*. The system parameter that you can change is the *tension* in the string.

The Theory

First work out the theory, then do the experiment. Recall the three essential elements of wave theory: (1) *Kinematics*: $v = \lambda f$. (2) *Dynamics*: $v = \sqrt{F/\mu}$. (3) *Harmonics*: $n\lambda/2 = L$.

The *string* parameters are length L , tension F , mass density μ , number of loops n.
The *wave* properties are velocity v , frequency f , wavelength λ .

1. Carefully *derive* the formula that gives F as a function of the four parameters: n , L , f , μ .

F =

.

2. Write the numerical values of the parameters:

   n = ____ . L = __________ *m.* f = _______ *Hz.* μ = ___________ *kg/m.*

3. Substitute these values into your F formula to find the tension value: F = ___________ *N* .

4. Compute the value of the hanging mass: m = __________ *kg* .

The Experiment

1. Set the tension in the string by hanging the amount of mass as predicted by your theory.
2. Turn on the oscillator.
3. Vary the frequency of the oscillator so that the 3-loop mode is established on the string *"loud"* and *clear*, i.e. fine tune the frequency until the loop amplitude is maximum.
4. Record the value of this 3-loop frequency: f_3 = ____________ *Hz* .
5. Compare this experimental value of f_3 with the *Design Specs* value of 57 *Hz*.

% difference between f_3 and 57 *Hz* is _________ % .